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THE PROJECTIVITY OF THE MODULI SPACE OF STABLE CURVES, II: THE STACKS $M_{g,n}$

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Introduction.

This paper is the second in a sequence of three papers by D. Mumford and myself, containing the results of my thesis and leading to a proof of the projectivity of the moduli space of stable curves. The story is as follows: after investigating the stack $M_{g,0}$ with Deligne, Mumford got interested in the question of whether or not it was projective. His original idea was to study the Torelli map:

$$t : M_{g,0} \to \begin{Bmatrix} \text{Satake's compactification of the} \\ \text{moduli space of abelian varieties.} \end{Bmatrix}$$

and use the fact that Satake's compactification was a projective variety (defined over $\mathbf{Z}$ by use of θ-functions in [12]). In my thesis, I then investigated the line bundles on $M_{g,0}$ and showed that the line bundle δ^{-1} (defined in section 4) was ample on all fibres of t. The map t, however, has only been constructed in characteristic 0. Seshadri then suggested that the problem could be attacked directly without the use of Jacobians by using instead the stability of Chow points of curves proven in [9], theorem 4.5. Mumford realized that it was necessary for this proof to introduce curves with basepoints, i.e. the stacks $M_{g,n}$ (cf. section 1 for definition). In this paper we study the stacks $M_{g,n}$, and certain maps between these stacks, that is:

1) contraction: $M_{g,n+1} \to M_{g,n}$

2) clutching: $\begin{cases} M_{g,n+2} \to M_{g+1,n} \\ M_{g_1,n_1+1} \times M_{g_2,n_2+1} \to M_{g_1+g_2,n_1+n_2} . \end{cases}$

In the first three sections of this paper we investigate these maps and prove that they are representable. The crucial point is to prove that $M_{g,n+1} \approx Z_{g,n}$, the universal curve over $M_{g,n}$, hence contraction is representable. The clutching maps factor through the contraction map.

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In sections 1 and 2 we prove that we have an isomorphism of functors $M_{g,n+1} \approx Z_{g,n}$, where $Z_{g,n}$ is the universal curve over $M_{g,n}$ (i.e. the functor of n-pointed stable curves with one extra section without any smoothness condition). The main steps of this proof are Lemma 1.6, Theorem 1.8, and the results in the appendix. We then use this result and an inductive argument to prove that $M_{g,n}$ is an algebraic stack, proper and smooth over Spec $(\mathbf{Z})$, and that the substack $S_{g,n}$ consisting of singular curves is a divisor with normal crossings relative to Spec (Z) (cf. Theorem 2.7.).

In section 3 we study the clutching morphisms β^0 and $\beta_{g_1,g_2,H,K}$, and prove that they are representable (Lemma 3.7), finite and unramified and almost always closed immersions (Corollary 3.9).

The clutching sequence Theorem 3.5 is used in section 4 to compute the pullback of the basic line bundles on $M_{g,n}$ by β. Actually what we are doing here is computing the self-intersection of the divisor at infinity $S_{g,n}$ of $M_{g,n}$ (section 4).

1. *n*-pointed stable curves.

Let S be a scheme, and let g, n be non-negative integers such that $2g-2+n > 0$.

DEFINITION 1.1. An n-pointed stable curve of genus g over S is a flat and proper morphism $\pi: C \to S$ together with n distinct sections $s_i: S \to C$ such that

i) The geometric fibres C_s of π are reduced and connected curves with at most ordinary double points.
ii) C_s is smooth at $P_i = s_i(s)$ $(1 \leqq i \leqq n)$.
iii) $P_i \neq P_j$ for $i \neq j$.
iv) The number of points where a nonsingular rational component E of C_s meets the rest of C_s plus the number of points P_i on E is at least 3.
v) $\dim H^1(C_s, \mathcal{O}_{C_s}) = g$.

Note that if $n=0$ and $g \geqq 2$, then C is a stable curve in the sense of [3].

Before we start with the technicalities, we briefly state some facts about the basic sheaves on a stable curve. Let $\pi: C \to S$ be a stable curve with sections s_i, $1 \leqq i \leqq n$. Since π is flat and the geometric fibres are reduced with only ordinary double points, π is locally a complete intersection morphism [8]. Therefore there is a canonical invertible dualizing sheaf $\omega_{C/S}$ on C. For reference, see [6], where $\omega_{C/S}$ is also denoted by $\pi^! \mathcal{O}_S$.

A way of constructing $\omega_{C/S}$ is via the theory of determinants [7]. The sheaf $\Omega_{C/S}$ of relative Kähler differentials is flat over S (see section 3, Proposition 3.2) and locally on C we can find a two-term free resolution

$$(0) \to \mathscr{E}^1 \to \mathscr{E}^0 \to \Omega_{C/S}|_U \to (0)\,.$$

This means that $\Omega_{C/S}$ considered as a complex supported only in degree 0 is a perfect complex, so we may form its determinant. We then have canonical isomorphisms

$$\Lambda^{\max}\mathscr{E}^0 \otimes \Lambda^{\max}(\mathscr{E}^1)\check{} \approx \det \Omega_{C/S}|_U \approx \omega_{C/S}|_U\,.$$

Since the fibres of π are reduced and one-dimensional, $\operatorname{rank}\mathscr{E}^0 = \operatorname{rank}\mathscr{E}^1 + 1 = k+1$. Let $\bar{v}$ be an element of $\Omega_{C/S,x}$, where $x \in U$ and let $v \in \mathscr{E}^0_x$ be an element which maps to $\bar{v}$. Choose a basis $w_1, w_2, \ldots, w_k$ of $\mathscr{E}^1$ and let $w'_1, w'_2, \ldots, w'_k$ be the dual basis. Considering the elements w_i as elements of $\mathscr{E}^0$ as well, we get an element

$$v \wedge w_1 \wedge w_2 \wedge \ldots \wedge w_k \otimes w'_1 \wedge w'_2 \wedge \ldots \wedge w'_k \in (\Lambda^{\max}\mathscr{E}^0 \oplus \Lambda^{\max}(\mathscr{E}^1)\check{}\,)_x$$

which is independent of the choice of $\bar{v}$ and the w_i's. Composing with the isomorphisms above, we see that we have a canonical homomorphism

$$\psi : \Omega_{C/S} \to \omega_{C/S}\,.$$

In general, this homomorphism is neither injective nor surjective, but it is an isomorphism near every point of C where π is smooth. Since $\Omega_{C/S}$ is flat over S, we have compactibility with any base change; i.e.,

a) For any morphism $T \to S$ we have a commutative diagram

$$\begin{array}{ccc} p_1^*(\Omega_{C/S}) & \xrightarrow{\sim} & \Omega_{C\times_S T/T} \\ {\scriptstyle p_1^*\psi}\downarrow & & \downarrow{\scriptstyle \psi} \\ p_1^*(\omega_{C/S}) & \xrightarrow{\sim} & \omega_{C\times_S T/T} \end{array}$$

b) If $S = \operatorname{Spec}(k)$, where k is an algebraically closed field, $f\colon \tilde{C} \to C$ is the normalization of C, and $x_1, \ldots, x_m, y_1, \ldots, y_m$ are the points of $\tilde{C}$ such that $z_i = f(x_i) = f(y_i)$, $1 \leqq i \leqq m$, are the double points of C, then $\omega_{C/S}$ is the sheaf of 1-forms η regular on $\tilde{C}$ except for simple poles at the x's and the y's and such that

$$\operatorname{Res}_{x_i}(\eta) + \operatorname{Res}_{y_i}(\eta) = 0 \quad \text{for } 1 \leqq i \leqq m\,.$$

c) If S is locally noetherian and of finite Krull dimension, and $\mathscr{F}$ is a coherent sheaf on C, then

$$\operatorname{Hom}_{\mathcal{O}_S}(R^1\pi_*\mathscr{F}, \mathcal{O}) \approx \operatorname{Hom}_{\mathcal{O}_C}(\mathscr{F}, \omega_{C/S})$$

(cf. [6, VII, Corollary 4.3.])

DEFINITION 1.2. We denote by $M_{g,n}$ the category of n-pointed stable curves. Morphisms in this category are diagrams of the form

$$\begin{array}{ccc} C' & \xrightarrow{f} & C \\ s'_i \uparrow\downarrow \pi' & & s_i \uparrow\downarrow \pi \\ S' & \xrightarrow{g} & S \end{array}$$

where

(i) $$fs'_i = s_i g \quad \text{for } 1 \leqq i \leqq n .$$

(ii) f, π' induce an isomorphism

$$C' \xrightarrow{\sim} C \times_S S' .$$

We denote by $Z_{g,n}$ the category of n-pointed stable curves with an extra section Δ. Morphisms in $Z_{g,n}$ are diagrams as above such that $f\Delta' = \Delta g$.

The category $M_{g,n}$ is a stack fibred in groupoids over the category of schemes. For a definition of stack, see [3, Definition 4.1]. In the next paragraph we prove that $M_{g,n}$ is a separated algebraic stack, smooth and proper over Spec $(\mathbf{Z})$.

The following definition plays a central role in this whole paper.

DEFINITION 1.3. A morphism of pointed stable curves over S:

$$\begin{array}{ccc} C & \xrightarrow{f} & C' \\ s_i \uparrow\downarrow \pi & & s'_i \uparrow\downarrow \pi' \\ S & = & S \end{array}$$

is called a contraction of

(i) C is an $n+1$-pointed curve, C' is an n-pointed curve and $fs_i = s'_i$ for $1 \leqq i \leqq n$.
(ii) If we consider the induced morphism on a geometric fibre C_s, we have one of two possible cases:

a) $f_s \colon C_s \to C'_s$ is an isomorphism.
b) There is a rational component $E \subset C_s$ such that $s_{n+1}(s) \in E$, $f_s(E) = x$ is a closed point of C'_s, and

$$f_s : C_s \setminus E \to C'_s \setminus \{x\}$$

is an isomorphism.

The picture in Figure 1 below shows the only two non-trivial contractions over Spec (k), where k is an algebraically closed field.

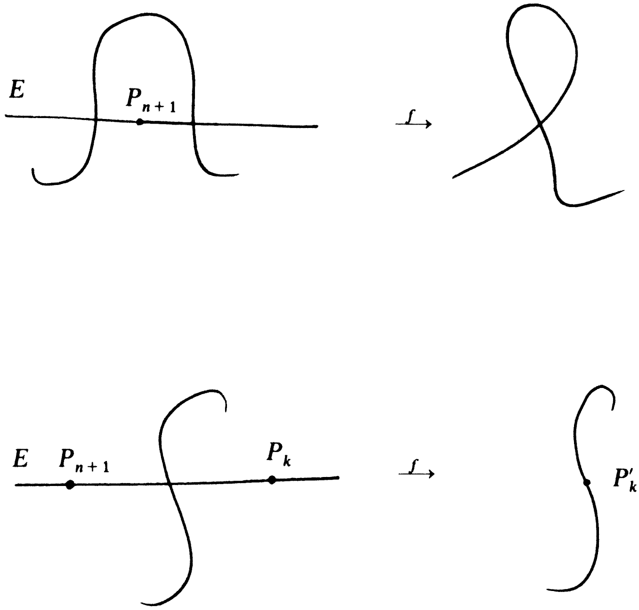

Figure 1.

REMARK. We leave to the reader to verify that when $S = \operatorname{Spec}(k)$, k an algebraically closed field, then for every C over S, there is one, and up to a unique isomorphism, only one contraction $C \to C'$. In fact, we have an equivalence of categories:

$$M_{g,n+1}(k) \xrightarrow{\sim} Z_{g,n}(k) .$$

In order to prove that there is an isomorphism of stacks $M_{g,n+1} \xrightarrow{\sim} Z_{g,n}$, we need the following results, which are corollaries of [5, III 4.6.1].

LEMMA 1.4. *Let Y be a locally noetherian scheme, $f\colon X \to Y$ a proper morphism, $\mathscr{F}$ a coherent sheaf on X, and y a point on Y. Suppose $f^{-1}(y) = X \times_Y \operatorname{Spec}(k(y))$ is an n-dimensional scheme and that*

$$H^n(f^{-1}(y), \mathscr{F} \otimes_{\mathcal{O}_Y} k(y)) = (0) .$$

Then:

i) *There exists a neighbourhood U of y in Y such that $R^n f_* \mathscr{F}|_U = (0)$.*
ii) *For each integer $p \geqq 0$ the canonical morphism*

$$(R^{n-1} f_* \mathscr{F})_y \to H^{n-1}(f^{-1}(y), \mathscr{F} \otimes_{\mathcal{O}_y} \mathcal{O}_y / m_y^{p+1})$$

is surjective.

PROOF. Consider the diagram

$$\begin{array}{ccccc} f^{-1}(y) & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}\,(k(y)) & \rightarrow & \mathrm{Spec}\,(\mathcal{O}_y) & \xrightarrow{i} & Y. \end{array}$$

Since i is flat, we can reduce the proof of the lemma to the case, where Y is affine and y is a closed point of Y. Since $R^n f_* \mathcal{F}$ is coherent, the first assertion is equivalent to $(R^n f_* \mathcal{F})_y = (0)$.

By [5, III 4.2.1], it suffices to prove that

$$H^n(f^{-1}(y), \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_y/m_y^{p+1}) = (0) \quad \text{for all } p .$$

It is true for $p=0$, so we proceed to prove it by induction.

