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THE PROJECTIVITY OF THE MODULI SPACE OF STABLE CURVES, III: THE LINE BUNDLES ON $M_{g,n}$, AND A PROOF OF THE PROJECTIVITY OF $\bar{M}_{g,n}$ IN CHARACTERISTIC 0

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Introduction.

In section 4 we construct the basic line bundles on $M_{g,n}$ and study their behaviour under pullback by contraction and clutching. This is where this paper originally was to end and then Mumford would use these results to prove projectivity as suggested by Seshadri. However Gieseker and Mumford discovered that a stable curve embedded in projective space of sufficiently high dimension is stable in the sense of [9]. Hence the projectivity can be proved in a much more natural way using the techniques of [9]. These results are published in [14].

Instead we have added to this paper sections 5, and 6, where we present the very first proof of the projectivity of the moduli space of stable curves. The result of Gieseker and Mumford is much sharper than Theorem 6.1. In fact the constant m occuring here can be chosen to equal 56/5 regardless of n, g and the characteristic.

None the less the results of this paper give considerable insight into the boundary of the moduli space and we believe this might have some general interest. Finally I want to express my gratitude to Professor Mumford for continuous support and encouragement throughout the writing of this paper.

4. Invertible sheaves on $M_{g,n}$ and their functorial properties.

Recall that a sheaf on the stack $M_{g,n}$ consists of the following data:

1) For every n-pointed stable curve $\pi\colon X \to S$ a sheaf $\mathscr{F}(\pi)$ on S.

2) For every morphism

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$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ S_1 & \xrightarrow{f} & S_2 \end{array}$$

an isomorphism $\varphi_{X_1,X_2}: f^*(\mathscr{F}(\pi_2)) \xrightarrow{\sim} \mathscr{F}(\pi_1)$ which for every composition

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow & & {\scriptstyle \pi_3}\downarrow \\ S_1 & \xrightarrow{f} & S_2 & \xrightarrow{g} & S_3 \end{array}$$

satisfies the cocycle condition

$$\begin{array}{ccc} f^*(g^*(\mathscr{F}(\pi_3))) & \xrightarrow{f^*(\varphi_{X_1,X_2})} & f^*(\mathscr{F}(\pi_2)) \\ \approx\Big| & & \Big\downarrow \varphi_{X_2,X_3} \\ (gf)^*\mathscr{F}(\pi_3) & \xrightarrow{\varphi_{X_1,X_3}} & \mathscr{F}(\pi_1) \end{array}$$

Moreover, because $M_{g,n}$ is an algebraic stack, to define a sheaf $\mathscr{F}$ it suffices to define $\mathscr{F}(\pi)$ for every $\pi: X \to S$ for which $S \to M_{g,n}$ is étale, plus φ's whenever these make sense.

Then there is one and only one way, up to canonical isomorphism, to extend this to a sheaf $\mathscr{F}$ on the whole stack $M_{g,n}$. This of course because descent data are effective for étale surjective morphisms.

Let $\pi: X \to S$; $s_i: S \to X$ $(1 \leqq i \leqq n)$ be an n-pointed stable curve of genus g. By Corollary 3.3, $\Omega_{X/S}$ is a perfect complex on X. Since π is flat, $R\pi_*\Omega_{X/S}$ is a perfect complex on S. We denote by $\Lambda_{g,n}(\pi)$ the invertible sheaf on S defined by

a) $\Lambda_{g,n}(\pi) = \det(R\pi_*\Omega_{X/S})$.

The operations Ω, $R\pi_*$ and det all commute with base change, so $\Lambda_{g,n}$ is an invertible sheaf on $M_{g,n}$. Similarly we define

b) $\lambda_{g,n}(\pi) = \det(R\pi_*\omega_{X/S})$

c) $\delta_{g,n} = \lambda_{g,n} \otimes \Lambda_{g,n}^{-1}$

d) $\varkappa_{g,n}^{(i)}(\pi) = s_i^*(\omega_{X/S})$

e) $\tilde{\Lambda}_{g,n} = \Lambda_{g,n} \otimes \bigotimes_{i=1}^{n} \varkappa_{g,n}^{(i)}$

f) $\tilde{\delta}_{g,n} = \lambda_{g,n} \otimes \tilde{\Lambda}_{g,n}^{-1}$.

Besides the above invertible sheaves we also have the divisors $S_{g,n}^H$ of Definition 3.8. When no confusion is possible we drop the subscripts g, n.

