MATH. SCAND. 52 (1983), 145–160

AN EXTENSION OF A THEOREM OF F. FORELLI

HIROSHI YAMAGUCHI

1. Introduction.

The classical F. and M. Riesz theorem is stated as follows: Let μ be a bounded regular measure on the circle group T. If

$$\begin{aligned}\hat{\mu}(n) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-inx}\,d\mu(x)\\ &= 0 \qquad (n<0)\,,\end{aligned}$$

μ is absolutely continuous with respect to the Lebesgue measure. The same result is satisfied for the reals R.

Forelli in [4] extended this theorem to the n-dimensional Euclidean space R^n. That is,

THEOREM 1.1 (cf. [4; Theorems 3 and 4], [10; 6.2.2. Theorem, p. 140]). *Suppose S is a compact set of unit vectors in the interior of R^n_+ and F is a Borel set in R^n with S-width zero. Let μ be a bounded regular measure on R^n such that $\hat{\mu}$ vanishes on R^n_-. Then we have $|\mu|(F)=0$.*

Moreover he proved the following in [4].

THEOREM 1.2 (cf. [4; Theorem 2], [10; 6.2.2. Theorem (b)]). *Suppose S is a compact set of unit vectors in R^n_+ and F is a Borel set in R^n with S-width zero. Let μ be a bounded regular measure on T such that $\hat{\mu}$ vanishes on Z^n_-. Then $|\mu|(\varphi(F))=0$, where φ is the canonical map from R^n onto T^n.*

On the other hand, deLeeuw and Glicksberg in ([2; Theorem 3.1, p. 186]) extended the F. and M. Riesz Theorem to a compact abelian group. In this paper we extend above Forelli's theorems to a LCA group. First we state our results..

THEOREM 1.3. *Let G be a* LCA *group and ψ a continuous homomorphism from $\hat{G}$ into R^n such that $\psi(\hat{G})$ contains e_k $(1\leqq k\leqq n)$. Let φ be the dual homomorphism*

Received December 28, 1981.

of ψ. For each $x \in G$, let S_x be a compact set of unit vectors in the interior of $\mathbf{R}^n_+$ and put $S=\{S_x\}_{x \in G}$. Let F be a Borel set in G with S-width zero in the direction of φ. Then we have $|\mu|(F)=0$ for every measure $\mu \in M(G)$, whose Fourier–Stieltjes transform vanishes on $\{\gamma \in \hat{G};\ \psi(\gamma) \in \mathbf{R}^n_-\}$.

COROLLARY 1.4. *Let G be a LCA group and ψ a continuous homomorphism from $\hat{G}$ into $\mathbf{R}^n$ such that $\psi(\hat{G})$ contains e_k $(1 \leqq k \leqq n)$. Let χ_k be an element in $\hat{G}$ such that $\psi(\chi_k)=e_k$, Λ a discrete subgroup of $\hat{G}$ generated by χ_k $(1 \leqq k \leqq n)$ and K the annihilator of Λ. Let S be a compact set of unit vectors in the interior of $\mathbf{R}^n_+$ and F a Borel set in $\mathbf{R}^n$ with S-width zero. Then $|\mu|(\varphi(F)+K)=0$ for every measure $\mu \in M(G)$, whose Fourier–Stieltjes transform vanishes on $\{\gamma \in \hat{G};\ \psi(\gamma) \in \mathbf{R}^n_-\}$, where φ is the dual homomorphism of ψ.*

REMARK 1.5. In Corollary 1.4, we note that $\varphi(F)+K$ is a Borel set in G on account of Proposition 2.6 in section 2.

Let G be a LCA group with the dual group $\hat{G}$. We denote by m_G the Haar measure on G. $M(G)$ is the Banach algebra of bounded regular measures on G under convolution multiplication and the total variation norm. $M_s(G)$ and $L^1(G)$ denote the closed subspace of $M(G)$ consisting of measures which are singular with respect to m_G and the closed ideal of $M(G)$ consisting of measures which are absolutely continuous with respect to m_G respectively. We denote by Trig (G) the space of all trigonometric polynomials on G. For a subset E of $\hat{G}$, $M_E(G)$ is the space of measures in $M(G)$, whose Fourier–Stieltjes transforms vanish off E. E° $(\mathring{E})$ and E^- $(\bar{E})$ mean the interior and the closure of E respectively. For a subgroup H of G, $H^\perp$ denotes the annihilator of H.

2. Definitions and several propositions.

Let Z be the integer group, $\mathbf{R}_+$ will denote the set of nonnegative real numbers, $\mathbf{R}_-$ the set of nonpositive real numbers, $\mathbf{Z}_+$ the set of nonnegative integers and $\mathbf{Z}_-$ the set of nonpositive integers respectively. For $x=(x_1,\dots,x_n)$, $(y_1,\dots,y_n) \in \mathbf{R}^n$, $\langle x,y\rangle$ denotes the scalar product, i.e. $\langle x,y\rangle=\sum_{i=1}^n x_i y_i$. Let u be an unit vector in $\mathbf{R}^n$ (that is $\langle u,u\rangle=1$). For a set E in $\mathbf{R}$, we define E_u as follows:

$$(2.1) \qquad E_u = \{x \in \mathbf{R}^n;\ \langle x,u\rangle \in E\}\ .$$

DEFINITION 2.1. Let S be a set of unit vectors in $\mathbf{R}^n$. A subset F of $\mathbf{R}^n$ is said to have S-width zero if to every $\delta>0$, there is a countable collection of pairs (E,u) with E an open set, u in S, $\bigcup E_u \supset F$, and $\sum m_{\mathbf{R}}(E)<\delta$.

Let e_i be the unit vector in $\mathbf{R}^n$ such that $e_i=(0,\overbrace{\dots,1}^{i},\dots,0)$ $(1 \leqq i \leqq n)$. Let

ψ be a continuous homomorphism from $\hat{G}$ into $\mathbf{R}^n$ such that $\psi(\hat{G})$ contains e_i $(i=1,2,\ldots,n)$ and φ the dual homomorphism of ψ, that is $(\varphi(t),\gamma) =\exp(i\langle t,\psi(\gamma)\rangle)$.

DEFINITION 2.2. For each $x \in G$, S_x is a set of unit vectors in $\mathbf{R}^n$. Put $S=\{S_x\}_{x\in G}$. A subset F of G is said to have S-width zero in the direction of φ, if $\{t \in \mathbf{R}^n;\ \varphi(t)+x \in F\}$ has S_x-width zero for each $x \in G$.

