MATH SCAND 52 (1983), 213–228

ON THE ZEROS OF THE DIRICHLET L-FUNCTIONS NEAR THE CRITICAL LINE

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1. Introduction.

Let χ be a Dirichlet character mod q and χ_0 the principal character mod q. Denote by $N_\chi(\alpha;\ T_1, T_2)$ the number of zeros of the Dirichlet L-function $L(s,\chi)$ in the rectangle $\alpha \leqq \sigma \leqq 1$, $T_1 \leqq t \leqq T_2$. Write briefly $N_\chi(\alpha;\ T) = N_\chi(\alpha;\ -T, T)$ and $N(\alpha;\ T) = N_{\chi_0}(\alpha;\ T)$. We shall give for the number

$$(1.1)\qquad N(q) = \sum_{\chi \bmod q} N_\chi(\alpha;\ T_1, T_2)$$

an upper bound which is interesting when α is near $\frac{1}{2}$. Results of this type were first obtained by Selberg. In [7] he proved namely that

$$(1.2)\qquad N(\alpha;\ T) \ll T^{1-(\alpha-\frac{1}{2})/4} \log T$$

for $\frac{1}{2} \leqq \alpha \leqq 1$ and $T \geqq 2$. In [8] he proved that if $|T_1|, |T_2| \leqq q^{\frac{1}{4}-\varepsilon}$, $T_2 - T_1 \geqq 1/\log q$ and $\frac{1}{2} + 1/\log q \leqq \alpha < 1$, then

$$(1.3)\qquad N'(q) \ll q^{1-\frac{2}{3}(\alpha-\frac{1}{2})\varepsilon} (T_2 - T_1) \log q\ ,$$

where $'$ means that χ_0 is omitted in the summation. Recently Jutila [3] proved that for $\frac{1}{2} \leqq \alpha \leqq 1$, $T \geqq 2$ and any fixed $\varepsilon > 0$,

$$(1.4)\qquad N(\alpha;\ T) \ll_\varepsilon T^{1-(1-\varepsilon)(\alpha-\frac{1}{2})} \log T,$$

which is a sharpened version of (1.2). We combine methods of Selberg [8], Jutila [3] and Ramachandra [6] to prove the following theorem.

THEOREM. *Let q be a positive integer, $\alpha \geqq \frac{1}{2}$, $T_2 - T_1 \gg 1/\log(q+1)$, $0 < 2\varepsilon \leqq c \leqq 1$ and*

$$(1.5)\qquad \max(|T_1|, |T_2|) \ll q^{1-c}(T_2 - T_1 + 1)\ .$$

Then, in the notation (1.1), *we have*

$$(1.6)\qquad N(q) \ll_\varepsilon q^{1-(c-\varepsilon)(\alpha-\frac{1}{2})} (T_2 - T_1) \times \times (T_2 - T_1 + 1)^{-(1-\varepsilon)(\alpha-\frac{1}{2})} \log q (T_2 - T_1 + 2)\ .$$

Received January 8, 1982.

COROLLARY 1. *If* $T \geqq 2$, *then*

$$\sum_{\chi \bmod q} N_\chi(\alpha;\, T) \ll_\varepsilon (qT)^{1-(1-\varepsilon)(\alpha-\frac{1}{2})} \log qT .$$

This is a generalization of (1.4) and for $\frac{1}{2} \leqq \alpha \leqq \frac{1}{2} + (24-\varepsilon)(\log\log qT)/\log qT$ it is sharper than the estimate of Montgomery ([4, Theorem 12.1]), which is a generalization of Ingham's [2] well-known theorem.

COROLLARY 2. *If* $T_2 - T_1 \gg 1/\log(q+1)$ *and*

$$\max(|T_1|, |T_2|) \ll q^{1-2\varepsilon} ,$$

then

$$N(q) \ll_\varepsilon (T_2 - T_1) q^{1-\varepsilon(\alpha-\frac{1}{2})} \log(q+1) .$$

This is both sharper and more general than (1.3). In particular it enables one to estimate nontrivially the number of zeros in a rectangle of height $1/\log q$ at a distance $q^{1-\varepsilon}$ (instead of $q^{\frac{1}{4}-\varepsilon}$) from the real axis.

Both of the corollaries are immediate consequences of the theorem.