Suppose $\pi: X \to S$ is smooth over each point of S of depth 0. Then, since $\Omega_{X/S}$ is flat over S, the associated points of $\Omega_{X/S}$ lie over the associated points of

S. Over the associated points of S, the canonical morphism $\Omega_{X/S} \to \omega_{X/S}$ is an isomorphism. Hence in this case $\Omega_{X/S} \to \omega_{X/S}$ is everywhere injective and the points in $\mathrm{Sup}(\pi_*(\omega_{X/S}/\Omega_{X/S}))$ have depth $\geqq 1$. Therefore we have

$$\delta(\pi) = \lambda \otimes \Lambda^{-1} \approx \mathcal{O}_S(\mathrm{Div}\, \pi_*(\omega_{X/S}/\Omega_{X/S})) .$$

$\pi\colon X \to S$ defines a morphism $S \to M_{g,n}$ and $\mathrm{Div}\, \pi_*(\omega_{X/S}/\Omega_{X/S})$ is simply the pullback to S of the divisor $S_{g,n} \subset M_{g,n}$, that is

$$\delta_{g,n} \approx \mathcal{O}_{M_{g,n}}(S_{g,n}) .$$

Roughly speaking, $\delta_{g,n}$ is the sheaf of functions on $M_{g,n}$ regular except for simple poles at infinity.

THEOREM 4.1. *Let $\pi = \pi_{n+1}\colon M_{g,n+1} \to M_{g,n}$, and let $S^i = S^{i,n+1}_{g,n+1}$ be the divisor on $M_{g,n+1}$, which is really the image of the i-th section $s_i\colon M_{g,n} \to M_{g,n+1}$. We have*

a) $\pi^*(\lambda_{g,n}) \cong \lambda_{g,n+1}$,

b) $\pi^*(\Lambda_{g,n}) \cong \Lambda_{g,n+1}(S^1 + S^2 + \dots + S^n)$,

c) $\pi^*(\varkappa^{(i)}_{g,n}) = \varkappa^{(i)}_{g,n+1}(-S^i)$,

d) $\varkappa^{(n+1)}_{g,n+1} \cong \omega_{M_{g,n+1}/M_{g,n}}(S^1 + S^2 + \dots + S^n)$,

and hence

e) $\pi^*(\tilde{\Lambda}_{g,n}) \otimes \omega_{M_{g,n+1}/M_{g,n}}(S^1 + \dots + S^n) \cong \tilde{\Lambda}_{g,n+1}$.

PROOF. Let $\pi\colon X \to S$, $s_i\colon S \to X$ $(1 \leqq i \leqq n)$ be an n-pointed stable curve of genus g such that the corresponding morphism $S \to M_{g,n}$ is étale. By duality we have

$$\lambda(\pi) = \det R\pi_*\omega_{X/S} \cong \det R\pi_*\mathcal{O}_X .$$

Consider the diagram

$$\begin{array}{ccccc} X' & \xrightarrow{q} & X \times_S X & \xrightarrow{p_2} & X \\ {\scriptstyle \pi'}\downarrow \uparrow{\scriptstyle s'_i} & & {\scriptstyle p_1}\downarrow \uparrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \pi} \\ X & = & X & \xrightarrow{\pi} & S \end{array}$$

where X' is the stabilization defined by Δ.

We have $Rq_*\mathcal{O}_{X'} \approx \mathcal{O}_{X \times_S X}$, so this proves a).

To prove b), notice that the divisor of singular curves on X is $\pi^{-1}(S_{g,n}(\pi)) + s_1 + \dots + s_n$. Since $\Lambda \approx \lambda \otimes \delta^{-1}$ we get b).

On X' we have a short-exact sequence of sheaves

$$(0) \to q^*\mathcal{O}_{X \times_S X}(-s_i) \to \mathcal{O}_{X'}(-s_i') \to \mathcal{O}_{X'}(-s_i')|_{q^{-1}(s_i \cap \Delta)} \to (0)\,.$$

s_i' is transversal to $q^{-1}(s_i \cap \Delta)$, so taking $s_i'^*$ of this sequence leaves it exact, i.e.

$$(0) \to \pi^*(\varkappa_n^{(i)}) \to \varkappa_{n+1}^{(i)} \to \mathscr{F} \to (0)\,,$$

where $\mathscr{F}$ is a sheaf with support on the divisor s_i on X and of rank 1. Hence $\operatorname{Div}(\mathscr{F}) = s_i$ and this proves c).