Let χ_i be an element in $\hat{G}$ such that $\psi(\chi_i)=e_i$ $(1\leqq i\leqq n)$ and put $\Lambda=\{m_1\chi_1 +\ldots+m_n\chi_n;\ m_i \in \mathbf{Z}\}$. Then Λ is a discrete subgroup of $\hat{G}$. Let K be the annihilator of Λ. We define a continuous homomorphism α: $\mathbf{R}^n\oplus K \mapsto G$ by

$$\alpha(t,u) = \varphi(t)+u \,. \tag{2.2}$$

We define a closed subgroup D of $\mathbf{R}^n\oplus K$ by

$$D = \ker(\alpha)\,. \tag{2.3}$$

The following propositions are proved in parallel with [11]. However we give the complete proofs.

PROPOSITION 2.3. *α is an onto continuous homomorphism.*

PROOF. For $x \in G$, there is $t_k \in (-\pi,\pi]$ such that $(x,\chi_k)=e^{it_k}$ $(k=1,2,\ldots,n)$. Put $t=(t_1,\ldots,t_n)$ and $u=x-\varphi(t)$. Then we have

$$\begin{aligned}(u,\chi_k) &= (x,\chi_k)(-\varphi(t),\chi_k)\\ &= e^{it_k}\cdot e^{-i\langle t,\psi(\chi_k)\rangle}\\ &= e^{it_k}\cdot e^{-it_k}\\ &= 1 \quad (k=1,2,\ldots,n)\,.\end{aligned}$$

Hence u belongs to $K=\Lambda^{\perp}$. Thus $\alpha(t,u)=x$. This completes the proof.

PROPOSITION 2.4.

$$D = \{(t,-\varphi(t)) \in \mathbf{R}^n\oplus K;\ t \in (2\pi\mathbf{Z})^n\}\,,$$

where

$$(2\pi\mathbf{Z})^n = \{(2\pi m_1,\ldots,2\pi m_n);\ (m_1,\ldots,m_n) \in \mathbf{Z}^n\}\,.$$

PROOF.

$$\begin{aligned}D &= \{(t,u) \in \mathbf{R}^n\oplus K;\ \varphi(t)+u=0\}\\ &= \{(t,-\varphi(t));\ \varphi(t) \in K\}\,.\end{aligned}$$

On the other hand, for $t=(t_1,\dots,t_n)\in \mathbf{R}^n$, we have

$$\begin{aligned}\varphi(t)\in K &\Leftrightarrow (\varphi(t),\gamma) = 1 && (\gamma\in\Lambda)\\ &\Leftrightarrow (\varphi(t), m_j\chi_j) = 1 && (m_j\in\mathbf{Z};\ 1\leqq j\leqq n)\\ &\Leftrightarrow e^{im_jt_j} = 1 && (m_j\in\mathbf{Z};\ 1\leqq j\leqq n)\\ &\Leftrightarrow t_j = 2\pi l_j && \text{for some } l_j\in\mathbf{Z}\ (1\leqq j\leqq n)\,.\end{aligned}$$

Thus this completes the proof.

PROPOSITION 2.5. $D^{\perp}=\{(\psi(\gamma),\gamma|_K);\ \gamma\in\hat{G}\}$.

PROOF. Let γ be in $\hat{G}$. For $(t,-\varphi(t))\in D$, we have

$$\begin{aligned}((t,-\varphi(t)),(\psi(\gamma),\gamma|_K)) &= \exp(i\langle t,\psi(\gamma)\rangle)(-\varphi(t),\gamma|_K)\\ &= \exp(i\langle t,\psi(\gamma)\rangle)(-\varphi(t),\gamma)\\ &= \exp(i\langle t,\psi(\gamma)\rangle)\exp(-i\langle t,\psi(\gamma)\rangle)\\ &= 1\,.\end{aligned}$$

Hence we have $(\psi(\gamma),\gamma|_K)\in D^{\perp}$.

Conversely, let (t,σ) be in $D^{\perp}$ $(t=(t_1,\dots,t_n),\ \sigma\in\hat{K})$. Let σ_* be an element in $\hat{G}$ such that $\sigma_*|_K=\sigma$. Then, for $m=(m_1,\dots,m_n)\in\mathbf{Z}^n$, we have

$$\begin{aligned}1 &= ((t,\sigma),(2\pi m,-\varphi(2\pi m))\\ &= \exp(i\langle t,2\pi m\rangle)(\sigma_*,-\varphi(2\pi m))\\ &= \exp(i\langle t,2\pi m\rangle)\exp(-i\langle\psi(\sigma_*),2\pi m\rangle)\\ &= \exp(i\langle t-\psi(\sigma_*),2\pi m\rangle)\,.\end{aligned}$$

Hence there exists $k=(k_1,\dots,k_n)\in\mathbf{Z}^n$ such that $t-\psi(\sigma_*)=k$. Put $\gamma=k_1\chi_1+\dots+k_n\chi_n+\sigma_*$. Then we have $\gamma|_K=\sigma_*|_K=\sigma$. Moreover,

$$\begin{aligned}\psi(\gamma) &= k_1\psi(\chi_1)+\dots+k_n\psi(\chi_n)+\psi(\sigma_*)\\ &= k+\psi(\sigma_*)\\ &= t\,.\end{aligned}$$

Thus we have $(t,\sigma)=(\psi(\gamma),\gamma|_K)$. This completes the proof.

PROPOSITION 2.6. *$\alpha((-\pi,\pi]^n\times K)=G$ and α is a homeomorphism on the interior of $(-\pi,\pi]^n\times K$. In particular, α is onto, open continuous homomorphism.*

PROOF. By the proof of Proposition 2.3, we can verify $\alpha((-\pi,\pi]^n\times K)=G$, and by Proposition 2.4, α is one-to-one on $(-\pi,\pi]^n\times K$. Let $\alpha(t_\delta,u_\delta)=\varphi(t_\delta)+u_\delta$ converge to $\alpha(t_0,u_0)$, where $t_\delta,t_0\in(-\pi,\pi)^n$ and $u_\delta,u_0\in K$. Since $\{t_\delta\}$ is bounded, there exist a subnet $\{t_\gamma\}$ of $\{t_\delta\}$ and $t_1\in[-\pi,\pi]^n$ such that $t_\gamma\xrightarrow{\gamma}t_1$. Then

$$u_\gamma\xrightarrow{\gamma}\varphi(t_0)-\varphi(t_1)+u_0\,.$$

Hence $\varphi(t_0)-\varphi(t_1)$ belongs to K. This means

$$\begin{aligned}1&=(\varphi(t_0-t_1),\gamma)\\&=\exp\left(i\langle t_0-t_1,\psi(\gamma)\rangle\right)\qquad\text{for all }\gamma\in\Lambda\,.\end{aligned}$$