Let $X_p = X \times_Y \mathrm{Spec}\,(\mathcal{O}_y/m_y^{p+1})$. Then X_{p-1} is a closed subscheme of X_p with the same underlying topological space. Hence by induction hypothesis we have

$$H^n(X_p, \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_y/m_y^p) = H^n(X_{p-1}, \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_y/m_y^p) = (0) .$$

On X_p we have an exact sequence of sheaves

$$(0) \rightarrow m_y^p \mathcal{F}/m_y^{p+1}\mathcal{F} \rightarrow \mathcal{F}/m_y^{p+1}\mathcal{F} \rightarrow \mathcal{F}/m_y^p\mathcal{F} \rightarrow (0) .$$

So from the long exact cohomology sequence it suffices to prove that for each p

$$(*) \qquad H^n(X_p, m_y^p \mathcal{F}/m_y^{p+1}\mathcal{F}) = (0) .$$

If we denote by Z the fibre $f^{-1}(y) = X_0$, then $m_y^p\mathcal{F}/m_y^{p+1}\mathcal{F}$ may be considered as an $\mathcal{O}_Z$-module and we have:

$$H^n(X_p, m_y^p \mathcal{F}/m_y^{p+1}\mathcal{F}) = H^n(Z, m_y^p \mathcal{F}/m_y^{p+1}\mathcal{F}) .$$

Let Q_p denote the kernel of the surjection

$$\mathcal{F}/m_y\mathcal{F} \otimes_{k(y)} m_y^p/m_y^{p+1} \rightarrow m_y^p\mathcal{F}/m_y^{p+1}\mathcal{F} .$$

We then have an exact sequence

$$\ldots \rightarrow H^n(Z, \mathcal{F}/m_y\mathcal{F} \otimes_{k(y)} m_y^p/m_y^{p+1}) \rightarrow H^n(Z, m_y^p\mathcal{F}/m_y^{p+1}\mathcal{F}) \rightarrow H^{n+1}(Z, Q_p) .$$

The sheaf $\mathcal{F}/m_y\mathcal{F} \otimes_{k(y)} m_y^p/m_y^{p+1}$ is just a direct sum of $\mathcal{F}/m_y\mathcal{F}$'s and therefore its nth cohomology group vanishes by the hypothesis. The last term vanishes since Z is n-dimensional. This proves the first assertion. For the second assertion we replace $\mathcal{F}$ by $\mathcal{G} = m_y^k\mathcal{F}$. By (*) we see that $\mathcal{G}$ satisfies the condition of the lemma so by the first assertion we have $(R^n f_* \mathcal{G})_y = (0)$.

From the long exact sequence of cohomology sheaves we get

$$(R^{n-1} f_* \mathcal{F})_y \rightarrow (R^{n-1} f_*(\mathcal{F}/m_y^p\mathcal{F}))_y \rightarrow (R^n f_*(m_y^p\mathcal{F}))_y = (0) .$$

But $R^{n-1}f_*(\mathscr{F}/m_y^p\mathscr{F})$ is a skyscraper sheaf, whose stalk at y is $H^{n-1}(f^{-1}(y), \mathscr{F}/m_y^p\mathscr{F})$. This completes the proof of the lemma.

COROLLARY 1.5. *Let S be a scheme, X and Y S-schemes and $f\colon X \to Y$ a proper S-morphism, whose fibres are at most one-dimensional. Let $\mathscr{F}$ be a coherent sheaf on X, flat over S, and consider the following two conditions*:

(1) *For each closed point $y \in Y$,*

$$H^1(f^{-1}(y), \mathscr{F} \otimes_{\mathcal{O}_Y} k(y)) = (0) .$$

(2) *For each closed point $y \in Y$, the sheaf $\mathscr{F} \otimes_{\mathcal{O}_Y} k(y)$ is generated by its global sections.*

Then if $\mathscr{F}$ satisfies (1), *we have*

i) $$R^1 f_* \mathscr{F} = (0) .$$

ii) $$f_* \mathscr{F} \text{ is } S\text{-flat} .$$

iii) *For any morphism $T \to S$ we have canonical isomorphisms*

$$f_* \mathscr{F} \otimes_{\mathcal{O}_S} \mathcal{O}_T \xrightarrow{\sim} (f \times 1)_* (\mathscr{F} \otimes_{\mathcal{O}_S} \mathcal{O}_T) .$$

If $\mathscr{F}$ satisfies both (1) *and* (2) *we have*

iv) *The canonical map $f^* f_* \mathscr{F} \to \mathscr{F}$ is surjective.*

PROOF. We may assume Y and S affine. Let $\mathfrak{U}$ be a finite affine covering of X. The first three assertions follows immediately by considering the sheaves $f_* \mathscr{C}_a^i(\mathfrak{U}, \mathscr{F})$ of alternating Čech cochains on Y. By the second part of the previous lemma and condition (2) it follows that, for each closed point $x \in X$, we have a surjection

$$f^* f_* \mathscr{F} \to \mathscr{F} \otimes_{\mathcal{O}_X} k(x) ,$$

so iv) follows by Nakayama's lemma.

Let $f\colon C \to C'$ be a contraction of an $n+1$-pointed stable curve over S, s_i $(1 \leqq i \leqq n+1)$ the sections of C over S, and $t_i = s_i f$ $(1 \leqq i \leqq n)$ the sections of C' over S. For every open U in C', a regular function on U with at least simple zeros at the sections t_i pulls back to a regular function on $f^{-1}(U)$ with at least simple zeros at the s_i's $(1 \leqq i \leqq n)$. Hence we have a canonical morphism of sheaves on C'

$$u\colon \mathcal{O}_{C'}(-t_1 - \ldots - t_n) \to f_* \mathcal{O}_C(-s_1 - \ldots - s_n) .$$

By definition, the geometric fibres of $C \to C'$ are at worst $\boldsymbol{P}^1$'s. Since they all have at least one rational point via one of the compositions $s_i \circ \pi'$ $(1 \leqq i \leqq n+1)$, all fibres are at worst $\boldsymbol{P}^1$'s. Each such fibre has at most one s_i $(1 \leqq i \leqq n)$ on it, so $\mathcal{O}_C(-s_1 - \ldots - s_n)$ satisfies condition (1) of Corollary 1.5. Therefore the formation of f_* commutes with base change over S. When $S = \operatorname{Spec}(k)$, u is easily seen to be an isomorphism. Since an extension of fields is faithfully flat, u is an isomorphism at every point of C', so by Nakayama's lemma u is always surjective. By Corollary 1.5 ii), $f_*\mathcal{O}_C(-s_1 - \ldots - s_n)$ is flat over S, so by the same reasoning u is an isomorphism. The inverse of the isomorphism u induces an isomorphism of sheaves on C':

$$\mathcal{H}om_{\mathcal{O}_{C'}}(\Omega_{C'/S}, f_*\mathcal{O}_C(-s_1 - \ldots - s_n)) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{C'}}(\Omega_{C'/S}, \mathcal{O}_{C'}(-t_1 - \ldots - t_n)).$$

By the general theory of sheaves of modules there is a canonical isomorphism

$$\mathcal{H}om_{\mathcal{O}_{C'}}(\Omega_{C'/S}, f_*\mathcal{O}_C(-s_1 - \ldots - s_n)) \xrightarrow{\sim} f_*\mathcal{H}om_{\mathcal{O}_C}(f^*\Omega_{C'/S}, \mathcal{O}_C(-s_1 - \ldots - s_n)).$$

Combining this with the natural map $f^*\Omega_{C'/S} \to \Omega_{C/S}$ we get a morphism

$$f_*\mathcal{H}om_{\mathcal{O}_C}(\Omega_{C/S}, \mathcal{O}_C(-s_1 - \ldots - s_n)) \to \mathcal{H}om_{\mathcal{O}_{C'}}(\Omega_{C'/S}, \mathcal{O}_{C'}(-t_1 - \ldots - t_n)).$$

For a stable curve $\pi\colon X \to S$, let $\mathcal{F}$ be the cokernel of the morphism $\psi\colon \Omega_{X/S} \to \omega_{X/S}$. We have

$$\operatorname{Ass} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) = \operatorname{Supp} \mathcal{F} \cap \operatorname{Ass} \mathcal{O}_X$$

[2, Chapt. IV, § 1, Prop. 10.]

Since X is flat over S, the associated points of $\mathcal{O}_X$ lie over the associated points of $\mathcal{O}_S$. However, π is smooth at the associated points of the fibres and $\mathcal{F}$ is supported on the closed set where π is not smooth, so $\check{\omega}_{X/S} \to \check{\Omega}_{X/S}$ is injective. Consider the diagram

$$\begin{array}{ccc} f_*\mathcal{H}om_{\mathcal{O}_C}(\Omega_{C/S}, \mathcal{O}_C(-s_1 - \ldots - s_n)) & \to & \mathcal{H}om_{\mathcal{O}_{C'}}(\Omega_{C'/S}, \mathcal{O}_{C'}(-t_1 - \ldots - t_n)) \\ \uparrow & & \uparrow \\ f_*(\omega_{C/S}(s_1 + \ldots + s_n)\check{\ }) & & \omega_{C'/S}(t_1 + \ldots + t_n)\check{\ } \end{array}$$

and let $\mathcal{P}$ denote the subsheaf of $\mathcal{H}om_{\mathcal{O}_{C'}}(\Omega_{C'/S}, \mathcal{O}_{C'}(-t_1 - \ldots - t_n))$ generated by $\omega_{C'/S}(t_1 + \ldots + t_n)\check{\ }$ and the image of $f_*(\omega_{C/S}(s_1 + \ldots + s_n)\check{\ })$. On the nontrivial fibres of $C \to C'$, $\omega_{C/S}(s_1 + \ldots + s_n)\check{\ }$ is non-canonically isomorphic to $\mathcal{O}_{P^1}$, so it satisfies both conditions of Corollary 1.5. Therefore, on the geometric fibres of $C' \to S$, the map $f_*(\omega_{C/S}(s_1 + \ldots + s_n)\check{\ }) \to \mathcal{P} \otimes_{\mathcal{O}_S} k$ factors through $\omega_{C'/S}(t_1 + \ldots + t_n)\check{\ }$. By Nakayama's lemma, then $\omega_{C'/S}(t_1 + \ldots + t_n)\check{\ } \approx \mathcal{P}$, so we get a global factorization which again by Nakayama's lemma is an isomorphism. Pullback by f gives us

$$f^*(\omega_{C'/S}(t_1 + \ldots + t_n)\check{\ }) \xleftarrow{\sim} f^*f_*(\omega_{C/S}(s_1 + \ldots + s_n)\check{\ }) \twoheadrightarrow \omega_{C/S}(s_1 + \ldots + s_n)\check{\ }.$$

By checking on the geometric fibres we see that the surjection on the right is an isomorphism. Since for locally free sheaves f^* commutes with dualization we get an isomorphism

$$f^*\omega_{C'/S}(t_1+\ldots+t_n) \xrightarrow{\sim} \omega_{C/S}(s_1+\ldots+s_n)\,.$$

By the general theory of sheaves there is an induced map

$$\omega_{C'/S}(t_1+\ldots+t_n) \to f_*\omega_{C/S}(s_1+\ldots+s_n)\,.$$

Again $\omega_{C/S}(s_1+\ldots+s_n)$ is trivial on the fibres of f so we may apply Corollary 1.5 and Nakayama's lemma to show that this map, too, is an isomorphism. We sum this up in

LEMMA 1.6 (MAIN LEMMA). *Consider a contraction $f\colon C \to C'$ as in Definition 1.3. We denote by $\mathscr{F}$ and $\mathscr{F}'$ the sheaves $\omega_{C/S}(s_1+\ldots+s_n)$ and $\omega_{C'/S}(s_1'+\ldots+s_n')$ respectively. Then for all $k>0$ we have:*

a) *There are canonical isomorphisms*

$$\mathscr{F}'^{\otimes k} \xrightarrow{\sim} f_*(\mathscr{F}^{\otimes k})$$

and

$$f^*\mathscr{F}'^{\otimes k} \xrightarrow{\sim} \mathscr{F}^{\otimes k}\,.$$

b) $$R^1f_*(\mathscr{F}^{\otimes k}) = (0)\,.$$

c) $$R^i\pi_*(\mathscr{F}^{\otimes k}) \approx R^i\pi'_*(\mathscr{F}'^{\otimes k}) \quad \textit{for } i\geqq 0\,.$$

PROOF. The isomorphism $f^*\mathscr{F}'^{\otimes k} \xrightarrow{\sim} \mathscr{F}^{\otimes k}$ follows from what we have just done. Pushing this down we at least get a map

$$\mathscr{F}'^{\otimes k} \to f_*f^*(\mathscr{F}'^{\otimes k}) \xrightarrow{\sim} f_*(\mathscr{F}^{\otimes k})\,.$$

Since $\mathscr{F}$ is trivial on the fibres of f, so is $\mathscr{F}^{\otimes k}$ and Corollary 1.5 applies to $\mathscr{F}^{\otimes k}$. On the geometric fibres the above composition is an isomorphism and $f_*(\mathscr{F}^{\otimes k})$ is flat over S. The same reasoning as before then proves a). b) is exactly the first assertion of Corollary 1.5 and c) follows from a) and the Leray spectral sequence which is degenerate by b).

DEFINITION 1.7. Let $\mathscr{F}$ be a coherent sheaf on a scheme X. We will say that $\mathscr{F}$ is *normally generated* if the canonical map

$$\Gamma(X,\mathscr{F})^{\otimes k} \to \Gamma(X,\mathscr{F}^{\otimes k})$$

is surjective for $k\geqq 1$.

For any pair of $\mathcal{O}_X$-modules $\mathscr{F},\mathscr{G}$, let $S(\mathscr{F},\mathscr{G})$ denote the cokernel of the map

$$\Gamma(X,\mathscr{F})\otimes\Gamma(X,\mathscr{G})\to\Gamma(X,\mathscr{F}\otimes\mathscr{G}).$$

Clearly, $\mathscr{F}$ is normally generated if and only if $S(\mathscr{F},\mathscr{F}^{\otimes k})=(0)$ for all $k\geqq 1$. We shall need the following result of [13].