To prove d) notice that the conormal bundle

$$\varkappa^{(n+1)}(\pi') \cong s_{n+1}'^*(\omega_{X'/X}(s_1' + \dots + s_n'))\,.$$

This is because the sections never cross. By Lemma 1.4 a) we have

$$\varkappa^{(n+1)}(\pi') \approx \Delta^* p_2^*(\omega_{X/S}(s_1 + \dots + s_n))\,.$$

But $p_2 \circ \Delta$ is the identity, so this is d).

THEOREM 4.2. *Let*

$$\alpha\colon M_{g-1,n+2} \to M_{g,n}\,,$$
$$\beta\colon M_{g_1,n_1+1} \times M_{g_2,n_2+1} \to M_{g,n}$$

be the clutching morphisms. Then

a) $\alpha^*\lambda_{g,n} \approx \lambda_{g-1,n+2}$, $\quad \beta^*\lambda_{g,n} \approx \lambda_{g_1,n_1+1} \otimes \lambda_{g_2,n_2+1}$,

b) $\alpha^*\tilde{\Lambda}_{g,n} \approx \tilde{\Lambda}_{g-1,n+2}$, $\quad \beta^*\tilde{\Lambda}_{g,n} \approx \tilde{\Lambda}_{g_1,n_1+1} \otimes \tilde{\Lambda}_{g_2,n_2+1}$,

c) $\alpha^*\tilde{\delta}_{g,n} \approx \tilde{\delta}_{g-1,n+2}$, $\quad \beta^*\tilde{\delta}_{g,n} \approx \tilde{\delta}_{g_1,n_1+1} \otimes \tilde{\delta}_{g_2,n_2+1}$.

PROOF. Consider a clutching diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & X' \\ {\scriptstyle s_1} \uparrow {\scriptstyle s_2} \uparrow {\scriptstyle \pi} \downarrow & & {\scriptstyle \pi'} \downarrow \uparrow {\scriptstyle s} \\ S & = & S \end{array}$$

On X' we have a short-exact sequence

$$(0) \to \mathcal{O}_{X'} \to p_*\mathcal{O}_X \xrightarrow{s_1^* - s_2^*} \mathcal{O}_S \to (0)\,.$$

Taking $\det R\pi'_*$ of this sequence yields a). b) follows by the clutching sequence Theorem 3.5, and c) is a consequence of a) and b).

Let α be the numerical function

$$\alpha(g) = \begin{cases} 3 & \text{if } g = 0 \\ 1 & \text{if } g = 1 \\ 0 & \text{if } g \geqq 2\,. \end{cases}$$

We want to see what happens to our line bundles under pullback by the full contraction morphism

$$\pi = \pi_{\alpha+1,\alpha+2,\ldots,\alpha+n} : M_{g,\alpha+n} \to M_{g,\alpha} .$$

DEFINITION. On $M_{g,n}$ we define the divisors

$$\nabla^k_{g,n} = \sum_{*H=k+1} S^H_{g,n}$$

$$D^l_{g,n} = \sum_{l\in H} S^H_{g,n} .$$

REMARK. Note that $S^H_{g,n}=\emptyset$, if $*(H\cap\{n;\ 1\leqq n\leqq\alpha(g)\}) \geqq 2$.
Consider the diagram

$$\begin{array}{ccc} M_{g,\alpha+n} & \xrightarrow{\pi_{\hat{i}}} & M_{g,\alpha+1} \\ \pi_i\downarrow & & \downarrow \\ M_{g,\alpha+n-1} & \to & M_{g,\alpha} \end{array}$$

$i>\alpha$ and $\pi_{\hat{i}}$ is short for $\pi_{\alpha+1,\alpha+2,\ldots,\hat{i},\ldots,\alpha+n}$.

LEMMA 4.3. *With notation as above*

$$\omega_{M_{g,\alpha+n}/M_{g,\alpha+n-1}}(S^{1,i}+S^{2,i}+\ldots+S^{\alpha+n,i}) \approx \pi_{\hat{i}}^*\omega_{M_{g,\alpha+1}/M_{g,\alpha}}\otimes\mathcal{O}(D^i) .$$

PROOF. We may well suppose that $i=\alpha+n$. Consider the commutative diagram

$$\begin{array}{ccccc} M_{g,\alpha+n} & \xrightarrow{\pi_{\alpha+1}} & M_{g,\alpha+n-1} & \xrightarrow{\pi_{\alpha+n-1}} & M_{g,\alpha+1} \\ \pi_{\alpha+n}\downarrow\uparrow s_i & & \pi_{\alpha+n-1}\downarrow\uparrow s_{i-1} & \downarrow & \downarrow \\ M_{g,\alpha+n-1} & \xrightarrow{\pi_{\alpha+1}} & M_{g,\alpha+n-2} & \longrightarrow & M_{g,\alpha} \end{array}$$