Hence we have $t_0-t_1\in(2\pi\mathbf{Z})^n$. On the other hand, since $t_0\in(-\pi,\pi)^n$ and $t_1\in[-\pi,\pi]^n$, we have $t_0=t_1$. Hence $u_\gamma\xrightarrow{\gamma}u_0$, and so (t_γ,u_γ) converges to (t_0,u_0). Thus α is a homeomorphism on the interior of $(-\pi,\pi]^n\times K$. Now we put

$$J_i=\{(t_1,\ldots,t_n);\ |t_j|\leqq\pi\ (j\neq i),\ t_i=\pi\}$$

$(i=1,2,\ldots,n)$. Then, by Proposition 2.4 we have

$$\alpha((-\pi,\pi)^n\times K)\cap\alpha(J_i\times K)=\varnothing$$

$(i=1,2,\ldots,n)$. Hence

$$\alpha((-\pi,\pi)^n\times K)=G\setminus\bigcup_{i=1}^{n}\alpha(J_i\times K)\,.$$

Therefore, since $\alpha(J_i\times K)=\varphi(J_i)+K$ are closed, $\alpha((-\pi,\pi)^n\times K)$ is open. Thus α is an open continuous homomorphism. This completes the proof.

PROPOSITION 2.7. *$G\cong\mathbf{R}^n\oplus K/D$. In particular,*

$$\{(\psi(\gamma),\gamma|_K);\ \gamma\in\hat{G}\}=D^\perp\cong\hat{G}\,.$$

PROOF. This is obtained from ([7; Theorem (5.27), p. 41]), Proposition 2.5, and Proposition 2.6.

PROPOSITION 2.8. *The following are satisfied.*

(I) $$\alpha(L^1(\mathbf{R}^n\oplus K))\subset L^1(G);$$

(II) $$\alpha(M_s(\mathbf{R}^n\oplus K))\subset M_s(G)\,.$$

PROOF. Let π_D be the natural homomorphism from $\mathbf{R}^n\oplus K$ onto $\mathbf{R}^n\oplus K/D$.

Then

$$\pi_D(L^1(\mathbf{R}^n \oplus K)) \subset L^1(\mathbf{R}^n \oplus K/D) .$$

Moreover, by Proposition 2.4, we have

$$\pi_D(M_s(\mathbf{R}^n \oplus K)) \subset M_s(\mathbf{R}^n \oplus K/D) .$$

On the other hand, for $\mu \in M(\mathbf{R}^n \oplus K)$, we have

$$\begin{aligned}\alpha(\mu)\hat{}(\gamma) &= \int_{\mathbf{R}^n \oplus K} (-\gamma, \alpha(t,u))\, d\mu(t,u) \\ &= \int_{\mathbf{R}^n \oplus K} \exp(-i\langle \psi(\gamma), t\rangle)(-\gamma|_K, u)\, d\mu(t,u) \\ &= \hat{\mu}(\psi(\gamma), \gamma|_K) \\ &= \pi_D(\mu)\hat{}(\psi(\gamma), \gamma|_K) .\end{aligned}$$

Hence, by Proposition 2.7, (I) and (II) are obtained. This completes the proof.

Next we define a continuous homomorphism $\alpha_1\colon \mathbf{R}^n \oplus \hat{G} \mapsto \mathbf{R}^n \oplus \hat{K}$ as follows:

$$\alpha_1(t,\gamma) = (t + \psi(\gamma), \gamma|_K) . \tag{2.4}$$

Then the following propositions are satisfied. (These propositions are also proved in parallel with [11].)

PROPOSITION 2.9.

(I) $\ker(\alpha_1) = \{((m_1, \ldots, m_n), -(m_1\chi_1 + \ldots + m_n\chi_n)) \in \mathbf{R}^n \oplus \hat{G};$
$(m_1, \ldots, m_n) \in \mathbf{Z}^n\};$

(II) $\alpha_1([-\tfrac{1}{2}, \tfrac{1}{2})^n \times \hat{G}) = \mathbf{R}^n \oplus \hat{K} .$

PROOF. (I):

$$\begin{aligned}\ker(\alpha_1) &= \{(t,\gamma) \in \mathbf{R}^n \oplus \hat{G};\ (t+\psi(\gamma), \gamma|_K) = 0\} \\ &= \{(t,\gamma) \in \mathbf{R}^n \oplus \hat{G};\ \gamma \in \Lambda,\ t = -\psi(\gamma)\} \\ &= \{((m_1, \ldots, m_n), -(m_1\chi_1 + \ldots + m_n\chi_n));\ (m_1, \ldots, m_n) \in \mathbf{Z}^n\} .\end{aligned}$$

(II): Let (t, σ) be an element in $\mathbf{R}^n \oplus \hat{K}$ ($t = (t_1, \ldots, t_n) \in \mathbf{R}^n$). We choose $\gamma \in \hat{G}$ such that $\gamma|_K = \sigma$. Then there is $(m_1, \ldots, m_n) \in \mathbf{Z}^n$ such that

$$t - \psi(\gamma) \in [m_1 - \tfrac{1}{2}, m_1 + \tfrac{1}{2}) \times \ldots \times [m_n - \tfrac{1}{2}, m_n + \tfrac{1}{2}) .$$

Put $\psi(\gamma)=(y_1,\ldots,y_n)$ and $s_k=-y_k+t_k-m_k$ $(k=1,2,\ldots,n)$. Then

$$s = (s_1,\ldots,s_n)\in[-\tfrac{1}{2},\tfrac{1}{2})^n$$

and

$$\begin{aligned}\alpha_1(s,\gamma+m_1\chi_1+\ldots+m_n\chi_n)\\ &= (s+\psi(\gamma)+m_1\psi(\chi_1)+\ldots+m_n\psi(\chi_n),\gamma|_K)\\ &= (t,\sigma)\,.\end{aligned}$$

Hence (II) is proved. This completes the proof.