GENERALIZED LEMMA OF CASTELNUOVO. *Suppose $\mathscr{L}$ is an invertible sheaf on a complete scheme X of finite type over a field k such that $\Gamma(X,\mathscr{L})$ has no base points and suppose $\mathscr{F}$ is a coherent sheaf on X such that*

$$H^i(X,\mathscr{F}\otimes\mathscr{L}^{-i}) = (0) \quad \textit{for } i\geqq 1.$$

Then

(a) $$H^i(X,\mathscr{F}\otimes\mathscr{L}^{j}) = (0) \quad \textit{for } i+j\geqq 0,\ i\geqq 1.$$

(b) $$S(\mathscr{F}\otimes\mathscr{L}^{i},\mathscr{L}) = (0) \quad \textit{for } i\geqq 0.$$

THEOREM 1.8. *Let C be an n-pointed stable curve over* Spec(k) *with distinguished k-valued points $P_1,\dots,P_n$. We denote by $\mathscr{L}$ the invertible sheaf*

$$\mathscr{L} = \omega_{C/k}(D),$$

where $D=P_1+P_2+\dots+P_n$.

Then we have

a) $H^1(C,\mathscr{L}^{\otimes m})=(0)$ *for* $m\geqq 2$.
b) $\Gamma(C,\mathscr{L}^{\otimes m})$ *is base-point-free for* $m\geqq 2$.
c) $\mathscr{L}^{\otimes m}$ *is normally generated for* $m\geqq 3$.

PROOF. By the Künneth theorem we may assume that k is algebraically closed. Let x be a node of C. We will call x a disconnecting node if $B_x(C)$, the blowing up of C with center x, is disconnected. We first prove the theorem in case C has no disconnecting nodes. Let x be a node of C. From the exact sequence

$$(0)\to\mathcal{O}_C\to p_*\mathcal{O}_{B_x(C)}\xrightarrow{-}k\to(0),$$

we get $\chi(\mathcal{O}_C)=\chi(\mathcal{O}_{B_x(C)})-1$. Since $B_x(C)$ is connected, we have

$$\dim H^1(B_x(C),\mathcal{O}_{B_x(C)}) = \dim H^1(C,\mathcal{O}_C)-1.$$

From this formula we see that a curve of genus 0 without disconnecting nodes is nonsingular, i.e., $\boldsymbol{P}^1$ and a curve of genus 1 is either nonsingular or a "ring" of $\boldsymbol{P}^1$'s as in Figure 2 below.

For $g=0$, $C=\boldsymbol{P}^1$ and the theorem is clear. When $g=1$ and C is nonsingular, the theorem is classical. Consider, then, a "ring" of $\boldsymbol{P}^1$'s. In a noncanonical way, $\omega_{C/k}\approx\mathcal{O}_C$, and since there are lots of distinguished points spread around, $\mathscr{L}$

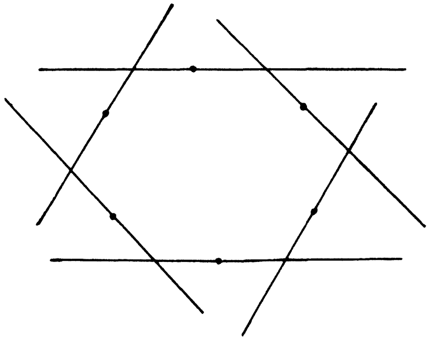

Figure 2.

restricted to each $\boldsymbol{P}^1$ has degree $\geqq 1$. Hence $\Gamma(C, \mathscr{L})$ is base-point-free and $H^1(C, \mathscr{L}) = (0)$. The theorem then follows from Castelnuovo's Lemma. When $g \geqq 2$, we have the following result. If E is an effective divisor on C, then

$$\Gamma(\omega_C(E)) \text{ is base-point-free for } \deg E = 0 \text{ or } \deg E \geqq 2\,,$$

$$H^1(\omega_C(E)) = 0 \quad \text{if } \deg E \geqq 1\,.$$

PROOF. Let x be a k-rational (closed) point of C. From the short exact sequence

$$(0) \to m_x\omega(E) \to \omega(E) \to \omega(E)\otimes k \to (0)$$

we get the long exact sequence

$$\to \Gamma(\omega(E)) \to \Gamma(\omega(E)\otimes k) \to H^1(m_x\omega(E)) \xrightarrow{\alpha} H^1(\omega(E)) \to (0)\,.$$

Hence $\Gamma(\omega(E))$ is base-point-free if and only if α is an isomorphism for all points $x \in C$. By duality, α is an isomorphism, if and only if

$$\dim \mathscr{H}om_{\mathcal{O}_C}(m_x, \mathcal{O}(-E)) = \dim \Gamma(C, \mathcal{O}(-E))\,.$$

But we have

$$\dim \mathscr{H}om_{\mathcal{O}_C}(m_x, \mathcal{O}(-E)) = \begin{cases} \dim \Gamma(C, \mathcal{O}(-E+x)), & x \text{ nonsingular} \\ \dim \Gamma(B_x(C), \mathcal{O}_{B_x(C)}(-E)), & x \text{ singular}\,. \end{cases}$$

The result follows, since $B_x(C)$ is connected.

This proves a) and b) of the theorem. To prove c), consider the diagram

$$\begin{array}{ccc} \Gamma(\omega^{km}(kmD))\otimes\Gamma(\omega)\otimes\Gamma(\omega^{m-1}(mD)) & \longrightarrow & \Gamma(\omega^{km}(kmD))\otimes\Gamma(\omega^{m}(mD)) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} \\ \Gamma(\omega^{km+1}(kmD))\otimes\Gamma(\omega^{m-1}(mD)) & \xrightarrow{\beta} & \Gamma(\omega^{(k+1)m}((k+1)mD))\,. \end{array}$$

By Castelnuovo's Lemma, α is surjective, since $\Gamma(\omega)$ is base-point-free and

$$H^1(\omega^{km-1}(kmD)) = (0) \quad \text{for } k \geqq 1 \text{ and } m \geqq 3 .$$

β is surjective, since $\Gamma(\omega^{m-1}(mD))$ is base-point-free for $m \geqq 2$ and

$$H^1(\omega^r((k-1)mD)) = (0) \quad \text{for } r \geqq 2 .$$

Hence γ is surjective for all $k \geqq 1$ and $m \geqq 3$.

We now prove the theorem by induction on the number of disconnecting nodes. Let x be a disconnecting node of C, C_1, and C_2 the two connected components of $B_x(C)$. If x_1 (respectively x_2) is the point of C_1 (respectively C_2) which maps to x, and if we take x_1 (respectively x_2) to be an extra distinguished point on C_1 (respectively C_2), we see that C_1 is an $l+1$-pointed stable curve and C_2 is an l_2+1-pointed stable curve, where $l_1+l_2=n$. Let $\mathscr{L}_1$ (respectively $\mathscr{L}_2$) be the sheaf $\omega_{C_1}(D_1)$ (respectively $\omega_{C_2}(D_2)$), where D_1 (respectively D_2) is the distinguished divisor of C_1 (respectively C_2). If i_1 and i_2 are the closed immersions of C_1 and C_2 into C, we have by property b) of the dualizing sheaves

$$i_\beta^*(\mathscr{L}^{\otimes m}) \approx \mathscr{L}_\beta^{\otimes m}, \quad \beta = 1,2 .$$

Moreover, both C_1 and C_2 have fewer disconnecting nodes than C, so the theorem holds for $\mathscr{L}_1$ and $\mathscr{L}_2$ by the induction hypothesis.

We have an exact sequence

$$(0) \to \Gamma(\mathscr{L}^m) \to \Gamma(\mathscr{L}_1^m) \oplus \Gamma(\mathscr{L}_2^m) \xrightarrow{\alpha_m} k(x) \to H^1(\mathscr{L}^m) \to H^1(\mathscr{L}_1^m) \oplus H^1(L_2^m) \to (0) .$$

For $m \geqq 2$, α_m is surjective by part b) of the theorem and $H^1(\mathscr{L}_\beta^m) = (0)$ for $\beta = 1,2$ by part a) of the theorem. This proves a) for $\mathscr{L}$. Part b) of the theorem is clear, since a section of $\mathscr{L}^m$ is the same as a pair of sections (s,t) with $s \in \Gamma(\mathscr{L}_1^m)$ and $t \in \Gamma(\mathscr{L}_2^m)$ such that $s(x_1) = t(x_2)$. To prove part c) let $m \geqq 3$ and $k \geqq 1$ and consider a section s of $\Gamma(\mathscr{L}^{(k+1)m})$ such that $s(y) = 0$ for all points $y \in C_2$. Let $s_1, \ldots, s_r$ be sections of $\Gamma(\mathscr{L}_1^{km})$ and $t_1, \ldots, t_r$ be sections of $\Gamma(\mathscr{L}_1^m)$ such that $s|_{C_1}$ is the image of $s_1 \otimes t_1 + \ldots + s_r \otimes t_r$ by the canonical map. Since $\Gamma(\mathscr{L}_2^m)$ is base-point-free we can find sections u of $\Gamma(\mathscr{L}_2^{km})$ and v of $\Gamma(\mathscr{L}_2^m)$ such that $u \otimes v\ (x_2) \neq (0)$. Hence there are scalars $a_1, \ldots, a_r$, $b_1, \ldots, b_r$ such that

$$a_i u(x_2) = s_i(x_1) \quad \text{and} \quad b_i v(x_2) = t_i(x_1), \quad 1 \leqq i \leqq r .$$

The sections $\bar{s}_i$ defined as s_i on C_1 and as $a_i u$ on C_2 and the sections $\bar{t}_i$ defined as t_i on C_1 and as $b_i v$ on C_2 are global sections of $\mathscr{L}^{km}$ and $\mathscr{L}^m$, respectively. By the canonical map the section

$$\bar{s}_1 \otimes \bar{t}_1 + \ldots + \bar{s}_r \otimes \bar{t}_r \in \Gamma(\mathscr{L}^{km}) \otimes \Gamma(\mathscr{L}^m)$$

maps to s because $\sum a_i b_i = 0$. This argument holds just as well for a section s of $\Gamma(\mathscr{L}^{(k+1)m})$ which vanishes on C_1. It follows that the image of the map

$$\Gamma(\mathscr{L}^{km}) \otimes \Gamma(\mathscr{L}^m) \to \Gamma(\mathscr{L}^{(k+1)m})$$

contains all sections which vanish at x. The theorem will follows, if we can show that there is at least one section in the image that does not vanish at x, but this is clear since $\mathscr{L}_C^m$ is base-point-free.

COROLLARY 1.9. *If C and $\mathscr{L}$ are as in Theorem* 1.8, *then $\mathscr{L}^m$ is very ample for* $k \geqq 3$.

PROOF. $\mathscr{L}$ restricted to each irreducible component has positive degree and is therefore ample. Furthermore, it is clear that a normally generated ample sheaf is very ample.

COROLLARY 1.10. *Let C be an $n+1$-pointed stable curve over a field k with distinguished points $P_1, \ldots, P_{n+1}$ and such that $2g-2+n>0$. Then the sheaf $\omega_{C/k}(P_1 + \ldots + P_n)$ satisfies* a), b), *and* c) *of Theorem* 1.8.

PROOF. Clear by Lemma 1.6 and the theorem.

COROLLARY 1.11. *Let $\pi\colon C \to S$ be an $n+1$-pointed stable curve with $2g-2+n>0$. Then the sheaf $\pi_*(\omega_{C/S}(s_1 + \ldots + s_n)^{\otimes m})$ is locally free of rank $(2g-2+n)m-g+1$ for $m \geqq 2$.*

PROOF. Clear, since C is flat over S and

$$R^1\pi_*(\omega_{C/S}(s_1 + \ldots + s_n)^{\otimes m}) = (0) \quad \text{for } m \geqq 2 .$$

2. Contraction and stabilization.

In this paragraph we will constantly make use of the following fact:

Let $\mathscr{F}$ be a coherent sheaf on a scheme Y, then we have a one to one correspondence:

$$\left\{\begin{matrix} S\text{-valued points} \\ \text{of } \mathscr{P}roj(\mathrm{Sym}\,\mathscr{F}) \end{matrix}\right\} \leftrightarrow \left\{\begin{array}{l} \text{Triples consisting of:} \\ \text{1) a map } f\colon S \to Y, \\ \text{2) an invertible sheaf } \mathscr{L} \text{ on } S \\ \text{3) an epimorphism } \alpha\colon f^*\mathscr{F} \to \mathscr{L} \end{array}\right\} \Big/ \sim$$

Two triples $(f, \mathscr{L}, \alpha)$ and $(f', \mathscr{L}', \alpha')$ are equivalent, if $f = f'$ and $\ker\alpha = \ker\alpha'$.

Proposition 2.1. *Given any $n+1$-pointed stable curve X over S with $2g-2+n>0$, there is one and up to canonical isomorphism only one contraction.*

Proof. Let $\pi: X \to S$ be the curve and define

$$\mathcal{S}^i = \pi_*(\omega_{X/S}(s_1+\ldots+s_n)^{\otimes i})$$

$$\mathcal{S} = \bigoplus_{i\geqq 0} \mathcal{S}^i .$$

By Corollary 1.11, $\mathcal{S}^i$ is a locally free sheaf on S for $i\geqq 2$.

We define

$$X^c = \mathcal{P}roj(\mathcal{S})$$

$$Y = \mathcal{P}roj(\operatorname{Sym}\mathcal{S}^3) .$$

By Theorem 1.8. and Corollary 1.5. we have a surjection:

$$\pi^*(\mathcal{S}^3) \to \omega_{C/S}(s_1+\ldots+s_n)^{\otimes 3}$$

i.e. a morphism $p: X \to Y$. But since $\omega_{X/S}(s_1+\ldots+s_n)^{\otimes 3}$ is normally generated by Corollary 1.9, X^c is exactly the image of this morphism, and X^c is flat over S, since $\mathcal{S}^i$ is flat for $i\geqq 2$.