(The diagram commutes since we renumber the sections.)
We define

$$\delta_j = \begin{cases} j & \text{if } j\leqq\alpha \\ j+1 & \text{if } j>\alpha ; \end{cases}$$

then

$$\pi^{-1}_{\alpha+1}(S^{j_1,j_2,\ldots,j_k,\alpha+n-1}_{g,\alpha+n-1}) = S^{\delta_{j_1},\delta_{j_2},\ldots,\delta_{j_k},\alpha+n}_{g,\alpha+n} + S^{\delta_{j_1},\ldots,\delta_{j_k},\alpha+1,\alpha+n}_{g,\alpha+n} .$$

Hence

$$\pi^{-1}_{\alpha+1}\Bigg(\sum_{\substack{*H=k\\ \alpha+n-1\in H}} S^H_{g,\alpha+n-1}\Bigg) = \sum_{\substack{*H=k\\ \alpha+n\in H}} S^H_{g,\alpha+n} + \sum_{\substack{*H=k+1\\ \alpha+n\in H\\ \alpha+1\in H}} S^H_{g,\alpha+n} - \sum_{\substack{*H=k\\ \alpha+n\in H\\ \alpha+1\in H}} S^H_{g,\alpha+n} .$$

Summing these relations over k gives

$$\pi_{\alpha+1}^{-1} D_{g,\alpha+n-1}^{\alpha+n-1} = D_{g,\alpha+n}^{\alpha+n} - S_{g,\alpha+n}^{\alpha+1,\alpha+n} .$$

By Lemma 1.6 a) we have

$$\pi_{\alpha+1}^{*}\omega_{M_{g,\alpha+n-1}/M_{g,\alpha+n-2}}(S^{1,\alpha+n-1}+\ldots+S^{\alpha+n-2,\alpha+n-1})$$

$$\approx \omega_{M_{g,\alpha+n}/M_{g,\alpha+n-1}}(S^{1,\alpha+n}+\ldots+S^{\alpha+n-1,\alpha+n}-S^{\alpha+1,\alpha+n}) ,$$

and the Lemma follows readily by induction.

The main formula of this section is

THEOREM 4.4.

$$\tilde{\Lambda}_{g,\alpha+n} \approx \pi^*(\tilde{\Lambda}_{g,\alpha}) \otimes \bigotimes_{i=\alpha+1}^{\alpha+n} \pi_{\hat{i}}^*(\omega_{M_{g,\alpha+1}/M_{g,\alpha}}) \otimes \mathcal{O}(E_{g,\alpha+n})$$

where

$$E_{g,n} = \nabla_{g,n}^{1} + 2\nabla_{g,n}^{2} + \ldots + (n-1)\nabla_{g,n}^{n-1} .$$

PROOF. In order to use induction we consider the composition

$$M_{g,\alpha+n} \xrightarrow{\pi_{\alpha+n}} M_{g,\alpha+n-1} \to M_{g,\alpha} .$$

By theorem 4.1 e) and the previous lemma we have

$$\tilde{\Lambda}_{g,\alpha+n} \approx \pi_{\alpha+n}^{*}(\tilde{\Lambda}_{g,\alpha+n-1}) \otimes \omega_{M_{g,n+2}/M_{g,\alpha+n-1}}(S^{1,\alpha+n}+\ldots+S^{\alpha+n-1,\alpha+n})$$

$$\approx \pi_{\alpha+n}^{*}(\tilde{\Lambda}_{g,\alpha+n-1}) \otimes \pi_{\alpha+n}^{*}(\omega_{M_{g,\alpha+1}/M_{g,\alpha}}) \otimes \mathcal{O}(D_{g,\alpha+n}^{\alpha+n}) .$$

$$\pi_{\alpha+n}^{-1}(\nabla^{k}) = \sum_{*H=k+1} S^{H} + \sum_{\substack{*H=k+2\\ \alpha+n\in H}} S^{H} - \sum_{\substack{*H=k+1\\ \alpha+n\in H}} S^{H} .$$

Multiplying each of these relations by k and summing over k gives the relation

$$\pi_{\alpha+n}^{-1}(E_{g,\alpha+n-1}) = E_{g,\alpha+n} - D_{g,\alpha+n}^{\alpha+n}$$

and the theorem follows by induction.