Now we shall outline the proof and introduce some notation. Let

$$1 < z_1 < z_2 ,$$

$$1 < v_1 < v_2 ,$$

$$\varkappa_n = \varkappa_n(v_1, v_2) = \begin{cases} 1, & 1 \leqq n \leqq v_1 , \\ \log(v_2/n)/\log(v_2/v_1), & v_1 < n \leqq v_2 , \\ 0, & n > v_2 , \end{cases}$$

$$\lambda_n = \lambda_n(z_1, z_2) = \mu(n)\varkappa_n(z_1, z_2) ,$$

where $\mu(n)$ is the Möbius function. Then the "mollifier"

$$M(s,\chi) = \sum_{n < z_2} \chi(n)\lambda_n n^{-s} \tag{1.7}$$

makes the quantity $|M(s,\chi)L(s,\chi) - 1|^2$ small on the average, but at the zeros of $L(s,\chi)$ this expression equals 1: this argument gives the density estimate. It turns out, as was observed by Jutila [3] in the case $\chi = \chi_0$, that the mean value of the same expression with $L(s,\chi)$ replaced by the smoothened partial sum

$$F(s,\chi) = \sum_{n < v_2} \chi(n)\varkappa_n n^{-s}$$

can be estimated satisfactorily. In order to make use of this fact in estimating the mean value of $|M(s,\chi)L(s,\chi) - 1|^2$, we appeal to an idea of Ramachandra

[6] (see Lemma 2 below). The problem then reduces to estimating sums of the type $\sum a_n^2 n^{-2\sigma}$ and $\sum b_n^2(x) n^{-1}$, where

$$a_n = a_n(z_1, z_2) = \sum_{d|n} \lambda_d \tag{1.8}$$

and

$$b_n(x) = b_n(x; v_1, v_2, z_1, z_2) = \sum_{d|n} \lambda_d \varkappa_{n/d} d^x; \tag{1.9}$$

in particular the numbers $b_n(0)$ are the coefficients of the Dirichlet polynomial $M(s,\chi)F(s,\chi)$. The estimates of these sums, given in Lemmas 4 and 5, are based on Lemma 3, which is in some respects more general than the corresponding lemma of Motohashi [5].

2. A formula for $M(s,\chi)L(s,\chi)$.

LEMMA 1. *Let*

$$L(s,\chi) = \Psi(s,\chi)L(1-s,\bar{\chi}) . \tag{2.1}$$

Then $\Psi(s,\chi)$ is holomorphic in the region $\sigma < 1$. If $A \leqq \sigma \leqq \frac{1}{2}$, then

$$\Psi(s,\chi) \ll_A (q(|t|+1))^{\frac{1}{2}-\sigma} ,$$

whether χ is primitive or not.

The proof of lemma 1 is well-known.

LEMMA 2. *If $X>1$, $0<y<\frac{1}{4}$, $\sigma \geqq \frac{1}{2}-y$, $h>2\sigma$ and $\chi \neq \chi_0$, then*

$$M(s,\chi)L(s,\chi) = \exp(-X^{-h}) + S(s,X,\chi) - I_0(s,X,\chi) - I_1(s,X,\chi) ,$$

where

$$S(s,X,\chi) = \sum_{n>z_1} \chi(n) a_n \exp(-(n/X)^h) n^{-s} , \tag{2.2}$$

$$I_j(s,X,\chi) = \frac{1}{2\pi i} \int_{C_j} \Psi(s+w,\chi) M(s+w,\chi) \times \tag{2.3}$$

$$\times (jL(1-s-w,\bar{\chi}) + (1-2j)F(1-s-w,\bar{\chi}))\Gamma\left(1+\frac{w}{h}\right) X^w w^{-1} dw; \quad j=0,1 ,$$

and the paths C_j are defined as follows:

$$\begin{aligned} C_0 &= \{w : |\mathrm{Im}\, w| < y,\ \mathrm{Re}\, w = \min(-y, \tfrac{1}{2} - \sigma)\} \\ &\quad \cup \{w : |\mathrm{Im}\, w| = y,\ \min(-y, \tfrac{1}{2} - \sigma) \leqq \mathrm{Re}\, w \leqq \tfrac{1}{2} - \sigma\} \\ &\quad \cup \{w : |\mathrm{Im}\, w| > y,\ \mathrm{Re}\, w = \tfrac{1}{2} - \sigma\}, \\ C_1 &= \{w : \mathrm{Re}\, w = -h/2\}. \end{aligned}$$

PROOF. By Mellin's transformation, we have

$$\sum_{n=1}^{\infty} \chi(n) a_n \exp(-(n/X)^h) n^{-s}$$
$$= \frac{1}{2\pi i} \int_{\mathrm{Re}\, w = 2} M(s+w, \chi) L(s+w, \chi) \Gamma\left(1 + \frac{w}{h}\right) X^w w^{-1}\, dw.$$

We shift the line of integration to the line C_1. The pole at $w = 0$ gives then the term $M(s, \chi) L(s, \chi)$. By (2.1)

$$L(s+w, \chi) = \Psi(s+w, \chi) \sum_{j=0}^{1} (jL(1-s-w, \bar{\chi}) + (1-2j) F(1-s-w, \bar{\chi})).$$

Thus the integral over C_1 equals $I_0(s, X, \chi) + I_1(s, X, \chi)$, since the integrand of $I_0(s, X, \chi)$ has no singularities between C_0 and C_1.

3. A lemma related to Selberg's sieve.

Let $\sigma_a(n)$ denote the sum of the ath powers of the divisors of n. It is easy to see that if $z > 1$, $a \geqq 1$, $b \gg 1$, and $c \ll 1$, then

$$\sum_{m < z} m^{-a} \sigma_{-b}^{c}(m) \ll \log z. \tag{3.1}$$

The following lemma is a modification of a lemma of Motohashi [5].