For uniqueness consider a diagram:

$$\begin{array}{ccc} X & \searrow & \\ {\scriptstyle p}\downarrow & & \\ X' & \xrightarrow{f} & X^c \\ {\scriptstyle s_i'}\uparrow\,{\scriptstyle \pi'}\downarrow & & \downarrow \\ S & = & S \end{array}$$

where q is a contraction. We have to prove that a map f exists making the diagram commutative.

By Lemma 1.6 c) we have an isomorphism:

$$\pi_*\omega_{X/S}(s_1+\ldots+s_n)^{\otimes k} \approx \pi'_*\omega_{X'/S}(s'_1+\ldots+s'_n)^{\otimes k} .$$

Hence by Corollary 1.5 and Theorem 1.8 a surjection:

$$\pi'^*(\pi_*\omega_{X/S}(s_1+\ldots+s_n)^{\otimes k}) \to \omega_{X'/S}(s'_1+\ldots+s'_n)^{\otimes k}$$

and this is f.

In the language of stacks, the proposition says that contraction is a 1-morphism of stacks:

$$c : M_{g,n+1} \to Z_{g,n} .$$

The rest of this paragraph will be devoted to the construction of an inverse to c which we call stabilization:

$$s : Z_{g,n} \to M_{g,n+1}$$

LEMMA 2.2. *Let S be a noetherian scheme, and π: $X \to S$ a flat family of reduced curves with at most ordinary double points. Let Δ: $S \to X$ be a section defined by an $\mathcal{O}_X$-ideal $\mathcal{J}$. Then*

1) *$\mathcal{J}$ is stably reflexive with respect to π [see appendix].*
2) *The subsheaf $\mathcal{J}'$ of the total quotient ring sheaf K_X consisting of sections that multiply $\mathcal{J}$ into $\mathcal{O}_X$ is isomorphic to the dual of $\mathcal{J}$, that is $\mathcal{J}' \approx \check{\mathcal{J}} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}, \mathcal{O}_X)$.*
3) *$\Delta^*(\check{\mathcal{J}}/\mathcal{O}_X)$ is an invertible sheaf on S.*

PROOF. The theorem is of local nature so let s be a point of S such that $x = \Delta(s)$ is an ordinary double point of the fibre $\pi^{-1}(s)$, $\mathfrak{o}$ the completion of the local ring $\mathcal{O}_{S,s}$, with residue field k. We consider the category Λ of Artin local $\mathfrak{o}$-algebras with residue field k and the functor G on Λ defined by:

$$G(A) = \left\{ \begin{array}{l} \text{Cartesian diagrams of the form} \\ \begin{array}{ccc} k[[x,y]]/(x\cdot y) & \xleftarrow{q} & B \\ {\scriptstyle g}\downarrow\uparrow & & {\scriptstyle h}\downarrow\uparrow \\ k & \xleftarrow{p} & A \end{array} \\ \text{where } g(x)=g(y)=0 \text{ and } gp=ph \end{array} \right\} \Big/ \begin{array}{l}\text{modulo} \\ \text{isomorphisms}\end{array}$$

It follows from the general theory of deformations that there exists a versal deformation

$$\begin{array}{ccc} k[[x,y]]/(x\cdot y) & \xleftarrow{q} & \mathcal{B} \\ {\scriptstyle g}\downarrow\uparrow & & {\scriptstyle h}\downarrow\uparrow \\ k & \xleftarrow{p} & \mathcal{A} \end{array}$$

where $\mathcal{B} = \mathfrak{o}[[s,t,x,y]]/(xy-st)$, $\mathcal{A} = \mathfrak{o}[[s,t]]$, and $h(x)=s$, $h(y)=t$.

This means that there exist two element b, c in $\mathfrak{o}$ such that $\hat{\mathcal{O}}_{X,x} \approx \mathfrak{o}[[x,y]]/(xy-bc)$ and $\hat{\mathcal{J}}$ corresponds to the ideal generated by $x-b$ and $y-c$. The Lemma now follows from Proposition 6 of the appendix and the example.

DEFINITION 2.3 (The stabilization morphism). Consider an S-valued point of $Z_{g,n}$ i.e. an n-pointed stable curve $\pi\colon X \to S$ together with n sections $s_1,\ldots,s_n$ and an extra section Δ. Let $\mathscr{J}$ be the $\mathcal{O}_X$-ideal defining Δ, and define the sheaf $\mathscr{K}$ on X via the exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{\delta} \mathscr{J}^{\vee} \oplus \mathcal{O}_X(s_1+s_2+\ldots+s_n) \to \mathscr{K} \to 0$$

where δ is the "diagonal" $\delta(t)=(t,t)$.

Then

$$X^s = \mathscr{P}roj(\mathrm{Sym}\,\mathscr{K}).$$

It is clear by Lemma 2.2 that for any $T \to S$ we have a canonical isomorphism

$$\psi_{T,S} : (X^s)_T \xrightarrow{\sim} (X_T)^s$$

and that the ψ's satisfy the "cocycle" condition for any pair of morphisms $U \to T$, $T \to S$.

THEOREM 2.4. *With notations as in Definition* 2.3 *the sections* $s_1,\ldots,s_n$ *and* Δ *have unique liftings* $s'_1, s'_2,\ldots,s'_{n+1}$ *to* X^s *making* X^s *an* $n+1$*-pointed stable curve and* $p\colon X^s \to X$ *a contraction*, i.e. *the assignment* X^s *to* X *is a* 1*-morphism of stacks*

$$s : Z_{g,n} \to M_{g,n+1}.$$

PROOF. The theorem is local on S. We must study the map p in the neighbourhood of points, where Δ meets non-smooth points of the fibre and in the neighbourhood of points, where Δ meets one of the other sections. Since π is smooth near the s_i's we may study these cases separately.

CASE I. Δ meets a non-smooth point x in the fibre.

In this case we have a completed fibre-product diagram

$$\begin{array}{ccc} X^s|_{p^{-1}(x)} & \hookrightarrow & X^s \\ \hat{p}\downarrow & & p\downarrow \\ \hat{X}|_x & \hookrightarrow & X \end{array}$$

$$\hat{\mathcal{O}}_x \approx \mathfrak{o}[[x,y]]/(xy-bc)$$

and we have an exact sequence

$$\hat{\mathcal{O}}_x \oplus \hat{\mathcal{O}}_x \xrightarrow{\alpha} \hat{\mathcal{O}}_x \oplus \hat{\mathcal{O}}_x \to (\mathscr{J}_x^{\vee})\hat{} \to 0,$$

where

$$\alpha = \begin{pmatrix} -b & y \\ -x & c \end{pmatrix}$$

(note that in this case we have $\mathscr{I}^\vee \approx \mathscr{K}$).

Therefore $\hat{X}^s|_{p^{-1}(x)}$ is covered by two affines $\operatorname{Spec} f(R_1)$ and $\operatorname{Spec} f(R_2)$, where

$$R_1 = \hat{\mathcal{O}}_x\{s\}/(x+bs, c-ys) = \mathfrak{o}[[y]]\{s\}/(ys-c)$$
$$R_2 = \hat{\mathcal{O}}_x\{t\}/(xt+b, ct-y) = \mathfrak{o}[[x]]\{t\}/(xt+b)\,.$$

Hence we see that $X^s \to S$ is a flat family of reduced curves with at most ordinary double points.

The surjection $\Delta^*(\mathscr{K}) \to \Delta^*(\mathscr{K}/\mathcal{O}_X) \cong \Delta^*(\mathscr{I}^\vee/\mathcal{O}_X)$ gives us a lifting of the section Δ. Recall Lemma 2.2 that $\Delta^*(\mathscr{I}^\vee/\mathcal{O}_X)$ is an invertible sheaf on S.

In the coordinates we have chosen $s'_{n+1}(p(x))$ is the point given by $s=t=-1$. In particular $X^s \to S$ is smooth at $s'_{n+1}(p(x))$.

Case II. In this case Δ is a divisor. Assuming only one section s we have

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(\Delta) \oplus \mathcal{O}_X(s) \to \mathscr{K} \to 0\,.$$

It is therefore clear that the fibres of p are at most projective lines, so by Corollary 1.5, $X^s \to S$ is a flat family of curves with at most ordinary double points.

The composition

$$\mathscr{I}^\vee \xrightarrow{(1,0)} \mathscr{I}^\vee \oplus \mathcal{O}_X(s_1 + \ldots + s_n) \to \mathscr{K} \to 0$$

gives us an injection $\mathscr{I}^\vee \hookrightarrow \mathscr{K}$ and it is clear that the cokernel is simply $\mathcal{O}_X(s_1 + s_2 + \ldots + s_n)|_{\bigcup_{i=1}^n s_i}$. Hence for each i we have surjections $s_i^*(\mathscr{K}) \to s_i^*\mathcal{O}_X(s_1 + \ldots + s_n)$ and this defines the liftings of the s_i's. The picture for $S = \operatorname{Spec}(k)$ looks as follows in Figure 3.

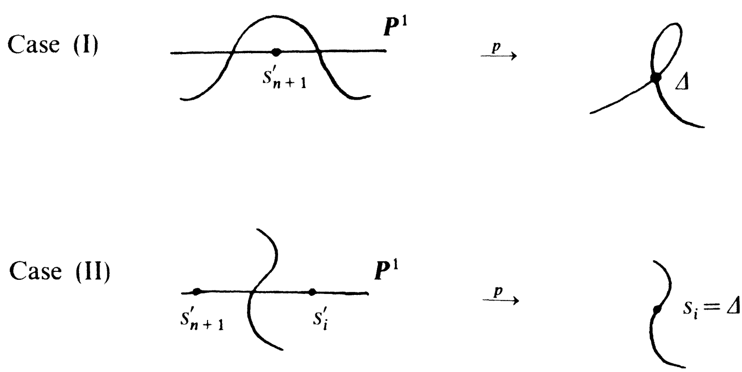

Figure 3.

LEMMA 2.5. *Consider a diagram*

$$t_i' \left[\begin{array}{ccc} Y & \xrightarrow{f} & X^s \\ {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow \\ X & = & X \\ \downarrow & & \downarrow \\ S & = & S \end{array}\right] s_i'$$

where q is a contraction and p is as in Theorem 2.4. Then there is a unique isomorphism $f\colon Y \to X^s$ making the diagram commutative.

PROOF. By Corollary 1.5 we have isomorphisms

$$\mathcal{O}_X \xrightarrow{\sim} q_*\mathcal{O}_Y$$

and

$$\mathcal{O}_X(-s_1 - \ldots - s_n) \xrightarrow{\sim} q_*\mathcal{O}_Y(-t_1 - \ldots - t_n)\,.$$

Let $x \in X$, $a \in \mathcal{J}_x$, $b \in q_*\mathcal{O}_Y(t_{n+1} - t_1 - \ldots - t_n)$. Then a may be considered as an element of $q_*\mathcal{O}_Y(-t_{n+1})$ and so

$$ab \in q_*\mathcal{O}_Y(-t_1 - \ldots - t_n) \approx \mathcal{O}_X(-s_1 - \ldots - s_n)\,,$$

i.e. we have a morphism of sheaves

$$q_*\mathcal{O}_Y(t_{n+1} - t_1 - \ldots - t_n) \to \mathcal{J}^{\vee}(-s_1 - \ldots - s_n)\,.$$

We leave to the reader to check that this is an isomorphism on the geometric fibres of π so by flatness it is an isomorphism. We may put this together in a commutative diagram with exact rows.

$$\begin{array}{ccccccccc} (0) \to & q_*\mathcal{O}_Y(-t_1 - \ldots - t_n) & \xrightarrow{(1,1)} & q_*\mathcal{O}_Y(t_{n+1} - t_1 - \ldots - t_n) \otimes q_*\mathcal{O}_Y & \xrightarrow{\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)} & q_*\mathcal{O}_Y(t_{n+1}) & \to (0) \\ & \wr\wr & & \wr\wr & & \wr\wr & \\ (0) \to & \mathcal{O}_X(-s_1 - \ldots - s_n) & \longrightarrow & \mathcal{J}^{\vee}(-s_1 - \ldots - s_n) \oplus \mathcal{O}_X & \longrightarrow & \mathcal{K} & \to (0)\,. \end{array}$$

By Corollary 1.5 the composition

$$q^*\mathcal{K} \to q^*q_*\mathcal{O}_Y(t_{n+1}) \to \mathcal{O}_Y(t_{n+1})$$

is a surjection and this defines a morphism

$$f\colon\ Y \to X^s\,.$$

f is easily seen to be an isomorphism on the geometric fibres of π so by flatness f is an isomorphism everywhere. For uniqueness we simply mention that an automorphism of $\boldsymbol{P}^1$ fixing three distinct points is the identity.

COROLLARY 2.6. *Contraction and stabilization are inverse to each other.*

THEOREM 2.7. *For all relevant* g, n, $M_{g,n}$ *is an algebraic stack, proper and smooth over* Spec ($\mathbf{Z}$). *The substack* $S_{g,n}$ *consisting of singular curves is a divisor with normal crossings relative to* Spec ($\mathbf{Z}$). (*We refer to* [3] *for definitions.*)

PROOF. For $g \geqq 2$, $n=0$ the result is proved in Theorem 5.2 of [3]. We consider first the cases $g=0$, $n=3$, and $g=1$, $n=1$, $M_{0,3} = \mathrm{Spec}\,(\mathbf{Z})$, so here is nothing to prove. For $g=1$, $n=1$ we make use of the clutching morphism of the next paragraph. Consider the 3-pointed elliptic curve E having three rational components as in Figure 4 below.