PROPOSITION 2.10. *The following are satisfied.*

(I) $\alpha_1((-\tfrac{1}{2},\tfrac{1}{2})^n\times\hat{G})$ *is an open set in* $\mathsf{R}^n\oplus\hat{K}$;

(II) α_1 *is a homeomorphism on* $(-\tfrac{1}{2},\tfrac{1}{2})^n\times\hat{G}$;

(III) α_1 *is an open continuous homomorphism.*

PROOF. (I): By Proposition 2.9, α_1 is one-to-one on $[-\tfrac{1}{2},\tfrac{1}{2})^n\times\hat{G}$ and $\alpha_1([-\tfrac{1}{2},\tfrac{1}{2})^n\times\hat{G})=\mathsf{R}^n\oplus\hat{K}$. Hence we have

$$\alpha_1((-\tfrac{1}{2},\tfrac{1}{2})^n\times\hat{G}) = \mathsf{R}^n\oplus\hat{K}\setminus\bigcup_{j=1}^{n}\alpha_1(I_j\times\hat{G})\,, \tag{1}$$

where

$$I_j = [-\tfrac{1}{2},\tfrac{1}{2})^{j-1}\times\{-\tfrac{1}{2}\}\times[-\tfrac{1}{2},\tfrac{1}{2})^{n-j}\qquad(1\leqq j\leqq n)\,.$$

CLAIM. $\alpha_1(I_j\times\hat{G})=\alpha_1(\bar{I}_j\times\hat{G})$, where

$$\bar{I}_j = [-\tfrac{1}{2},\tfrac{1}{2}]^{j-1}\times\{-\tfrac{1}{2}\}\times[-\tfrac{1}{2},\tfrac{1}{2}]^{n-j}\,.$$

Let (t,γ) be an element in $\bar{I}_j\times\hat{G}$ $(t=(t_1,\ldots,t_{j-1},-\tfrac{1}{2},t_{j+1},\ldots,t_n))$. We define $\tilde{t}_k$ $(k\neq j)$ as follows:

$$\tilde{t}_k = \begin{cases}t_k-1 & \text{if } t_k=\tfrac{1}{2}\\ t_k & \text{if } t_k\in[-\tfrac{1}{2},\tfrac{1}{2})\,.\end{cases}$$

Put $m_k=\tilde{t}_k-t_k$ $(k\neq j)$ and

$$s = (\tilde{t}_1,\ldots,\tilde{t}_{j-1},-\tfrac{1}{2},\tilde{t}_{j+1},\ldots,\tilde{t}_n)\,.$$

Then $s\in I_j$ and

$$\alpha_1\left(s,-\left(\sum_{k\neq j}m_k\chi_k\right)+\gamma\right) = \left(s-\left(\sum_{k\neq j}m_k\psi(\chi_k)\right)+\psi(\gamma),\gamma|_K\right)$$

$$= (t+\psi(\gamma), \gamma|_K)$$
$$= \alpha_1(t, \gamma|_K) .$$

Thus Claim is proved. On the other hand, since

$$\alpha_1(\bar{I}_j \times \hat{G}) = \bar{I}_j \times \{0\} + \{(\psi(\gamma), \gamma|_K) ; \ \gamma \in \hat{G}\}$$
$$= \bar{I}_j \times \{0\} + D^{\perp} ,$$

$\alpha_1(\bar{I}_j \times \hat{G})$ is a closed subset of $\mathbf{R}^n \oplus \hat{K}$. Hence, by (1) and Claim, $\alpha_1((-\frac{1}{2}, \frac{1}{2})^n \times \hat{G})$ is an open set in $\mathbf{R}^n \oplus \hat{K}$.

(II): Suppose

$$\alpha_1(t_\alpha, \gamma_\alpha) \xrightarrow{\alpha} \alpha_1(t_0, \gamma_0)$$

$((t_\alpha, \gamma_\alpha), (t_0, \gamma_0) \in (-\frac{1}{2}, \frac{1}{2})^n \times \hat{G})$. Let $\{\alpha_1(t_\delta, \gamma_\delta)\}$ be any subnet of $\{\alpha_1(t_\alpha, \gamma_\alpha)\}$. Then there exist a subnet $\{t_\beta\}$ of $\{t_\delta\}$ and $t_1 \in [-\frac{1}{2}, \frac{1}{2}]^n$ such that $t_\beta \xrightarrow{\beta} t_1$. Since

$$(t_\beta + \psi(\gamma_\beta), \gamma_\beta|_K) \xrightarrow{\beta} (t_0 + \psi(\gamma_0), \gamma_0|_K) ,$$

$(\psi(\gamma_\beta), \gamma_\beta|_K)$ converges to $(t_0 - t_1 + \psi(\gamma_0), \gamma_0|_K)$. By Proposition 2.5, we have $(t_0 - t_1 + \psi(\gamma_0), \gamma_0|_K) \in D^{\perp}$. Hence by Proposition 2.5, there exists $\gamma_1 \in \hat{G}$ such that

$$(\psi(\gamma_1), \gamma_1|_K) = (t_0 - t_1 + \psi(\gamma_0), \gamma_0|_K) .$$

This means that $\gamma_0 - \gamma_1 \in \Lambda$, and so $\psi(\gamma_0) - \psi(\gamma_1) \in \mathbf{Z}^n$. Hence we have $t_0 = t_1$ because $t_0 \in (-\frac{1}{2}, \frac{1}{2})^n$ and $t_1 \in [-\frac{1}{2}, \frac{1}{2}]^n$. Therefore we get $(t_\beta, \gamma_\beta) \xrightarrow{\beta} (t_0, \gamma_0)$. This proves (II).

(III): (III) is easily obtained from (I) and (II).
This completes the proof.

The following three remarks are easily obtained from the definition.

REMARK 2.11. Let S be a set of unit vectors in $\mathbf{R}^n$ and E_k a subset of $\mathbf{R}^n$ with S-width zero ($k = 1, 2, 3, \ldots$). Then $\bigcup_1^\infty E_k$ is also a set with S-width zero.

REMARK 2.12. Let G be a LCA group and φ a continuous homomorphism from $\mathbf{R}^n$ into G. For each $x \in G$, S_x is a set of unit vectors in $\mathbf{R}^n$. Put $\mathbf{S} = \{S_x\}_{x \in G}$. Let E_k be a subset of G with $\mathbf{S}$-width zero in the direction of φ ($k = 1, 2, 3, \ldots$). Then $\bigcup_1^\infty E_k$ is a set with $\mathbf{S}$-width zero in the direction of φ.

REMARK 2.13. Let S be a set of unit vectors in $\mathbf{R}^n$ and F a subset of $\mathbf{R}^n$ with S-width zero. Then $F + a$ is also a set with S-width zero for every $a \in \mathbf{R}^n$.

3. Key lemmas.

DEFINITION 3.1. For $0<\varepsilon<\frac{1}{3}$, we define a function $\Delta_\varepsilon(x,\sigma)$ on $\mathsf{R}^n\oplus\hat{K}$ by

$$\Delta_\varepsilon(x,\sigma) = \prod_{i=1}^{n} \max\left(1-\frac{1}{\varepsilon}|x_i|,0\right)$$

for $\sigma=0$ and $\Delta_\varepsilon(x,\sigma)=0$ for $\sigma\neq 0$ $(x=(x_1,\dots,x_n)\in\mathsf{R}^n)$.