We end this section with an inequality which is crucial in the next paragraph. Suppose $g<2$ so that $\alpha>0$ and consider the map

$$\pi_{\hat{j}} : M_{g,\alpha+n} \to M_{g,\alpha+1} .$$

On $M_{g,\alpha+1}$ we have $E=E_{g,\alpha+1}=\sum_{p=1}^{\alpha} S_{g,\alpha+1}^{p,\alpha+1}$ and

$$\sum_{j=\alpha+1}^{\alpha+n} \pi_{\hat{j}}^{-1}(E_{g,\alpha+1}) = \sum_{p=1}^{\alpha} D^{p} \cap E .$$

Since $D^p \cap D^q = \emptyset$ for $p < q \leqq \alpha$, we have

LEMMA 4.5.

$$E_{g,\alpha+n} \geqq \sum_{j=\alpha+1}^{\alpha+n} \pi_j^{-1}(E_{g,\alpha+1}) \, .$$

5. Ampleness of $\tilde{\Lambda}$ on the fibres of the full contraction morphism.

Let $\pi\colon C \to S$ be a smooth stable curve with $\alpha(g)$ sections. Then π corresponds to a morphism which we also call π

$$\pi : S \to M_{g,\alpha} \setminus S_{g,\alpha} \, .$$

We define the scheme $B_n(C/S)$ via the cartesian diagram

$$\begin{array}{ccc} B_n(C/S) & \to & M_{g,\alpha+n} \\ \downarrow & & \downarrow \\ S & \longrightarrow & M_{g,\alpha} \end{array}$$

$B_{n+1}(C/S)$ is an $n+\alpha$-stable curve over $B_n(C/S)$, and we have the cartesian diagram

$$\begin{array}{ccc} B_{n+1}(C/S) & \to & M_{g,\alpha+n+1} \\ {\scriptstyle \pi_{\alpha+n+1}(C/S)}\downarrow & & \downarrow \\ B_n(C/S) & \underset{\pi_{\alpha+n+1}(C/S)}{\to} & M_{g,\alpha+n} \end{array}$$

On $B_n(C/S)$ we have the line bundle

$$\tilde{\Lambda}_n(C/S) = \tilde{\Lambda}(\pi_{\alpha+n+1}(C/S)) = \pi_{\alpha+n+1}(C/S)^*(\tilde{\Lambda}_{g,\alpha+n}) \, .$$

PROPOSITION 5.1. *$B_n(C/S) \to S$ is a smooth and proper morphism of relative dimension n.*

PROOF. We may suppose $S = \operatorname{Spec}(k)$, k an algebraically closed field. $B_1(C/k) = C$ is smooth and proper of dimension 1 over k, so we may proceed by induction.

Let x be a point in $B_n(C/k)$. Then x corresponds to a curve E/k with $\alpha+n$ distinguished points $P_1, \ldots, P_{\alpha+n}$. One of the components of E is C.

If $\alpha+1 \leqq j \leqq \alpha+n$, the fibre of the contraction

$$\pi_j : B_n(C/k) \to B_{n-1}(C/k)$$

over the point $\pi_j(x)$ is the curve E' obtained from E by contracting the point P_j and the point $x \in E' = \pi_j^{-1}(\pi_j(x))$ is the image of P_j.

If $E = C$, clearly π_j is smooth at x. Otherwise $x \in S^{H_1} \cap S^{H_2} \cap \ldots \cap S^{H_q}$, and we may suppose that H_1 does not contain any of the other H_j's. Then if $j \in H_1$ and $j > \alpha$, the map

$$\pi_j : B_n(C/k) \to B_{n-1}(C/k)$$

is smooth at x and the proposition follows by induction. See Definition 3.8 for a definition of S^H.

In the rest of this section we fix an algebraically closed field k and consider all of our schemes to be defined over k.

Let C be a smooth stable curve of genus g and with distinguished points P_j ($1 \leqq j \leqq \alpha(g)$). We consider a morphism $\pi: S \to B_n(C)$, where S is another nonsingular irreducible complete curve. π corresponds to an $\alpha + n$-pointed stable curve $\pi: \hat{C} \to S$. Each one of the sections of π gives rise to a morphism $t_i: S \to C$. This is the composition

$$S \xrightarrow{\pi} B_n(C) \xrightarrow{\pi_i(C)} B_1(C) = C.$$

For $1 \leqq j \leqq \alpha$ we have $t_j(S) = P_j \in C$.