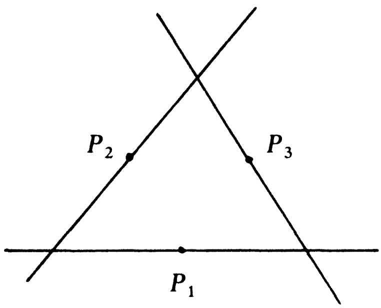

Figure 4.

Clearly E has no non-trivial automorphisms leaving the dstinguished points fixed, and so clutching defines a closed immersion $M_{1,1} \hookrightarrow M_{2,2}$.

Assuming for the moment that the theorem is proved for $M_{2,2}$, we see that $M_{1,1}$ in $M_{2,2}$ is the intersection of four branches of $S_{2,2}$, and $S_{1,1}$ is the intersection with a fifth branch. See the example at the end of section 3. This proves the theorem for $M_{1,1}$. We then proceed by induction with respect to n, having in mind that $M_{g,n+1}$ is the universal n-pointed curve. The divisor

$$S_{g,n+1} = \pi^{-1}(S_{g,n}) \cup \bigcup_{i=1}^{n} S_{g,n+1}^{\{i,n+1\}},$$

where a "point" in $S_{g,n+1}^{\{i,n+1\}}$ corresponds to a curve of the type in Figure 5. Since $\pi\colon M_{g,n+1} \to M_{g,n}$ is smooth near the sections $S_{g,n+1}^{\{i,n+1\}}$, we only have to prove that $\pi^{-1}(S_{g,n})$ has normal crossings. Near a singular point of $\pi^{-1}(S_{g,n})$ the morphism π looks formally like

$$\mathfrak{o}[[t_1 \dots t_k]] \to \mathfrak{o}[[x, y, t_1, \dots \hat{t}_i \dots, t_k]],$$

where $t_i \to x \cdot y$. $S_{g,n}$ has local equation $t_1 \cdot t_2 \dots t_k$ so $\pi^{-1}(S_{g,n})$ has local equation $x \cdot y \cdot t_1 \dots \hat{t}_i \dots t_k$.

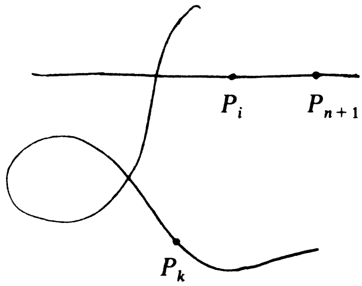

Figure 5.

3. The clutching morphism.

In this section we will study families of curves without stability conditions.

DEFINITION 3.1. A prestable curve is a flat and proper morphism $\pi: X \to S$ such that the geometric fibres of π are reduced curves with at most ordinary double points. No connectedness is assumed.

Recall that for any morphism $X \to S$ Lichtenbaum and Schlessinger [8] have defined the notion of a cotangent complex $L.(X/S)$. In general this complex is defined only locally on X, but it is unique up to homotopy, so it defines cotangent sheaves. For any coherent $\mathscr{F}$ on X, we have for $0 \leqq i \leqq 2$

$$T^i(X/S, \mathscr{F}) = H^i(\mathscr{H}om_{\mathcal{O}_X}(L.(X/S), \mathscr{F}))$$
$$T_i(X/S, \mathscr{F}) = H_i(L.(X/S) \otimes_{\mathcal{O}_X} \mathscr{F}).$$

PROPOSITION 3.2. *If $\pi: X \to S$ is a prestable curve, then*

a) $T_0(X/S, \mathcal{O}_X) \cong \Omega^1_{X/S}$ *is flat over S,*
b) $T_i(X/S, \mathscr{F}) \cong \mathscr{T}or_i^X(\Omega_{X/S}, \mathscr{F})$,
c) $T_2(X/S, \mathscr{F}) = (0)$ *for all $\mathscr{F}$.*

PROOF. Since a reduced curve with at most ordinary double points is locally a complete intersection and the morphism π is flat, it follows that π is locally a complete intersection morphism. That means that we have factorisations

$$\begin{array}{ccccc} X & \supset & U & \overset{i}{\hookrightarrow} & A^n_V \\ \pi\downarrow & & \downarrow & & \\ S & \supset & V & & \end{array}$$

where the ideal defining U, $\mathscr{I}$, is generated by a regular sequence. By [8] we have $T^2 \equiv T_2 \equiv 0$ and

$$(*) \qquad (0) \to i^*(\mathscr{I}) \xrightarrow{d} i^*\Omega^1_{A^n_V/V} \to (0)$$

is a cotangent complex for the morphism $U \to V$.

To prove a) and b) we have to show that d is universally injective. $i^*(\mathscr{I})$ and
$i^*\Omega^i_{A^n_V/V}$ are flat over $\mathcal{O}_V$ and the cotangent complex commutes with base change, since π is flat so in fact all we have to prove is that d is injective.

Let $\mathscr{K} = T_1(U/V, \mathcal{O}_U) = \ker d$. By the Jacobian criterion of smoothness, $\mathscr{K}_x = (0)$ for all $x \in U$, where π is smooth. So if $\Delta \subset U$ is the closed subset of U, where π is not smooth, we have

$$\operatorname{Supp}(\mathscr{K}) \subset \Delta .$$

Since $i^*(\mathscr{I})$ is locally free and $\mathscr{K} \subset i^*(\mathscr{I})$, we have

$$\operatorname{Ass}(\mathscr{K}) \subset \operatorname{Ass}(U) .$$

Since U is flat over V we have [2, Chap. IV, 2.6.2]

$$\operatorname{Ass}(U) = \bigcup_{y \in \operatorname{Ass}(V)} \operatorname{Ass}(U_y) .$$

Therefore since the fibres of π are reduced

$$\Delta \cap \operatorname{Ass}(U) = \varnothing .$$

Hence $\mathscr{K} = (0)$ and d is injective.

Corollary 3.3. *Let $\pi: X \to S$ be a prestable curve, $\mathscr{E}_0$ a locally free sheaf on X, and $d_0: \mathscr{E}_0 \to \Omega^1_{X/S}$ a surjection. Then $\mathscr{E}_1 = \ker(d_0)$ is locally free and $d_1: \mathscr{E}_1 \to \mathscr{E}_0 \to (0)$ is a cotangent complex for the morphism π.*

Let $\pi: X \to S$ be a prestable curve, and $s_1, s_2: S \to X$ two non-crossing sections such that π is smooth at all points $s_i(t)$ $(t \in S)$.

Theorem 3.4. *With the notations above there is a diagram*

$$\begin{array}{ccc} X & \xrightarrow{p} & X' \\ {\scriptstyle s_i}\uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi'}\downarrow\uparrow{\scriptstyle s} \\ S & = & S \end{array}$$

such that

(1) *$ps_1 = ps_2$ and p is universal with respect to this property.*

(2) *p is a finite morphism.*

(3) *If t is a geometric point of S, the fibre X'_t is obtained from X_t by identifying the two points $s_1(t)$ and $s_2(t)$ in such a way that the image point is an ordinary double point.*

(4) *As a topological space, X' is the quotient of X under the equivalence relation $s_1(t) \sim s_2(t)$ for all $t \in S$.*

(5) *If U is open in X' and $V = p^{-1}(U)$, then*

$$\Gamma(U, \mathcal{O}_{X'}) = \{h \in \Gamma(V, \mathcal{O}_X) \mid s_1^*(h) = s_2^*(h)\} .$$

(6) *The morphism $\pi'\colon X' \to S$ is flat, so by* (3), *π' is again a prestable curve.*

PROOF. Properties (4) and (5) determine X' as a ringed space. To show that X' is a scheme satisfying (1), (4), and (5), all we have show is that X' is locally affine.

If $x \in X'$, and $p^{-1}(x)$ does not meet any of the sections, we can clearly find an affine neighbourhood of x.

Suppose therefore that $x = p_1 s_1(t)$. Since π is flat and any curve is projective, we may assume that π is locally projective. Hence we can find an affine $U \subset X$ containing $s_1(t)$ and $s_2(t)$. Let $V \subset S$ be an open affine contained in $s_1^{-1}(U) \cap s_2^{-1}(U)$ and containing t. The restriction of π to $W = U \cap \pi^{-1}(V)$ yields an affine morphism $W \to V$ with two disjoint regular sections. Localizing further, if necessary, we may assume that $W = \operatorname{Spec}(B)$, $V = \operatorname{Spec}(A)$, and that the sections are defined by two principal ideals (f_1) and (f_2) of B.

We have two split-exact sequences of A-modules

$$(0) \to B \xrightarrow{f_i} B \to A \to (0), \qquad i = 1, 2 .$$

Since the sections do not cross, we have $(f_1) + (f_2) = B$, so $(f_1) \cap (f_2) = (f_1, f_2)$, and the ring of invariants is given by

$$B' = A \oplus (f_1 f_2) B .$$

B' is isomorphic to B as an A-module, hence B' is flat over A. It is easy to check that $\operatorname{Spec}(B')$ satisfies (1), (4), (6) of the theorem for the morphism $W \to V$.

Suppose $B = A[x_1, \ldots, x_n]$ and write

$$x_i = a_{1i} + f_1 b_{1i} = a_{2i} + f_2 b_{2i}$$

with $a_{ij} \in A$, $b_{ij} \in B$.

The elements $y_i = x_i^2 - (a_{1i} + a_{2i}) x_i$ are in B', hence

$$A[y_1, \ldots, y_n] = B'' \subset B' \subset B .$$

B is a finite B''-module, and since A is noetherian, B' is a finitely generated A-algebra and B is a finite B'-module. This proves (2).

To show (3), note first that the construction of B' commutes with base change, hence we may assume that $A = k$ is an algebraically closed field. If $\hat{B}$ denotes the completion of B with respect to the ideal $(f_1 \cdot f_2)$ we have an isomorphism.

$$k[[x]] \oplus k[[y]] \approx \hat{B}$$

sending x to f_1 and y to f_2. The completion of B' with respect to the ideal $f_1 \cdot f_2 B$, corresponds then to the kernel of the map

$$k[[x]] \oplus k[[y]] \to k$$

sending (a, b) to $\bar{a} - \bar{b}$ and this is just $k[[x,y]]/(x \cdot y)$.

THEOREM 3.5. (The clutching sequence). *We consider the diagram as in* 3.4

$$\begin{array}{ccc} X & \xrightarrow{p} & X' \\ {\scriptstyle s_i}\uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi'}\downarrow\uparrow{\scriptstyle s} \\ S & = & S \end{array}$$

and denote $\varkappa^{(i)}$ *the conormal bundle of* S *in* X *via the section* s_i.

$$\varkappa^{(i)} = s_i^*(\Omega_{X/S}) \approx s_i^*(\omega_{X/S}) \approx s_i^*(\mathcal{O}_X(-D_i)),$$

where D_i *is the divisor on* X *defined by* s_i.

Then on X' *we have a short exact sequence* (*the clutching sequence*)

$$(0) \to s_*(\varkappa^{(1)} \otimes \varkappa^{(2)}) \to \Omega_{X'/S} \to p_*\Omega_{X/S} \to (0).$$

PROOF. It turns out to be a bit messy to define the map $s_*(\varkappa^{(1)} \otimes \varkappa^{(2)}) \to \Omega_{X'/S}$ so we take double coverings

$$\begin{array}{ccc} Y & \xrightarrow{q} & Y' \\ {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle \sigma'} \\ X & \xrightarrow{p} & X' \end{array}$$

where $Y = X \coprod X$,

$$Y' = X \coprod X \Big/ \begin{pmatrix} s_1 \text{ in first factor} \sim s_2 \text{ in second} \\ s_2 \text{ in first factor} \sim s_1 \text{ in second} \end{pmatrix}.$$

The section $s \colon S \to X'$ lifts to two sections t_1 and t_2 in Y'.

The picture is given by Figure 6.

Assuming everything to be affine we have

$$S = \operatorname{Spec}(T), \quad X = \operatorname{Spec}(R), \quad Y = \operatorname{Spec}(R \oplus R).$$

The $\mathbf{Z}/2\mathbf{Z}$ action simply interchanges the factors. The four sections of Y are defined by non-zero-divisors $(f_i, 1)$ and $(1, f_i)$, $i = 1, 2$.

The affine sets

$$Y_1 = \operatorname{Spec}(R_{f_1} \oplus R_{f_2})$$

$$Y_2 = \operatorname{Spec}(R_{f_2} \oplus R_{f_1})$$

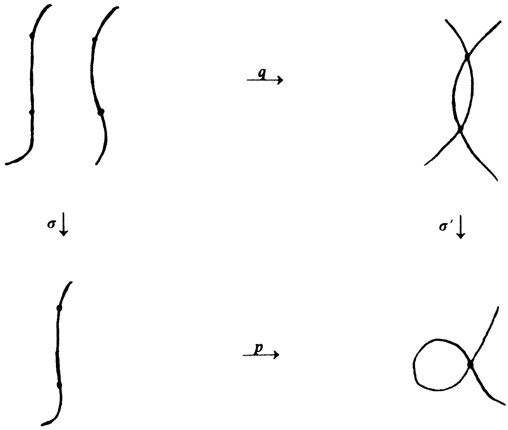

Figure 6.

are invariant sets for the map q. Clutching the sections of Y_1 and Y_2 gives us an affine covering Y'_1 and Y'_2 of Y', where

$$Y'_i = \operatorname{Spec}(S_i) \qquad i=1,2$$

and

$$S_1 = \{(u,v) \in R_{f_1} \oplus R_{f_2} \mid s_2^*(u) = s_1^*(v)\}$$

$$S_2 = \{(u,v \in R_{f_2} \oplus R_{f_1} \mid s_1^*(u) = s_2^*(v)\} .$$

S_1 is an R_{f_1}-algebra via $u \to (u, \pi^* s_2^* u)$ and an R_{f_2}-algebra via $v \to (\pi^* s_1^* v, v)$.
Hence we have a homomorphism

$$R_{f_1} \otimes_T R_{f_2} \to S_1 .$$

We leave to the reader to check that

$$(0) \to (f_2 \otimes f_1) \to R_{f_1} \otimes_T R_{f_2} \to S_1 \to (0)$$

We leave to the reader to check that

$$(0) \to (f_2 \otimes f_1) \to R_{f_1} \otimes_T R_{f_2} \to S_1 \to (0)$$

is exact.