Let G be compact abelian group and ψ a homomorphism from $\hat{G}$ into R^n such that $\psi(\hat{G})$ contains e_k $(k=1,2,\dots,n)$. Let φ be the dual homomorphism of ψ. For $0<\varepsilon<\frac{1}{3}$, we define a subset V_ε of $\mathsf{R}^n\oplus\hat{K}$ by

$$V_\varepsilon = \{(t_1,\dots t_n,\sigma)\in\mathsf{R}^n\oplus\hat{K};\ |t_i|<\varepsilon\ (1\leqq i\leqq n),\ \sigma=0\}\ .$$

Then $V_\varepsilon\cap D^\perp=\{0\}$. Since G is compact, by Proposition 2.7,

$$D^\perp = \{(\psi(\gamma),\gamma|_K)\ ;\ \gamma\in\hat{G}\}$$

is a discrete subgroup of $\mathsf{R}^n\oplus\hat{K}$. For $\mu\in M(G)$, by regarding μ as a measure in $M(\mathsf{R}^n\oplus K/D)$ (cf. Proposition 2.7), we define a function Φ_μ^ε on $\mathsf{R}^n\oplus\hat{K}$ as follows:

$$\Phi_\mu^\varepsilon(t,\sigma) = \sum_{\gamma\in\hat{G}} \hat{\mu}(\gamma)\Delta_\varepsilon((t,\sigma)-(\psi(\gamma),\gamma|_K))\ . \tag{3.1}$$

Then, by ([6; A.7.1. Theorem, p. 421]),

$$\Phi_\mu^\varepsilon\in M(\mathsf{R}^n\oplus K)\hat{}\quad\text{and}\quad \|(\Phi_\mu^\varepsilon)\check{}\,\| = \|\mu\|\ , \tag{3.2}$$

where $(\Phi_\mu^\varepsilon)\check{}$ is the inverse Fourier transform of Φ_μ^ε.

The following two lemmas can be proved in parallel with [10]. However we give the complete proofs.

LEMMA 3.2. *Let G be a compact abelian group and ψ a homomorphism from $\hat{G}$ into R^n such that $\psi(\hat{G})$ contains e_i $(1\leqq i\leqq n)$. For $0<\varepsilon<\frac{1}{3}$, the following are satisfied*:

(I) $\Phi_\mu^\varepsilon\in \mathrm{L}^1(\mathsf{R}^n\oplus K)\hat{}$ if $\mu\in L^1(G)$;

(II) $\Phi_\mu^\varepsilon\in M_s(\mathsf{R}^n\oplus K)\hat{}$ if $\mu\in M_s(G)$.

PROOF. (I) is obtained from ([6; A.7.1. Theorem, p. 421]).

(II): Let $\mu\in M_s(G)$ and put $A=\|\mu\|$. Then, by (3.2), $\|(\Phi_\mu^\varepsilon)\check{}\,\|=A$. Let ε' be any positive number and K' a compact set in $\mathsf{R}^n\oplus\hat{K}$. Since $D^\perp\cap K'$ is a compact set in $D^\perp$, by ([3; Theorem 1]), there exists

$$p(y) = \sum c_i(-y, \gamma_i) \in \mathrm{Trig}\,(G)$$

with $(\psi(\gamma_i), \gamma_i|_K) \in D^\perp \setminus (D^\perp \cap K')$ such that

$$\|p\|_\infty \leqq 1 \quad \text{and} \quad |\sum c_i\hat{\mu}(\gamma_i)| > A - \varepsilon' .$$

Now we define $\tilde{p}(t,u) \in \mathrm{Trig}\,(\mathbf{R}^n \oplus K)$ by

$$\tilde{p}(t,u) = \sum c_i \exp\left(-i\langle t, \psi(\gamma_i)\rangle\right)\cdot(-u, \gamma_i|_K) .$$

Then $\|\tilde{p}\|_\infty \leqq 1$. Since $\Phi_\mu^\varepsilon(\psi(\gamma), \gamma|_K) = \hat{\mu}(\gamma)$, we have

$$\begin{aligned} |\sum c_i \Phi_\mu^\varepsilon(\psi(\gamma_i), \gamma_i|_K)| &= |\sum c_i\hat{\mu}(\gamma_i)| \\ &> A - \varepsilon' . \end{aligned}$$

Hence, by ([3; Theorem 1]), we have $\Phi_\mu^\varepsilon \in M_s(\mathbf{R}^n \oplus K)\hat{}$. This completes the proof.

Next we consider the case that G is a general LCA group and ψ is a continuous homomorphism from $\hat{G}$ into $\mathbf{R}^n$ such that $\psi(\hat{G})$ contains e_i $(1 \leqq i \leqq n)$. Let K and Λ be as in section 2 and φ the dual homomorphism of ψ. Let $\bar{G}$ be the Bohr compactification of G and $\bar{K}$ the closure of K in $\bar{G}$. Then $\bar{K}$ is annihilator of Λ in $\bar{G}$. We define ψ_*: $\hat{\bar{G}} = \hat{G}_d \mapsto \mathbf{R}^n$ by $\psi_*(\gamma) = \psi(\gamma)$, and let φ_* be the dual homomorphism of ψ_*. We define a continuous homomorphism α_* from $\mathbf{R}^n \oplus \bar{K}$ into $\bar{G}$ by $\alpha_*(t,u) = \varphi_*(t) + u$. Then, as seen in section 2, α_* is an onto, open continuous homomorphism and

$$D_*(=\ker(\alpha_*)) = \{(2\pi m, -\varphi_*(2\pi m));\ m \in \mathbf{Z}^n\} .$$

Moreover, by Proposition 2.7,

$$\hat{G}_d = D_*^\perp = \{(\psi_*(\gamma), \gamma|_{\bar{K}});\ \gamma \in \hat{G}_d\} ,$$

where $\hat{G}_d$ is the group $\hat{G}$ with the discrete topology.