We wish to study the pullback to S of the line bundle $\tilde{\Lambda}_n(C)$ on $B_n(C)$. By Theorem 4.4 this is

$$\pi^*(\tilde{\Lambda}_n(C)) = \tilde{\Lambda}(\pi) \approx \bigotimes_{i=\alpha+1}^{\alpha+n} t_i^*(\omega_C) \otimes \mathcal{O}_S(\pi^{-1}(E)).$$

By Lemma 4.5 we have

$$\pi^{-1}(E) \geqq \pi^{-1}\left(\sum_{j=\alpha+1}^{\alpha+n} \pi_j^{-1}(E_{g,\alpha+1})\right) = \sum_{i=1}^{\alpha}\left(\sum_{j=\alpha+1}^{\alpha+n} t_j^{-1}(P_i)\right).$$

Hence we have

PROPOSITION 5.2. *Let $\pi: S \to B_n(C)$ be as above, then*

$$\deg \pi^*(\tilde{\Lambda}_n(C)) = \deg \tilde{\Lambda}(\pi) \geqq (2g-2+\alpha) \sum_{i=\alpha+1}^{\alpha+n} \deg(t_i).$$

In particular,

$$\deg \pi^*(\tilde{\Lambda}_n(C)) \geqq \sum_{i=1}^{\alpha+n} \deg(t_i).$$

DEFINITION 5.3. Let X be a complete scheme and S an integral curve in X. We denote by $m_P(S)$ the multiplicity of a point P on the curve, and by $m(S)$ the number

$$m(S) = \sup_{P \in S} \{m_P(S)\} .$$

LEMMA 5.4. *For all stable nonsingular C and for all integral curves $S \subset B_n(C)$*

$$\deg \tilde{\Lambda}_n(C)|_S \geqq m(S) .$$

PROOF. We prove the lemma by induction with respect to n. For $n=1$ $S=C$ $=B_1(C)$, and by Theorem 4.4

$$\tilde{\Lambda}_1(C) \approx \omega_C \otimes \mathcal{O}_C(E) .$$

In this case $E=\sum_{j=1}^{\alpha} P_j$, so $\deg \mathcal{O}_C(E)=\alpha$. Hence

$$\deg \tilde{\Lambda}_1(C) = 2g-2+\alpha \geqq 1 = m(S) .$$

Assume that the lemma is true for all $k<n$ and fix notation

$S \subset B_n(C)$ an integral curve ,
$\hat{S}$ the normalization of S ,

$$\pi : S \to B_n(C)$$
$$t_i : S \to C \quad (1 \leqq i \leqq \alpha+n) ,$$
$$\hat{t}_i : \hat{S} \to C \quad (1 \leqq i \leqq \alpha+n) .$$

If all the t_i's are constant maps, at least three of them are equal. If r is the maximum number of equal maps we get a factorization via the clutching morphism

$$S \hookrightarrow B_{n-r+1}(C) \times B_{r-2}(\boldsymbol{P}_k^1) \hookrightarrow B_n(C)$$

and the lemma follows from Theorem 4.2 b).

Suppose then that t_i is not a constant map. Then on the one hand we have

$$\deg \tilde{\Lambda}_n(C)|_S \geqq \sum_{j=\alpha+1}^{\alpha+n} \deg \hat{t}_j \geqq \deg \hat{t}_i .$$

On the other hand, if P is a point of S, P' its image in C^n, and S' the image of S in C^n, we have

$$\deg \hat{t}_i \geqq m_{P'}(S') \geqq m_P(S) .$$

The second inequality is obvious. To prove the first inequality, let H be the divisor $p_i^{-1}(t_i(P))$. Then by the very definition of multiplicity we have

$$m_{P'}(S') \leqq (H,S')_{P'} \leqq (H,S') = \deg \hat{t}_i ,$$

and the lemma follows by induction.

THEOREM 5.5. *For all n and all nonsingular stable curves C, $\tilde{\Lambda}_n(C)$ is ample.*

PROOF. For this it is enough to state *Seshadri's ampleness criterion*:
Let L be an invertible sheaf on a complete scheme X, then L is ample if and only if there is an $\varepsilon > 0$ such that $\deg L|_S \geqq \varepsilon \cdot m(S)$ for every integral curve S in X.