Similarly we have the exact sequence

$$(0) \to (f_1 \otimes f_2) \to R_{f_2} \otimes_T R_{f_1} \to S_2 \to (0) .$$

By flatness $f_1 \otimes f_2$ is not a zero-divisor in $R_{f_1} \otimes R_{f_2}$, so we have

$$(*) \qquad (0) \to (f_2 \otimes f_1)/(f_2 \otimes f_1)^2 \to \Omega_{R_{f_1} \otimes R_{f_2}/T} \otimes_{R_{f_1} \otimes R_{f_2}} S_1 \to \Omega_{S_1/T} \to 0 .$$

Using the canonical isomorphism

$$\Omega_{R_{f_1}\otimes R_{f_2}/T} \approx R_{f_1}\otimes_T \Omega_{R_{f_2}/T} \oplus \Omega_{R_{f_1}/T}\otimes_T R_{f_2},$$

where $d(u\otimes v) = (u\otimes dv,\ du\otimes v)$.

We may write (*) in the form

$$(**)\quad (0) \to (f_2\otimes f_1)/(f_2\otimes f_1)^2 \xrightarrow{d} (S_1\otimes_{R_{f_2}} \Omega_{R_{f_2}/T}) \oplus (\Omega_{R_{f_1}/T}\otimes_{R_{f_1}} S_1) \to \Omega_{S_1/T} \to (0),$$

where $d(f_2\otimes f_1) = ((f_2,0)\otimes df_1, df_2\otimes(0,f_1))$.

Hence in $\Omega_{S_1/T}$ we have

$$(f_2,0)d(0,f_1) = -(0,f_1)d(f_2,0).$$

From the canonical isomorphism

$$\Omega_{R_{f_1}\oplus R_{f_2}/T} \approx \Omega_{R_{f_1}/T}\oplus\Omega_{R_{f_2}/T}$$

(module multiplication componentwise), we see from (**) that the natural map

$$\Omega_{S_1/T} \to \Omega_{R_{f_1}\oplus R_{f_2}/T}$$

is surjective and that the kernel is generated by the element $(0,f_1)d(f_2,0) = -(f_2,0)d(0,f_1)$ that is we have a right exact sequence

$$f_1R_{f_2}/f_1^2R_{f_2}\otimes f_2R_{f_1}/f_2^2R_{f_1} \xrightarrow{\alpha_1} \Omega_{S_1/T} \to \Omega_{R_{f_1}\oplus R_{f_2}/T} \to (0),$$

where $\alpha_1(u\otimes v) = (0,u)d(v,0)$.

The kernel of $\Omega_{S_1/T} \to \Omega_{R_{f_1}\oplus R_{f_2}/T}$ is a flat T-module and the left hand side of the sequence above is a locally free rank 1 T-module. Hence α_1 is injective.

On Y'_2 we have a similar exact sequence

$$(0) \to (f_1R_{f_2}/f_1^2R_{f_2})\otimes(f_2R_{f_1}/f_2^2R_{f_1}) \xrightarrow{\alpha_2} \Omega_{S_2/T} \to \Omega_{R_{f_2}\oplus R_{f_1}/T} \to (0),$$

where $\alpha_2(u\otimes v) = (u,0)d(0,v)$.

Since both α_1 and α_2 vanish on $Y'_1\cap Y'_2$, they patch up to give a global map

$$(0) \to \underbrace{(t_1)_*(\varkappa^{(1)}\otimes\varkappa^{(2)})\oplus(t_2)_*(\varkappa^{(1)}\otimes\varkappa^{(2)})}_{\sigma^*s_*(\varkappa^{(1)}\otimes\varkappa^{(2)})} \xrightarrow{\alpha} \Omega_{Y'/S} \to q_*\Omega_{Y/S} \to (0)$$

By definition we see that α is $\mathbf{Z}/2\mathbf{Z}$ invariant and is therefore induced by a map

$$\alpha' : s_*(\varkappa^{(1)}\otimes\varkappa^{(2)}) \to \Omega_{X'/S}.$$

Since étale morphisms are faithfully flat

$$(0) \to s_*(\varkappa^{(1)}\otimes\varkappa^{(2)}) \to \Omega_{X'/S} \to p_*\Omega_{X/S} \to (0)$$

is exact.

REMARK. Let $\mathscr{I}$ be the sheaf of ideals on X' defining the section s. We have natural maps

$$p_1 : \mathscr{I}/\mathscr{I}^2 \to p_*(\mathscr{I}_1/\mathscr{I}_1^2) = s_*\varkappa^{(1)}$$
$$p_2 : \mathscr{I}/\mathscr{I}^2 \to p_*(\mathscr{I}_2/\mathscr{I}_2^2) = s_*\varkappa^{(2)} .$$

One checks that these maps give an isomorphism

$$\mathscr{I}/\mathscr{I}^2 \approx s_*(\varkappa^{(1)} \oplus \varkappa^{(2)}) .$$

Hence we have an isomorphism

$$\Lambda^2(\mathscr{I}/\mathscr{I}^2) \approx s_*(\varkappa^{(1)} \otimes \varkappa^{(2)})$$

defined by

$$\bar{u} \wedge \bar{v} \mapsto p_1\bar{u} \otimes p_2\bar{v} - p_1\bar{v} \otimes p_2\bar{u} .$$

The map α' is the composition

$$s_*(\varkappa^{(1)} \otimes \varkappa^{(2)}) \approx \Lambda^2(\mathscr{I}/\mathscr{I}^2) \xrightarrow{\beta} \Omega_{X'/S} ,$$

where $\beta(\bar{u} \wedge \bar{v}) = udv$.

DEFINITION 3.6. Let

$$H = \{h_1, h_2, h_3, \ldots, h_{n_1}\}, \quad h_1 < h_2 < \ldots < h_{n_1}$$

and

$$K = \{k_1, k_2, \ldots, k_{n_2}\}, \quad k_1 < k_2 < \ldots < k_{n_2}$$

be disjoint subsets of $\{1, 2, \ldots, n\}$, $n_1 + n_2 = n$.

Let g_1 and g_2 be two non negative integers with $g_1 + g_2 = g$. Then for each quadruple g_1, g_2, H, K we have a morphism of stacks

$$\gamma_{g_1, g_2, H, K} : M_{g_1, n_1+1} \times M_{g_2, n_2+1} \to M_{g, n+2} .$$

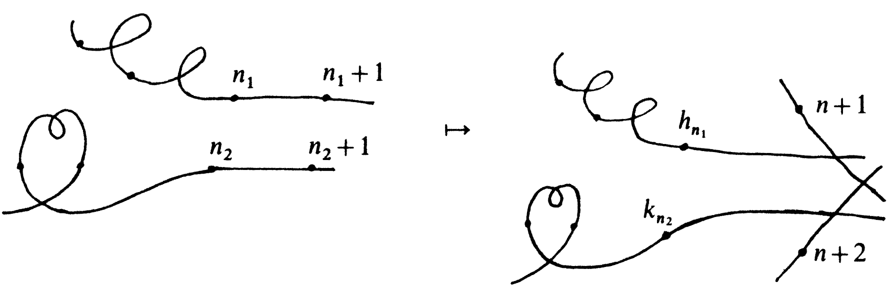

Figure 7.

This is obtained by attaching a pair of projective lines and renumbering the sections as best described by the picture in Figure 7.

We define

$$\gamma_0 : M_{g-1,n+2} \to M_{g,n+2}$$

as described by Figure 8.

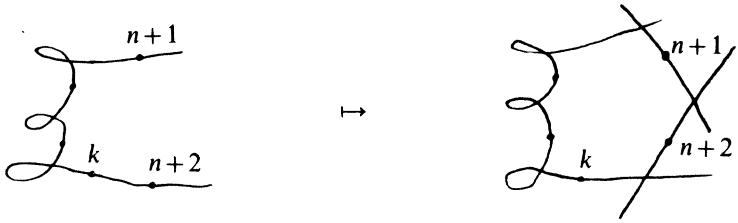

Figure 8.

Contracting the two last sections gives a morphism

$$\pi_{n+1,n+2} : M_{g,n+2} \to M_{g,n} .$$

We denote by β the composition $\beta = \pi_{n+1,n+2} \circ \gamma$.

THEOREM 3.7. *γ is a closed immersion and β is finite.*

PROOF. Clearly β is quasifinite and $\pi_{n+1,n+2}$ is proper, so it suffices to prove that γ is a closed immersion.

Let $\pi: C \to S$ be an $n+2$-pointed stable curve with sections $s_1, \dots, s_{n+2}$.

Contracting s_{n+1}, s_{n+2} and both s_{n+1} and s_{n+2} gives us a diagram

$$\begin{array}{ccccc} C & \xrightarrow{p} & C' & & \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p''} & & \\ C'' & \xrightarrow{p'''} & C''' & \xrightarrow{\pi'''} & S . \end{array}$$

On C' and C'' we have extra sections $\Delta' = p \circ s_{n+2}$ and $\Delta'' = p' \circ s_{n+1}$, respectively. On C''' we have two extra sections namely

$$p''ps_{n+1} = p'''p's_{n+1} = p'''\Delta'''$$

and

$$p''ps_{n+2} = p'''p's_{n+2} = p''\Delta' .$$

We define three closed subschemes of S via the cartesian diagrams:

$$\begin{array}{ccc} T' & \hookrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle \Delta'} \\ C'_{\text{sing}} & \hookrightarrow & C' \end{array} \qquad \begin{array}{ccc} T'' & \hookrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle \Delta''} \\ C''_{\text{sing}} & \hookrightarrow & C'' \end{array} \qquad \begin{array}{ccc} T''' & \hookrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle p''\Delta'} \\ S & \underset{p'''\Delta''}{\hookrightarrow} & C''' \end{array}$$

and finally

$$T = T' \times_S T'' \times_S T''' .$$

Note that for any stable curve $X \to S$, X_{sing} is defined by the sheaf of ideals image of $\Omega_{X/S} \otimes \check{\omega}_{X/S}$ in $\mathcal{O}_X$.

Let Δ''' denote the section of $C''' \times_S T$, and let $\mathcal{I}'$, $\mathcal{I}''$, and $\mathcal{I}'''$ be the sheaf of ideals defining the sections Δ', Δ'', and Δ''' over T, respectively.

By uniqueness of stabilization we have

$$E_1 = p^{-1}(\Delta') \approx \operatorname{Proj}(\mathcal{S}ym(\mathcal{I}'^{\vee}/\mathcal{I}'\mathcal{I}'^{\vee}))$$

$$E_2 = p'^{-1}(\Delta'') \approx \operatorname{Proj}(\mathcal{S}ym(\mathcal{I}''^{\vee}/\mathcal{I}''\mathcal{I}''^{\vee}))$$

$$E_3 = p'''^{-1}(\Delta''') \approx E_4 = p'''^{-1}(\Delta''') \approx \operatorname{Proj}(\mathcal{S}ym(\mathcal{I}'''^{\vee}/\mathcal{I}'''\mathcal{I}'''^{\vee})) .$$

Over T we have the picture in Figure 9.

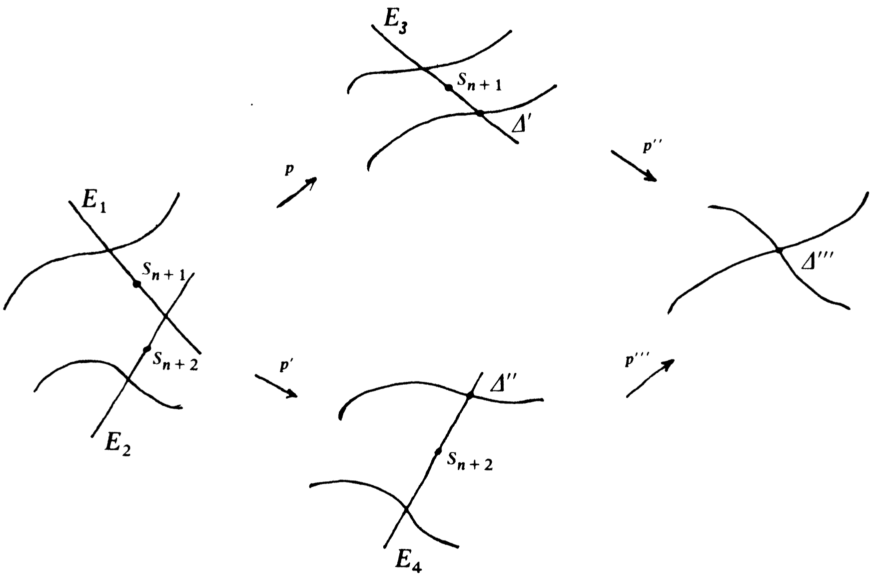

Figure 9.

Consider the general situation where $X \to S$ is a stable curve with a section Δ. Let s be a point in S and $x = \Delta(s)$ a point in X such that X_s has a double point with rational tangents at x. If $\mathcal{I}$ is the ideal defining the section in the local ring $R = \mathcal{O}_{X,x}$, we have a map

$$\mathcal{I} \otimes_R \mathcal{I}^{\vee} \to R .$$

We denote by $\mathcal{I}\mathcal{I}^{\vee}$ the image of this map. Clearly $\mathcal{I} \subset \mathcal{I}\mathcal{I}^{\vee}$.