Let ε be a positive real number such that $0 < \varepsilon < \frac{1}{3}$ and μ a measure in $M(G)$. Regarding μ as a measure in $M(\bar{G})$, we define a function ${}_*\Phi_\mu^\varepsilon$ on $\mathbf{R}^n \oplus \hat{\bar{K}}_d$ as follows:

$$(3.3) \qquad {}_*\Phi_\mu^\varepsilon(t,\sigma) = \sum_{\gamma \in \hat{G}_d} \hat{\mu}(\gamma)\Delta_\varepsilon((t,\sigma) - (\psi(\gamma), \gamma|_{\bar{K}})) .$$

Then, by Lemma 3.2, ${}_*\Phi_\mu^\varepsilon$ belongs to $M(\mathbf{R}^n \oplus \bar{K})\hat{}$. Noting that $(\mathbf{R}^n \oplus \bar{K})\hat{} \cong \mathbf{R}^n \oplus \hat{\bar{K}}_d$, we define a function Φ_μ^ε on $\mathbf{R}^n \oplus \hat{K}$ by

$$(3.4) \qquad \Phi_\mu^\varepsilon(t,\sigma) = {}_*\Phi_\mu^\varepsilon(t,\sigma) .$$

Then the following is satisfied.

LEMMA 3.3. *Let G be a* LCA *group and ψ a continuous homomorphism from $\hat{G}$ into $\mathbf{R}^n$ such that $\psi(\hat{G})$ contains e_i $(1 \leqq i \leqq n)$. Let ε be a positive real number such that $0<\varepsilon<\frac{1}{3}$. Then the following are satisfied:*

(I) $\Phi_\mu^\varepsilon \in M(\mathbf{R}^n \oplus K)\hat{}$ *if* $\mu \in M(G)$;

(II) $\|(\Phi_\mu^\varepsilon)\check{}\| = \|\mu\|$;

(III) $\Phi_\mu^\varepsilon \in L^1(\mathbf{R}^n \oplus K)\hat{}$ *if* $\mu \in L^1(G)$;

(IV) $\Phi_\mu^\varepsilon \in M_s(\mathbf{R}^n \oplus K)\hat{}$ *if* $\mu \in M_s(G)$.

PROOF. (I): By Lemma 3.2 and the construction of Φ_μ^ε, it is sufficient to prove that Φ_μ^ε is continuous on $\mathbf{R}^n \oplus \hat{K}$. Put

$$I = \{(t,\gamma) \in \mathbf{R}^n \oplus \hat{G};\ |t| \leqq \tfrac{3}{2}\varepsilon\}$$

and

$$I^\circ = \{(t,\gamma) \in \mathbf{R}^n \oplus \hat{G};\ |t| < \tfrac{3}{2}\varepsilon\},$$

where $|t| = \max_{1 \leqq i \leqq n} |t_i|$ $(t=(t_1,\dots,t_n))$. We define a continuous homomorphism α_1 from $\mathbf{R}^n \oplus \hat{G}$ into $\mathbf{R}^n \oplus \hat{K}$ and a function Ψ_μ^ε on $\mathbf{R}^n \oplus \hat{G}$ as follows:

$$\alpha_1(t,\gamma) = (t+\psi(\gamma), \gamma|_K); \tag{1}$$

$$\Psi_\mu^\varepsilon(t,\gamma) = {}_\varepsilon\Delta(t)\hat{\mu}(\gamma),$$

where

$$_\varepsilon\Delta(t) = \prod_{k=1}^n \max\left(1-\frac{1}{\varepsilon}|t_k|, 0\right).$$

CLAIM.

$$\Psi_\mu^\varepsilon(t,\gamma) = \Phi_\mu^\varepsilon(\alpha_1(t,\gamma)) \qquad \text{for } (t,\gamma) \in [-\tfrac{1}{2},\tfrac{1}{2})^n \times \hat{G}.$$

Indeed,

$$\begin{aligned}
&\Phi_\mu^\varepsilon(\alpha_1(t,\gamma)) \\
&= \Phi_\mu^\varepsilon(t+\psi(\gamma), \gamma|_K) \\
&= {}_*\Phi_\mu^\varepsilon(t+\psi_*(\gamma), \gamma|_{\bar{K}}) \\
&= \sum_{\tau \in \hat{G}_d} \hat{\mu}(\tau)\Delta_\varepsilon((t+\psi_*(\gamma), \gamma|_{\bar{K}}) - (\psi_*(\tau), \tau|_{\bar{K}})) \\
&= \sum_{m=(m_1,\dots,m_n) \in \mathbf{Z}^n} \hat{\mu}(\gamma + m_1\chi_1 + \dots + m_n\chi_n)\Delta_\varepsilon\,(t-m, 0)
\end{aligned}$$

$$= \hat{\mu}(\gamma)\Delta_\varepsilon(t,0) \qquad (t-m \in [-\varepsilon,\varepsilon]^n \Leftrightarrow m=0)$$
$$= \hat{\mu}(\gamma)_\varepsilon\Delta(t)$$
$$= \Psi^\varepsilon_\mu(t,\gamma)\,.$$

Thus Claim is proved. Hence, by Proposition 2.9.(II) and Claim, Φ^ε_μ vanishes on $\alpha_1([-\frac{5}{4}\varepsilon,\frac{5}{4}\varepsilon]^n \times \hat{G})^c$. Therefore, in order to prove the continuity of Φ^ε_μ, we may only prove that Φ^ε_μ is continuous on the open set $\alpha_1(I^\circ)$. Suppose $\alpha_1(t_\delta,\gamma_\delta)$ converges to $\alpha_1(t_0,\gamma_0)$ $((t_0,\gamma_0) \in I^\circ)$. Then, by Proposition 2.10.(II), (t_δ,γ_δ) converges to (t_0,γ_0). Hence,

$$\begin{aligned}\lim_\delta \Phi^\varepsilon_\mu(\alpha_1(t_\delta,\gamma_\delta)) &= \lim_\delta \Psi^\varepsilon_\mu(t_\delta,\gamma_\delta)\\ &= \Psi^\varepsilon_\mu(t_0,\gamma_0)\\ &= \Phi^\varepsilon_\mu(\alpha_1(t_0,\gamma_0))\,.\end{aligned}$$

This proves (I).

(II). By Lemma 3.2, we have

$$\begin{aligned}\|(\Phi^\varepsilon_\mu)^\vee\| &= \|({}_*\Phi^\varepsilon_\mu)^\vee\|_{M(\mathbf{R}^n \oplus \bar{K})}\\ &= \|\mu\|_{M(\bar{G})}\\ &= \|\mu\|\,.\end{aligned}$$

(III). Let μ be a measure in $L^1(G)$. Then there exists a sequence $\{\mu_n\}$ in $L^1(G)$ such that $\hat{\mu}_n$ has a compact support and $\lim_n \|\mu_n-\mu\|=0$. We note that $\Phi^\varepsilon_{\mu_n}$ has a compact support (cf. Proposition 2.10 and Claim). Hence, by (I), $\Phi^\varepsilon_{\mu_n} \in L^1(G)\hat{}$. By (II),

$$\lim_n \|(\Phi^\varepsilon_\mu)^\vee - (\Phi^\varepsilon_{\mu_n})^\vee\| = \lim_n \|\mu-\mu_n\| = 0\,.$$

Hence $\Phi^\varepsilon_\mu \in L^1(G)\hat{}$.