6. Proof of projectivity in characteristic 0.

In this section let k be the field of complex numbers. All schemes and morphisms will be defined over k. In particular by $M_{g,n}$ in this section we will mean

$$M_{g,n} \times_{\mathrm{Spec}\,(\mathbf{Z})} \mathrm{Spec}\,(k) .$$

THEOREM 6.1. *For all pairs g,n with $2g-2+n>0$, the stack $M_{g,n}$ is coarsely represented by a normal projective variety $\bar{M}_{g,n}$. More precisely, there is a morphism*

$$\Phi : M_{g,n} \to \bar{M}_{g,n}$$

such that

1) *Φ induces an isomorphism*

$$\Phi(k): \begin{Bmatrix} \text{isomorphism of classes} \\ \text{of objects in } M_{g,n}(k) \end{Bmatrix} \xrightarrow{\sim} \bar{M}_{g,n}(k) .$$

2) *$\bar{M}_{g,n}$ is normal and proper over* Spec (k).
3) *There exists an integer N such that $\lambda^{\otimes N}$ and $\tilde{\delta}^{\otimes N}$ are pullbacks of invertible sheaves on $\bar{M}_{g,n}$, which we also write $\lambda^{\otimes N}$ and $\tilde{\delta}^{\otimes N}$. Finally there exists a number $m>0$ such that if $N \mid a$, $N \mid b$ and $a \geqq mb > 0$, then,*

$$\lambda^{\otimes a} \otimes (\tilde{\delta}^{-1})^{\otimes b} \quad \textit{is ample on } \bar{M}_{g,n} .$$

REMARK. It is not hard to see that 1) and 2) imply that Φ is universal for all morphisms from $M_{g,n}$ to schemes. To prove the theorem we introduce some auxiliary schemes. Fix an integer $e \geqq 3$ and let $\{C, x_1, x_2, \ldots, x_n)$ be an n-pointed stable curve of genus g. We define

$$d = e(2g-2+n) = \deg(\omega_C(x_1+\ldots+x_n)^{\otimes e}),$$

$$P(t) = dt-(g-1) = \text{Hilbert polynomial of } \omega_C(x_1+\ldots+x_n)^{\otimes e},$$

$$\nu = P(1) = h(\omega_C(x_1+\ldots+x_n)^{\otimes e}).$$

Consider the Hilbert scheme $\mathrm{Hilb}^P_{\boldsymbol{P}^{\nu-1}}$ of subschemes of $\boldsymbol{P}^{\nu-1}$ with Hilbert polynomial $P(t)$ and let

$$H_{g,n} \subset \mathrm{Hilb}^P_{\boldsymbol{P}^{\nu-1}} \times (\boldsymbol{P}^{\nu-1})^n$$

be the locally closed subscheme representing $n+1$ tuples $(C, x_1, \ldots, x_n)$, where the x_i's are distinct smooth points on C, C with these x_i's is an n-pointed stable curve and C is embedded in $\boldsymbol{P}^{\nu-1}$ in such a way that $\mathcal{O}(1)|_C \approx \omega_C(x_1+\ldots+x_n)^{\otimes e}$. Thus $H_{g,n}$ represents the functor

$$H_{g,n}(S) = \left\{\begin{array}{l}\text{pairs } (\{\pi\colon X \to S,\ s_i\colon S \to X\}, \alpha) \\ \text{consisting of an } n\text{-pointed stable} \\ \text{curve over } S \text{ and an } S\text{-isomorphism} \\ \alpha\colon \boldsymbol{P}(\pi_*\omega_{X/S}(s_1+\ldots+s_n)^{\otimes e} \xrightarrow{\sim} \boldsymbol{P}_S^{\nu-1}\end{array}\right\} \Big/ \begin{array}{l}\text{modulo} \\ \text{isomorphisms}\end{array}.$$

$\mathrm{PGL}(\nu-1)$ acts on $H_{g,n}$ and $M_{g,n}$ is the stack theoretic quotient of $H_{g,n}$ by $\mathrm{PGL}(\nu-1)$. According to ([15, Theorem 6.1]), $H_{g,n}$ has a finite normal Galois covering $H^*_{g,n}$ (not at all unique) on which $\mathrm{PGL}(\nu-1)$ acts freely, commuting with the Galois group Γ and with local cross sections in the Zariski topology, and $H^*_{g,n} \to H_{g,n}$ is a $\mathrm{PGL}(\nu-1)$ morphism. $H^*_{g,n}$ is a principal fibre bundle over a normal variety $X_{g,n}$ with group $\mathrm{PGL}(\nu-1)$, and the action of Γ descends to $X_{g,n}$. On $H_{g,n}$ we have the universal family of n-pointed stable curves in $\boldsymbol{P}^{\nu-1}$. Pulling this back to $H^*_{g,n}$ and dividing by $\mathrm{PGL}(\nu-1)$ we see that $X_{g,n}$ has a family of n-pointed stable curves on it. Then we have a commutative diagram