If we take completions with respect to the maximal ideal $m = m_x$ in R we have by general theory of Zariski rings

$$(\mathscr{I}\mathscr{I}^{\vee})^{\wedge} \approx \mathscr{I}^{\vee}.(\mathscr{I}^{\vee})^{\wedge} \approx \mathscr{I}^{\wedge}(\mathscr{I}^{\wedge})^{\vee}.$$

So we have a diagram

$$\begin{array}{ccccc} \mathscr{I}^{\wedge} & \subset & \mathscr{I}^{\wedge}(\mathscr{I}^{\wedge})^{\vee} & = & (\mathscr{I}\mathscr{I}^{\vee})^{\wedge} \\ \cup & & \cup & & \\ \mathscr{I} & \subset & \mathscr{I}\mathscr{I}^{\vee}. & & \end{array}$$

By the example in the appendix $\mathscr{I}^{\wedge} = \mathscr{I}^{\wedge}(\mathscr{I}^{\wedge})^{\vee}$ and so since $\hat{R}$ is a faithfully flat R-algebra $\mathscr{I} = \mathscr{I}\mathscr{I}^{\vee}$. The exact sequence

$$(*) \qquad (0) \to R/\mathscr{I} \to \mathscr{I}^{\vee}/\mathscr{I}\mathscr{I}^{\vee} \to \mathscr{I}^{\vee}/R \to (0)$$

shows that $\mathscr{I}^{\vee}/\mathscr{I}\mathscr{I}^{\vee}$ is locally free of rank 2. If X_s does not have rational tangents at x, there is an étale neighbourhood $S' \to S$ of S such that the sequence (*) is exact on $X_{S'}$; hence (*) is exact in any case, and the E's are flat over T. p and p' induce proper maps $E_1 \to E_3$ and $E_2 \to E_4$. On the geometric fibres they are isomorphism, so by Nakayama's lemma they are closed immersions. By flatness they are isomorphisms.

The isomorphisms $E_1 \approx E_3 \approx E_4 \approx E_2$ shows that they all have exactly three sections. Hence they are all isomorphic to $\boldsymbol{P}^1_T$. In particular we have three extra sections t_1, t_2, and t_3 in C_T, that is in Figure 10.

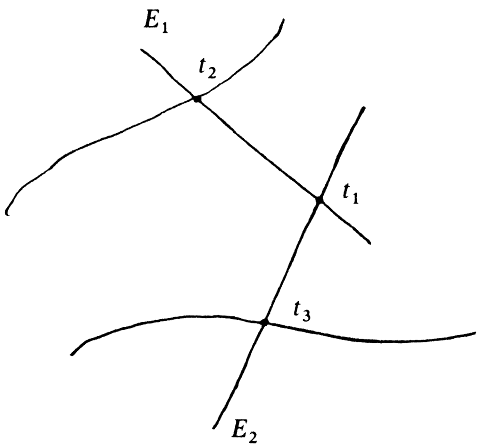

Figure 10.

The three schemes $C_T \setminus t_1(T)$, E_1, and E_2 patched along their common open sets yields a stable curve $\tilde{C}_T$ over T with $n+4$ sections. Contracting the two last sections in $\tilde{C}_T$ gives us a morphism:

$$T \to M = M_{g-1,n+2} \cup \bigcup M_{g_1,n_1+1} \times M_{g_2,n_2+1}$$

and the diagram

$$\begin{array}{ccc} T & \hookrightarrow & S \\ \downarrow & & \downarrow \\ M & \xrightarrow{\gamma} & M_{g,n+2} \end{array}$$

commutes by the universal property (1) of the clutching construction. It is cartesian by the very definition of T.

DEFINITION 3.8. Let $H = \{h_1, h_2, \ldots, h_{n_1}\}$ and $K = \{k_1, k_2, \ldots, k_{n_2}\}$ be complementary subsets of $\{1, 2, \ldots, n\}$ with $h_1 < h_2 < \ldots < h_{n_1}$ and $k_1 < k_2 < \ldots < k_{n_2}$. Let g_1 and g_2 be integers with $g = g_1 + g_2$. The finite morphisms

$$\beta_0 : M_{g-1,n+2} \to M_{g,n}$$

and

$$\beta_{g_1,g_2,H,K} : M_{g_1,n_1+1} \times M_{g_2,n_2+1} \to M_{g,n}$$

define closed substacks $S^0_{g,n}$ and $S_{g_1,g_2,H,K}$ of $M_{g,n}$, and these are the irreducible components of $S_{g,n}$. When $g_1 = 0$ we write for short $S_{g_1,g_2,H,K} = S^H_{g,n}$, or simply S^H. If also $g_2 = 0$, then H should contain at most one of the integers 1, 2, 3.

COROLLARY 3.9.

a) *The clutching morphism β is finite and unramified.*
b) *When $g_1 \neq g_2$ or $n \neq 0$, $\beta_{g_1 g_2 H,K}$ is a closed immersion.*

PROOF. Let X be a scheme and π: $C \to X$ an n-pointed stable curve and let D be the curve over C obtained from $C \times_X C$ by stabilization. Since the geometric fibres of π are reduced with ordinary double points, $C_{\text{sing}} \to X$ is unramified. Define the scheme T by the fibre product

$$\begin{array}{ccc} T & \longrightarrow & X \\ \downarrow & & \downarrow \\ M & \xrightarrow{\beta} & M_{g,n} \end{array}$$

Then $T \to X$ factors as follows.

$$T \xhookrightarrow{\gamma} D_{\text{sing}} \times_C C_{\text{sing}} \to C_{\text{sing}} \to X$$

and this proves a); b) is clear.

We conclude this section with a picture of the zoo of 2-pointed stable curves of genus 2. $S_{2,2}$ and its intersections define a stratification of $M_{2,2}$ into locally closed non-singular connected strata as follows

dimension	5	4	3	2	1	0
number of components	1	4	13	24	23	10 .

The 23 components of dimension 1 correspond to the curves in Figure 11.

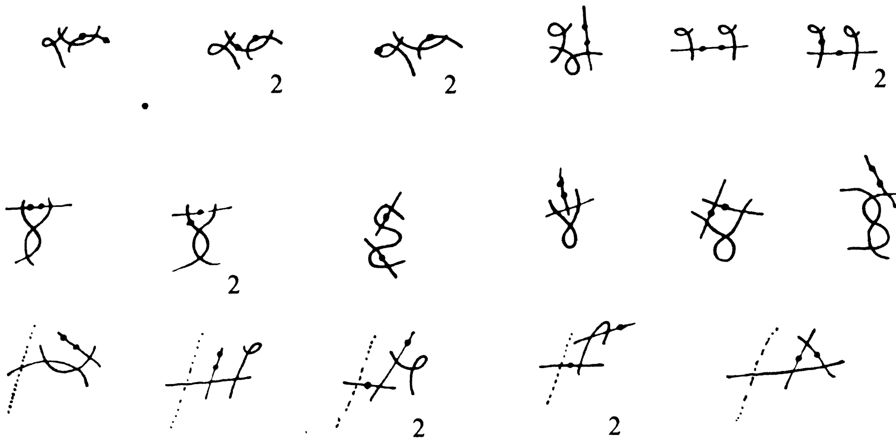

Figure 11.

The 10 components of dimension 0 correspond to the curves in Figure 12.

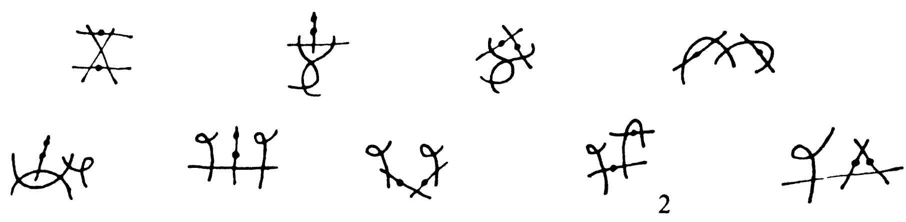

Figure 12.

The last components here are isomorphic to $M_{1,1}$ and $S_{1,1}$ respectively. (A dotted line stands for an elliptic curve. The number 2 indicates that by ordering the points we actually have 2 components of this type).

Appendix. Stably reflexive modules.

Let R and S be noetherian rings and $h\colon S \to R$ a ring homomorphism making R into a flat S-algebra.

Definition 1. A noetherian R-module M is stably reflexive with respect to the homomorphism h, or if there can be no doubt about the homomorphism we say simply that M is stably reflexive with respect to S, if M satisfies the equivalent properties of the theorem below.

THEOREM 2. *The following are equivalent.*

1) a) *For all* $i>0$ *and all* S*-modules* N

$$\mathrm{Ext}^i_R(M, R\otimes_S N) = (0).$$

b) *The canonical map*

$$\varphi_{M,N} : M^\vee \otimes_S N \to \mathrm{Hom}_R(M, R\otimes_S N)$$

is an isomorphism.

a$^\vee$) *For all* $i>0$ *and all* S*-modules* N

$$\mathrm{Ext}^i_R(M^\vee, R\otimes_S N) = (0).$$

b$^\vee$) *The canonical map*

$$\psi_{M,N} : M\otimes_S N \to \mathrm{Hom}_R(M^\vee, R\otimes_S N)$$

is an isomorphism.

2) *There exists an infinite complex of finite locally free* R*-modules*

$$(*) \qquad \ldots \to E^{-2} \xrightarrow{d^{-2}} E^{-1} \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to \ldots$$

such that for all S*-modules* N,

$$E^{\cdot}\otimes_S N \text{ and } E^{\vee\cdot}\otimes_S N \text{ are acyclic and } M\approx \operatorname{im} d^0 .$$

3. *There exists an infinite acyclic complex* (*) *such that if* $B^i = \operatorname{im} d^i$, *then*
a) B^i *and* $B^{i\vee}$ *are* S*-flat,*
b) $\mathrm{Ext}^i_R(B^j, R) = \mathrm{Ext}^i_R(B^{j\vee}, R) = (0)$ *for* $i>0$,
c) $M = B^0$.

PROOF. We do a cyclic proof in the order

$$3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 3.$$

$3 \Rightarrow 2$ is immediate.

By 2) we have exact sequences

$$(*)_N \qquad (0) \to M\otimes_S N \to E^1\otimes_S N \to E^2\otimes_S N \to$$

$$(*^\vee)_N \qquad (0) \to M^\vee\otimes_S N \to E^{0\vee}\otimes_S N \to E^{-1\vee}\otimes_S N \to \ldots$$

Since the canonical map

$$\psi_{F,N} : F \otimes_S N \to \mathrm{Hom}_R(F^\vee, R \otimes_S N)$$

is an isomorphism when F is a finite locally free R-module, 1) follows from the diagrams:

$$\begin{array}{llccccccc}
(*)_N & (0) \to & M \otimes_S N & \to & E^1 \otimes_S N & \to & E^2 \otimes_S N & \to \ldots \\
 & & \downarrow & & \downarrow \wr & & \downarrow \wr & \\
(**)_N & (0) \to & \mathrm{Hom}_R(M^\vee, R \otimes_S N) & \to & \mathrm{Hom}_R(E^{1\vee}, R \otimes_S N) & \to & \mathrm{Hom}_R(E^{2\vee}, R \otimes_S N) & \to \ldots \\
 & & & & & & & \\
(*^\vee)_N & (0) \to & M^\vee \otimes_S N & \to & E^{0\vee} \otimes_S N & \to & E^{-1\vee} \otimes_S N & \to \ldots \\
 & & \downarrow & & \downarrow \wr & & \downarrow \wr & \\
(**^\vee)_N & (0) \to & \mathrm{Hom}_R(M, R \otimes_S N) & \to & \mathrm{Hom}_R(E^0, R \otimes_S N) & \to & \mathrm{Hom}_R(E^{-1}, R \otimes_S N) & \to \ldots
\end{array}$$

Note that $(**)_N$ and $(**^\vee)_N$ are left exact by $(*)_S$ and $(*^\vee)_S$ and general facts about Hom.

To show 1) $\Rightarrow$ 3) note first that b$^\vee$) with $N = S$ tells us that $M \approx M^{\vee\vee}$, and therefore the definition is completely symmetric with respect to M and $M^\vee$.

Since M is noetherian, so is $M^\vee$ and we can find locally free resolutions of finite R-modules of the form:

$$\ldots \to E^{-2} \xrightarrow{d^{-2}} E^{-1} \xrightarrow{d^{-1}} E^0 \to M \to (0)$$
$$\ldots \to E^{3\vee} \xrightarrow{d^{2\vee}} E^{2\vee} \xrightarrow{d^{1\vee}} E^{1\vee} \to M^\vee \to (0)\,.$$

By 1) $(**)_N$ and $(**^\vee)_N$ are exact and in the two diagrams above the left vertical arrows are isomorphisms. Hence we have $(*)_N$ and $(*^\vee)_N$ exact for all N.

Let $N \to N'$ be an injection of S-modules then by $(*)_N$ and $(*)_{N'}$

$$\begin{array}{ccccc}
(0) \to & M \otimes_S N & \to & E^1 \otimes_S N \\
 & \downarrow & & \downarrow \cap \\
(0) \to & M \otimes_S N' & \to & E^1 \otimes_S N'\,.
\end{array}$$

Now R is S-flat so E^1 is S flat and we see from the diagram that M is S-flat too. By the remark above, $M^\vee$ is S-flat as well.

Lemma. *Let* $(0) \to M'' \to M \to M' \to (0)$ *be a short exact sequence of R-modules such that M and M' satisfies* 1), *then M'' satisfies* 1).