(IV). This can be proved as same as in Lemma 3.2(II). This completes the proof.

4. Proofs of Theorem 1.3 and Corollary 1.4.

LEMMA 4.1. *Let G be a* LCA *group and ψ a continuous homomorphism from $\hat{G}$ into* $\mathbf{R}^n$ *such that $\psi(\hat{G})$ contains e_i $(1 \leqq i \leqq n)$. Let ε be a positive real number such that $0<\varepsilon<\frac{1}{3}$ and α the homomorphism defined in section 2. Then we have $\alpha((\Phi^\varepsilon_\mu)^\vee)=\mu$ for all $\mu \in M(G)$.*

PROOF. For $\gamma \in \hat{G}$, we have

$$\begin{aligned}\alpha(((\Phi_\mu^\varepsilon)\check{}\,)\hat{}\,(\gamma)) &= \int_{\mathbf{R}^n \oplus K} (-\alpha(t,u),\gamma)\, d(\Phi_\mu^\varepsilon)\check{}\,(t,u) \\ &= \int_{\mathbf{R}^n \oplus K} (-\varphi(t),\gamma)(-u,\gamma)\, d(\Phi_\mu^\varepsilon)\check{}\,(t,u) \\ &= \int_{\mathbf{R}^n \oplus K} \exp(-i\langle t, \psi(\gamma)\rangle)(-u,\gamma|_K)\, d(\Phi_\mu^\varepsilon)\check{}\,(t,u) \\ &= \Phi_\mu^\varepsilon(\psi(\gamma),\gamma|_K) \\ &= \hat{\mu}(\gamma)\,.\end{aligned}$$

Hence we have $\alpha((\Phi_\mu^\varepsilon)\check{}\,)=\mu$ and the proof is complete.

Now we prove Theorem 1.3. We may prove only that $\mu(E)=0$ for each Borel set E in F. By the definition, we note that E has also S-width zero in the direction of φ. Let χ_* be an element in $\hat{G}$ such that $\psi(\chi_*)=(1,1,\ldots,1)$. Considering $\chi_*\mu$ instead of μ, we may assume that μ satisfies

(1) $$\hat{\mu}(\gamma) = 0 \quad \text{on } \{\gamma \in \hat{G};\ \psi(\gamma) \in \mathbf{R}^n_{-1}\}\,,$$

where $\mathbf{R}^n_{-1}=\{x=(x_1,\ldots,x_n) \in \mathbf{R}^n;\ x_k \leqq 1\ (1 \leqq k \leqq n)\}$. Let ε be a positive real number such that $0<\varepsilon<\frac{1}{3}$. Then, by the construction of Φ_μ^ε and (1), we have

(2) $$\Phi_\mu^\varepsilon(y,\sigma) = 0 \quad \text{for } y \in \mathbf{R}^n_-\,.$$

Indeed, suppose

$$\Phi_\mu^\varepsilon(\alpha_1(t,\gamma)) = \Phi_\mu^\varepsilon(t+\psi(\gamma),\gamma|_K) \neq 0\,.$$

Then, by Claim in Lemma 3.3 and (1), we have $\psi(\gamma) \notin \mathbf{R}^n_{-1}$ and $t \in (-\varepsilon,\varepsilon)^n$, and so $\psi(\gamma)+t \notin \mathbf{R}^n_-$. Since $\alpha_1([-\frac{1}{2},\frac{1}{2})^n \times \hat{G})=\mathbf{R}^n \oplus \hat{K}$, (2) is proved.

Let π be the natrual homomorphism from $\mathbf{R}^n \oplus K$ onto K and put $\eta=\pi(|(\Phi_\mu^\varepsilon)\check{}\,|)$. Then, by ([13, Corollary 1.6]), there exists a family $\{\lambda_s\}_{s\in K}$ of measures in $M(\mathbf{R}^n \oplus K)$ with the following properties:

(3) $h \mapsto \lambda_h(f)$ is a η-measurable function for each bounded Borel measurable function f on $\mathbf{R}^n \oplus K$,

(4) $\operatorname{supp}(\lambda_h) \subset \mathbf{R}^n \times \{h\}$,

(5) $\|\lambda_h\| \leqq 1$,

(6) $(\Phi_\mu^\varepsilon)\check{}\,(g) = \int_K \lambda_h(g)\, d\eta(h)$ for each bounded Borel function g on $\mathbf{R}^n \oplus K$.

By (4), there exists a measure $\nu_h \in M(\mathbf{R}^n)$ such that $d\lambda_h(x,u)=d\nu_h(x)\times d\delta_h(u)$, where δ_h is the Dirac measure at h. Then, by (2) and ([13, Lemma 2.1]), we have

$$\hat{\nu}_h(y) = 0 \quad \text{on } \mathbf{R}^n_- \quad \text{a.a. } h\ (\eta)\,. \tag{7}$$

On the other hand, by Lemma 4.1 and (6), we have

$$\begin{aligned} \mu(E) &= (\Phi^{\varepsilon}_{\mu})\check{}\,(\alpha^{-1}(E)) \\ &= \int_K \lambda_h(\alpha^{-1}(E))\,d\eta(h) \end{aligned} \tag{8}$$

and

$$\alpha^{-1}(E) \cap \mathbf{R}^n \times \{h\} = \{x \in \mathbf{R}^n\,;\ \varphi(x)+h \in E\} \times \{h\}\,. \tag{9}$$

Now we put $E(h)=\{x \in \mathbf{R}^n;\ \varphi(x)+h \in E\}$. Then, since E has $\boldsymbol{S}$-width zero in the direction of φ, $E(h)$ has S_h-width zero. Hence, by (7), (9), and ([10; 6.2.2. Theorem (a), p. 140]), we have

$$\begin{aligned} \lambda_h(\alpha^{-1}(E)) &= \nu_h(E(h)) \\ &= 0 \quad \text{a.a. } h\ (\eta)\,. \end{aligned}$$

Thus, by (8), we have $\mu(E)=0$. This completes the proof of Theorem 1.3.