$$\begin{array}{ccc} H^*_{g,n} & \longrightarrow & H_{g,n} \\ \downarrow & & \downarrow \\ X_{g,n} & \xrightarrow[q]{} & M_{g,n} \end{array}$$

where $q\colon X_{g,n} \to M_{g,n}$ is finite and surjective; hence $X_{g,n}$ is proper over k, since $M_{g,n}$ is. Therefore by [15, Remark 6.1], it suffices to prove that $\lambda^{P_m} \otimes \delta^{-1}$ is ample on $X_{g,n}$ for some m, and then $\bar{M}_{g,n} = X_{g,n}/\Gamma$ will have all the required properties.

By contraction, $X_{g,n}$ has a family of stable curves over it, so by ([1, Theorem 1.1, and Lemma 1.4]) there is a morphism

$$t: X_{g,n} \to S_g = \left\{ \begin{array}{l} \text{Satake compactification of} \\ \text{the moduli space of} \\ \text{polarized abelian} \\ \text{varieties of dimension } g \end{array} \right.$$

such that:

1) If x is a point in $X_{g,n}$ corresponding to an n-pointed stable curve C, then $t(x)$ corresponds to the abelian part of the generalized jacobian of C.
2) If S_g is embedded in projective space by modular forms of weight m, then

$$t^*(\mathcal{O}(1)) \approx \lambda^{\otimes m}.$$

Let x be a point of $X_{g,n}$ corresponding to a stable curve C and let $C_1, \ldots, C_k$ be the non-rational components of the normalization of C. Let

$$m = n+2g-2-2\sum_{i=1}^{k}(g_i-1)-(\#\text{ of elliptic } C_i)$$

where $g_i = \text{genus } C_i$.

Let X be the variety:

$$X = \coprod_{\substack{n_1+\ldots+n_k+n'_1+\ldots+n'_l+1=m \\ n_i \geq 1 \ (0 \text{ if } C_i \text{ ellipic}) \\ n'_i \geq 0}} \left(\coprod B_{n_1}(C_1) \times \ldots \times B_{n_k}(C_k) \times B_{n'_1}(\boldsymbol{P}^1) \times \ldots \times B_{n'_l}(\boldsymbol{P}^1)\right)$$

in which the inner sum contains one copy for each choice of pairing all but n of the base points, leading to a stable curve by clutching.

Note that each point of X defines k different n_i-pointed stable curves of genus g_i (n_i+1-pointed when $g_i=1$) and l different n'_i+3 pointed stable curves of genus 0. Identifying all but n of these in pairs in such a way that the result is connected and taking into account the definition of m, one sees that this gives an n-pointed stable curve of genus g.

Note that we get in this way all stable curves C' such that the normalizations of C and C' minus their rational components are isomorphic. By Torelli's theorem, we get all C' such that the abelian part of the jacobians of C and C' are isomorphic.

Clutching the points in the various configurations corresponding to each component of X, we get a morphism $\beta: X \to M_{g,n}$. Let $q_x: t^{-1}(t(x)) \to M_{g,n}$ be the restriction of q and let $X' = \text{Isom}(\beta, q_x)$.

Then we get a diagram

$$\begin{array}{ccc} X' & \xrightarrow{p_1} & t^{-1}(t(x)) \\ {\scriptstyle p_2}\downarrow & & {\scriptstyle q_x}\downarrow \\ X & \xrightarrow{\beta} & M_{g,n}, \end{array}$$

where p_2 is finite and p_1 is finite and surjective.

Let δ be the invertible sheaf on X defined on each component as

$$p_1^*\tilde{\delta}_{g_1,n_1}\otimes\ldots\otimes p_k^*\tilde{\delta}_{g_{n_k},n_k}\otimes p_{k+1}^*\tilde{\delta}_{0,n_1'}\otimes\ldots\otimes p_{k+1}^*\tilde{\delta}_{0,n_l'}\,.$$

Then by Theorem 4.2 we have

$$p_1^*(\tilde{\delta}_{g,n}|_{t^{-1}t(x)}) = p_2^*(\tilde{\delta})\,.$$

Since p_2 is finite and p_1 is finite surjective, it follows from Theorem 5.5 that $\tilde{\delta}_{g,n}^{-1}$ is ample on $t^{-1}(t(x)$ hence there is an $m>0$ such that $\lambda^{\otimes m}\otimes\tilde{\delta}^{-1}$ is ample on $X_{g,n}$.

This proves Theorem 6.1.

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