PROOF. 1a) is clear and from the two diagrams

$$\begin{array}{ccccccccc}
\to \operatorname{Tor}_1^S(M''^{\vee},N) \to & M'^{\vee}\otimes_S N & \to & M^{\vee}\otimes_S N & \to & M''^{\vee}\otimes_S N & \to & (0) \\
 & \downarrow\sim & & \downarrow\sim & & \downarrow\varphi_{M'',N} & & \\
(0) \to & \operatorname{Hom}(M',R\otimes_S N) & \to & \operatorname{Hom}(M,R\otimes_S N) & \to & \operatorname{Hom}(M'',R\otimes_S N) & \to & (0)
\end{array}$$

$$\begin{array}{ccccccccc}
(0) \to & M''\otimes_S N & \to & M\otimes_S N & \to & M'\otimes_S N & \to & (0) \\
 & \downarrow\psi_{M'',N} & & \downarrow\sim & & \downarrow\sim & & \\
(0) \to & \operatorname{Hom}(M''^{\vee},R\otimes_S N) & \to & \operatorname{Hom}(M^{\vee},R\otimes_S N) & \to & \operatorname{Hom}(M'^{\vee},R\otimes_S N) & \to & \operatorname{Ext}_R^1(M''^{\vee},R\otimes_S N) \to
\end{array}$$

we see that the canonical maps $\varphi_{M'',N}$ and $\psi_{M'',N}$ are isomorphisms and that $\operatorname{Ext}_R^1(M''^{\vee},R\otimes_S N)=0$ for all N. For $i>1$ we have exact sequences

$$(0) = \operatorname{Ext}_R^{i-1}(M'^{\vee},R\otimes_S N) \to \operatorname{Ext}_R^i(M''^{\vee},R\otimes_S N) \to \operatorname{Ext}_R^i(M^{\vee},R\otimes_S N) \to$$

and the lemma is proved.

Back to proving 3). Combining the resolution $\ldots \to E^{-1} \to E^0 \to M \to (0)$ and the exact sequence $(*)_S$ we get an infinite acyclic complex $(*)$. If B^i denotes the image of d^i it follows by the Lemma that B^i satisfies 1) for $i\leqq 0$ and $(\operatorname{im} d^{i^{\vee}})$ satisfies 1) for $i\geqq 0$. However it is clear that $(\operatorname{im} d^{i^{\vee}})^{\vee} \approx B^i$ for $i\geqq 0$ so B^i satisfies 1) for $i>0$ too.

COROLLARY 3. *Given a short exact sequence of finite R-modules:*

$$(0) \to M'' \to M \to M' \to (0)$$

with M' stably reflexive with respect to S. Then M'' is stably reflexive with respect to S if and only if M is.

PROPOSITION 4. *If M is S-stably reflexive, then for any homomorphism $S \to T$ the $R_{(T)}$-module $M_{(T)}$ is T-stably reflexive and*

$$(*) \qquad M^{\vee}\otimes_S T \xrightarrow{\sim} \operatorname{Hom}_{R_{(T)}}(M_{(T)}, R\otimes_S T) = (M_{(T)})^{\vee}.$$

PROPOSITION 5. *Given S, R and M as before, then M is stably reflexive with respect to S if and only if for all prime ideals $p\subset R$, M_p is stably reflexive with respect to $S_{h^{-1}(p)}$.*

PROOF. For noetherian M it is well known that all the functors occuring in property 1) of Theorem 2, commute with localization.

PROPOSITION 6. *Suppose S and R are local noetherian rings and $h\colon S \to R$ a local homomorphism. The following properties are equivalent:*

a) *M is S-stably reflexive,*
b) *$\hat{M}$ is S-stably reflexive,*
c) *$\hat{M}$ is $\hat{S}$-stably reflexive.*

PROOF. Bourbaki [2, Chap. III, 5.4.4.]

REMARK. In view of these propositions, it is clear that stable reflexivity is a property of coherent sheaves with respect to a morphism. Moreover the property is local in the Zariski topology as well as in the étale topology.

We leave as exercise the following "local criterion of stable reflexivity".

PROPOSITION 7. *Given S, R, and M as above and let $\mathscr{I}$ be an ideal of S contained in the radical of S. Denote by S_k, R_k, and M_k $S/\mathscr{I}^k$, $R \otimes_S S/\mathscr{I}^k$, and $M \otimes_S S/\mathscr{I}^k$, respectively. Then M is stably reflexive with respect to S, if and only if, for each k, M_k is stably reflexive with respect to S_k.*

Example of a stably reflexive module.

Let S be a ring, b and c elements of S, and let R be the ring:

$$R = S[x,y]/(xy-bc) .$$

Note that every element $u \in R$ can be written uniquely in the form

$$u = \ldots + u_{-n}x^n + \ldots + u_{-2}x^2 + u_{-1}x + u_0 + u_1 y + u_2 y^2 + \ldots$$

with the u_n's all in S, and all but a finite number equal to zero. It will be convenient to write the elements of R as columns

$$u = \begin{bmatrix} \vdots \\ u_{-2} \\ u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ \vdots \end{bmatrix}.$$

Let E be R^2, α, β, p, and q endomorphisms of E given by:

$$\alpha = \begin{pmatrix} -b & y \\ -x & c \end{pmatrix}, \quad \beta = \begin{pmatrix} -c & y \\ -x & b \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

LEMMA. *The diagram*

$$\begin{array}{cccccccccc} \to E & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & E & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & E & \xrightarrow{\alpha} \\ {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle p}\downarrow & \\ \to E & \xrightarrow{t_\beta} & E & \xrightarrow{t_\alpha} & E & \xrightarrow{t_\beta} & E & \xrightarrow{t_\alpha} & E & \xrightarrow{t_\beta} \end{array}$$

commutes and has exact rows.

PROOF. Commutativity is straightforward to check. Considering E as a free S-module the elements of E can be written as columns

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \\ \vdots \\ v_{-1} \\ v_0 \\ v_1 \\ \vdots \end{bmatrix}.$$

We may then regard α and β as infinite matrixes:

$$\alpha = \left[\begin{array}{ccc|c|ccc} -b & 0 & 0 & bc & 0 & 0 & 0 \\ 0 & -b & 0 & & bc & 0 & 0 \\ 0 & 0 & -b & & 0 & 1 & 0 \\ & & & & & & 1 \\ & & & & & & \\ -1 & & & & & & \\ 0 & -1 & 0 & & c & 0 & 0 \\ 0 & 0 & -bc & & 0 & c & 0 \\ 0 & 0 & 0 & -bc & 0 & 0 & c \end{array}\right]$$

$$\beta = \left[\begin{array}{ccc|c|ccc} -c & 0 & 0 & bc & 0 & 0 & 0 \\ 0 & -c & 0 & & bc & 0 & 0 \\ 0 & 0 & -c & & 0 & 1 & 0 \\ & & & & & & 1 \\ & & & & & & \\ -1 & & & & & & \\ 0 & -1 & 0 & & b & 0 & 0 \\ 0 & 0 & -bc & & 0 & b & 0 \\ 0 & 0 & 0 & -bc & 0 & 0 & b \end{array}\right]$$

Let Λ be the S-homomorphism of E into E given by the matrix

$$\Lambda = \left[\begin{array}{ccc} \begin{array}{c|c|c}0&0&0\\\hline 0&0&0\\\hline 0&0&0\end{array} & -1 & \begin{array}{c|c|c}0&0&0\\\hline -1&0&0\\\hline 0&0&0\end{array} \\ 0 & & \\ \begin{array}{c|c|c}0&0&0\\\hline 0&0&1\\\hline 0&0&0\end{array} & 1 & \begin{array}{c|c|c}0&0&0\\\hline 0&0&0\\\hline 0&0&0\end{array}\end{array}\right]$$

It is then straightforward to check that

$$\alpha\Lambda+\Lambda\beta = 1 = \beta\Lambda+\Lambda\alpha .$$

Define:

$$P = \left\{\begin{array}{ccc|c|cccccc|c|ccc}
 & & & & & & & -1 & & & & & & \\
 & & & & & & & b & -1 & & & & & \\
 & & & & & & & & b & -1 & & & & \\ \hline
 & & & 1 & & & & & & & -1 & & & \\ \hline
 & & & & 1 & & & & & & & -1 & & \\
 & & & & & 1 & & & & & & & -1 & \\
 & & & & & & 1 & & & & & & & -1 \\
1 & & & & & & & & & & & & & \\
 & 1 & & & & & & -b & & & & & & \\
 & & 1 & & & & & & -b & & & & & \\ \hline
 & & & & -c & & & & & & 1 & & & \\ \hline
 & & & & & -c & & & & & & 1 & & \\
 & & & & & & -c & & & & & & 1 & \\
 & & & & & & & & & & & & & 1
\end{array}\right\}$$

$$Q = \left\{\begin{array}{ccc|c|cccccc|c|ccc}
 & & & & & & & 1 & & & & & & \\
-b & & & & & & & & 1 & & & & & \\
-b^2 & -b & & & & & & & & 1 & & & & \\ \hline
 & & & 1 & c & c^2 & c^3 & & & & 1 & c & c^2 & c^3 \\ \hline
 & & & & 1 & c & c^2 & & & & & 1 & c & c^2 \\
 & & & & & 1 & c & & & & & & 1 & c \\
 & & & & & & 1 & & & & & & & 1 \\
-1 & & & & & & & & & & & & & \\
-b & -1 & & & & & & & & & & & & \\
-b^2 & -b & -1 & & & & & & & & & & & \\ \hline
 & & & & c & c^2 & c^3 & & & & 1 & c & c^2 & c^3 \\ \hline
 & & & & & c & c^2 & & & & & 1 & c & c^2 \\
 & & & & & & c & & & & & & 1 & c \\
 & & & & & & & & & & & & & 1
\end{array}\right\}$$

Then $PQ = QP = 1$. Moreover if $\xi_1 = x - b$, $\xi_2 = y - c$, then

$$P \cdot \alpha = \begin{pmatrix} \xi_1 & \xi_2 \\ 0 & 0 \end{pmatrix}.$$

This shows that the ideal $\mathcal{J}$ in R generated by ξ_1 and ξ_2 is isomorphic to the image of α.

Let $t = x + y - b - c$; then t is not a zero-divisor, and if $\varepsilon_1 = (x-c)/t$ *and* $\varepsilon_2 = (y-b)/t$, then the fractional ideal $\mathcal{J}'$ generated by ε_1 and ε_2 is isomorphic to the image of β, which again is isomorphic to the image of ${}^t\alpha$, that is $\mathcal{J}' \approx \mathcal{J}^{\check{}}$. One may check that this isomorphism is the right one, i.e. for $s \in \mathcal{J}'$, $t \in \mathcal{J}$, $s(t) = s \cdot t$.

Since $\varepsilon_1 + \varepsilon_2 = 1$, it is clear that $\mathcal{J}^{\check{}}/R$ is generated by a single element say ε_1 also $(x-b)\cdot\varepsilon_1 = x \cdot 1$ and $(y-c)\cdot\varepsilon_1 = -c \cdot 1$ so the map $R \to \mathcal{J}^{\check{}}/R$ sending 1 to ε_1 factors through S, and it is easy to check that this is an isomorphism.

$$S \approx \mathcal{J}^{\check{}}/R .$$

Summarizing all this we have:

1) *The ideal $\mathcal{J} \subset R$ is stably reflexive with respect to S.*
2) *The fractional ideal $\mathcal{J}'$ consisting of all elements of the total quotient ring of R that maps $\mathcal{J}$ into R is isomorphic to the algebraic dual of $\mathcal{J}$.*
3) *$\mathcal{J}^{\check{}}/R$ is a free S-module of rank* 1.

REFERENCES

1. S. Arakelov, *Families of algebraic curves with fixed degeneracies*, Izv. Akad. Nauk. SSSR Ser. Mat. 35 (1971), 1269–1293.
2. N. Bourbaki, *Éléments de Mathématique. Algèbre Commutative* (Act. Sci. Ind. 1293), Hermann, Paris, 1961.
3. P. Deligne and D. Mumford, *The irreducibility of the space of curves of a given genus*, Inst. Hautes Études Sci. Publ. Math. 36 (1969), 75–109.
4. P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques* in *Modular functions of one variables* II, eds. W. Kuyk and P. Deligne (Proc. Internat. Summer School, Univ. Antwerp, 1972) (Lecture Notes in Math. 349), pp. 143–316. Springer-Verlag, Berlin - Heidelberg - New York, 1973.
5. A Grothendieck and J. Dieudonné, *Éléments de géométrie algèbrique*, Inst. Hautes Études Sci. Publ. Math. 11 (1961), 81–148.
6. R. Hartshorne, *Residues and duality* (Lecture Notes in Math. 20), Springer-Verlag, Berlin - Heidelberg - New York, 1966.
7. F. F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves* I, Math. Scand. 39 (1976), 19–55.
8. S. Lichtenbaum and M. Schlessinger, *The cotangent complex of a morphism*, Trans. Amer. Soc. 128 (1967), 41–70.

9. D. Mumford, *Geometric invariant theory* (Ergebnisse Math. 34) Springer-Verlag, Berlin - Heidelberg - New York, 1965.
10. D. Mumford, *Introduction to Algebraic Geometry*, Lecture Notes, Harvard University, 1967.
11. D. Mumford, *Picard groups of moduli problems*, in *Arithmetical Algebraic Geometry*, ed. O. F. G. Schilling (Proc. Conf. Purdue Univ. 1963) pp. 33–81. Harper and Row Publ., New York, 1965.
12. D. Mumford, *On the equations defining Abelian varieties* III, Invent. Math. 3 (1967), 215–244.
13. D. Mumford, *Varieties defined by quadratic equations. With an appendix by G. Kempf*, in *Questions on algebraic varieties*, (C.I.M.E., III Ciclo, Varenna, 1969) pp. 29–100, Edizioni Cremonese, Roma, 1970.
14. C. S. Seshadri, *Quotient spaces modulo reductive algebraic groups*, Ann. of Math. 95 (1972), 511–556.

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