Next we prove Corollary 1.4. We put $\boldsymbol{S}=\{S_x\}_{x\in G}$ ($S_x=S$ for all $x \in G$). Then, by Theorem 1.3, we may prove only that $\varphi(F)+K$ has $\boldsymbol{S}$-width zero in the direction of φ. For $x \in G$, we choose $t_0 \in (-\pi,\pi]^n$ and $u_0 \in K$ such that $x=\varphi(t_0)+u_0$. Then we have

$$\begin{aligned} &\{t \in \mathbf{R}^n;\ \varphi(t)+x \in \varphi(F)+K\} \\ &\quad= \{t \in \mathbf{R}^n;\ \varphi(t) \in \varphi(F)-\varphi(t_0)+K-u_0\} \\ &\quad= \{t \in \mathbf{R}^n;\ \varphi(t) \in \varphi(F-t_0)+K\} \\ &\quad\subset F-t_0+(2\pi Z)^n, \qquad (\varphi(s) \in K \Leftrightarrow s \in (2\pi Z)^n)\,. \end{aligned}$$

Hence, by Remarks 2.11 and 2.13, $\{t \in \mathbf{R}^n;\ \varphi(t)+x \in \varphi(F)+K\}$ has S-width zero, so that $\varphi(F)+K$ has $\boldsymbol{S}$-width zero on the direction of φ. This completes the proof of Corollary 1.4.

COROLLARY 4.2. *Let G be a* LCA *group and ψ a continuous homomorphism from $\hat{G}$ into $\mathbf{R}^n$ such that $\psi(\hat{G})$ contains e_k $(1\leqq k\leqq n)$. Let φ be the dual homomorphism of ψ. Let S be a compact set of unit vectors in the interior of $\mathbf{R}^n_+$ and F a Borel set in $\mathbf{R}^n$ with S-width zero. Then $|\mu|(\varphi(F))=0$ for every measure $\mu \in M(G)$ whose Fourier–Stieltjes transform vanishes on $\{\gamma \in \hat{G};\ \psi(\gamma) \in \mathbf{R}^n_-\}$.*

PROOF. This obtained from Corllary 1.4.

COROLLARY 4.3 (Theorem 1.2). *Let S be a compact set of unit vectors in the interior of* $\mathbf{R}^n_+$ *and F a Borel set in* $\mathbf{R}^n$ *with S-width zero. Let φ be the canonical map from* $\mathbf{R}^n$ *onto T^n. Then we have* $|\mu|(\varphi(F))=0$ *for each measure $\mu \in M(T^n)$, whose Fourier–Stieltjes transform vanishes on* $\mathbf{Z}^n_-$.

PROOF. Let ψ be the homomorphism from $\mathbf{Z}^n$ into $\mathbf{R}^n$ such that $\psi(m)=m$. Then φ is the dual homomorphism of ψ. Hence, by Corollary 4.2, we obtain the corollary. This completes the proof.

DEFINITION 4.4. Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. Let φ be the dual homomorphism of ψ. A Borel set E in G is called a null set in the direction of φ, if $\{t \in R;\ \varphi(t)+x \in E\}$ is a set of Lebesgue measure zero for each $x \in G$.

COROLLARY 4.5 (cf. [2; Theorem 3.1, p. 186]). *Let G be a* LCA *group, ψ a nontrivial continuous homomorphism from $\hat{G}$ into R and φ the dual homomorphism of ψ. Let E ($\subset G$) be a null set in the direction of φ and $\mu \in M(G)$ a φ-analytic measure* (i.e. $\hat{\mu}(\gamma)=0$ *for* $\gamma \in \hat{G}$ *with* $\psi(\gamma)<0$). *Then we have* $|\mu|(E)=0$.

PROOF. Since ψ is nontrivial, there exists $\chi_0 \in \hat{G}$ such that $\psi(\chi_0)>0$. Let α be a positive number such that $\alpha\psi(\chi_0)=1$. We define a continuous homomorphism ψ_α from $\hat{G}$ into R by $\psi_\alpha(\gamma)=\alpha\psi(\gamma)$. Then $\psi_\alpha(\hat{G})$ contains 1. Let φ_α be the dual homomorphism of ψ_α. Then, since $\varphi_\alpha(t)=\varphi(\alpha t)$, E is a null set in the direction of φ_α. Moreover we may assume that $\hat{\mu}(\gamma)=0$ on $\{\gamma \in \hat{G};\ \psi_\alpha(\gamma)\leqq 0\}$ by considering $\chi_0\mu$ instead of μ. Let $S_x=\{1\}$ and put $\boldsymbol{S}=\{S_x\}_{x\in G}$. Then E is a set with $\boldsymbol{S}$-width zero in the direction of φ_α. Hence, by Theorem 1.3, we have $|\mu|(E)=0$. This completes the proof.

REMARK 4.6. In order to prove ([2; Theorem 3.1, p. 186]), deLeeuw and Glicksberg have used the fact that φ-analytic measures are quasi-invariant under φ.

REFERENCES

1. N. Bourbaki, *Éléments de Mathématique, Intégration vectorielle*, Livre VI, chap. 6 (Act. Sci. Ind. 1281), Hermann, Paris, 1959.
2. K. deLeeuw and I. Glickberg, *Quasi-invariance and analyticity of measures on compact groups*, Acta Math. 109 (1963), 179–205.

3. R. Doss, *On the transform of a singular or absolutely continuous measure*, Proc. Amer. Math. Soc. 19 (1968), 361–363.
4. F. Forelli, *Measures orthogonal to polydisc algebras*, J. Math. Mech. 17 (1968), 1073–1086.
5. I. Glicksberg, *Fourier–Stieltjes transforms with small supports*, Illinois J. Math. 9 (1965), 418–427.
6. C. C. Graham and O. C. McGehee, *Essays in commutative harmonic analysis*, Springer-Verlag, Berlin - Heidelberg - New York, 1979.
7. E. Hewitt and K. Ross, *Abstract harmonic analysis*, Vol. I, Grundlehren Math. Wiss. 115, Springer-Verlag, Berlin - Heidelberg - New York, 1963.
8. L. Pigno and S. Saeki, *Fourier–Stieltjes transforms which vanish at infinity*, Math. Z. 141 (1975), 83–91.
9. W. Rudin, *Fourier analysis on groups* (Interscience Tracts in Pure and Applied Mathematics 12) Interscience Publishers, London, 1962.
10. W. Rudin, *Function theory in polydiscs*, W. A. Benjamin, Inc. New York, Amsterdam, 1969.
11. H. Yamaguchi, *Some multipliers on the space consisting of measures of analytic type*, Hokkaido Math. J. 11 (1982), 173–200.
12. H. Yamaguchi, *On the product of a Riesz set and a small p set*, Proc. Amer. Math. Soc. 81 (1981), 273–278.
13. H. Yamaguchi, *Idempotent multipliers on the space of analytic singular measures*, preprint.

DEPARTMENT OF MATHEMATICS
JOSAI UNIVERSITY
SAKADO, SAITAMA
JAPAN