LEMMA 3. *For* $z > 1$, *let*

$$\begin{cases} 1 \ll (\omega - 1) \log z \ll 1, \\ 1 \ll (\omega' - 1) \log z \ll 1 \end{cases} \tag{3.2}$$

and define

$$D_r = \sum_{n=1}^{\infty} \Bigg(\sum_{\substack{d \mid n \\ d < z}} \mu(d^r (n/d)^{1-r}) d^{\omega - \omega'} \log^k (z/d) \Bigg)^2 n^{-\omega}.$$

Then

$$D_r \ll_k (\log z)^{2k},$$

for any positive integer k *and for* $r = 0$ *or* $r = 1$.

PROOF. We have

$$D_r = \sum_{d_1, d_2 < z} [d_1, d_2]^{-\omega} \left(\prod_{j=1}^{2} d_j^{\omega - \omega'} \mu(d_j (d_1, d_2)^{r-1}) \log^k (z/d_j) \right) \times$$
$$\times \sum_{\substack{n=1 \\ (n^{1-r}, d_1 d_2 (d_1, d_2)^{-2}) = 1}}^{\infty} \mu^2(n^{1-r}) n^{-\omega} .$$

Here the series over n equals

$$\zeta(\omega) \left(\zeta(2\omega) \prod_{j=1}^{2} \sigma_{-\omega}(d_j/(d_1, d_2)) \right)^{r-1} ,$$

since we may assume that $d_j/(d_1, d_2)$ is square-free. Writing $(d_1, d_2) = d$, $d_1 = de_1$, $d_2 = de_2$ and summing first with respect to e_1 and e_2, we have further

$$D_r = \zeta(\omega)(\zeta(2\omega))^{r-1} E_r ,$$

where

$$E_r = \sum_{d<z} \mu^2(d^r) d^{\omega - 2\omega'} \sum_{\substack{e_1, e_2 < z/d \\ (e_1, e_2) = 1 \\ (e_1, d^r) = (e_2, d^r) = 1}} \prod_{j=1}^{2} F_{e_j, r}(z/d)$$

and

$$F_{u,r}(x) = u^{-\omega'} \mu(u) \log^k (x/u) \sigma_{-\omega}^{r-1}(u) .$$

The inner double sum in the expression for E_r is rewritten as

$$\sum_{\substack{a < z/d \\ (a, d^r) = 1}} \mu(a) \left(\sum_{\substack{u < z/ad \\ (u, d^r) = 1}} F_{au, r}(z/d) \right)^2 .$$

Hence, writing $ad = m$, we have

$$E_r = \sum_{m<z} m^{\omega - 2\omega'} \mu^2(m^r) \sum_{a|m} \mu(a) a^{-\omega} \sigma_{-\omega}^{2r-2}(a) (R_{a^{1-r}m^r, r}(z/m))^2 ,$$

where

$$R_{v,r}(x) = \sum_{\substack{u<x \\ (u,v)=1}} F_{u,r}(x) .$$

Let $\chi_0^{(v)}$ be the principal character mod v. The generating function of the arithmetic function $u^{-\omega'} \mu(u) \sigma_{-\omega}^{r-1}(u) \chi_0^{(v)}(u)$ is $P_{v,r}(s + \omega')$, where

$$P_{v,r}(s) = (\zeta(s))^{-1}(G(s))^{1-r}H_{v,r}(s) ,$$

$$G(s) = \prod_p (1+((p^s-1)(p^\omega+1))^{-1}) ,$$

$$H_{v,r}(s) = \prod_{p|v} (1-p^{-s}(1+p^{-\omega})^{r-1})^{-1} .$$

Indeed,

$$\sum_{u=1}^{\infty} u^{-s}\mu(u)\sigma^{r-1}_{-\omega}(u)\chi_0^{(v)}(u) = K_r(s)H_{v,r}(s) ,$$

where

$$K_r(s) = \prod_p (1-p^{-s}(1+p^{-\omega})^{r-1}) ,$$

whence

$$\begin{aligned} K_0(s) &= \prod_p (1-p^{-s})\left(1+\frac{p^{-s}(1-(1+p^{-\omega})^{-1})}{1-p^{-s}}\right) \\ &= (\zeta(s))^{-1}G(s) \end{aligned}$$

and

$$K_1(s) = (\zeta(s))^{-1} .$$

Hence we have the representation of $R_{v,r}(x)$ as an integral

$$R_{v,r}(x) = \frac{k!}{2\pi i}\int_{2-i\infty}^{2+i\infty} P_{v,r}(s+\omega')x^s s^{-k-1}\,ds .$$

We move the integration to the path C defined by

$$C = \{s=\sigma+it : \sigma=1-\omega'-\Delta/\log(|t|+2)\} ,$$

where Δ is a small positive constant. This shows that

$$R_{v,r}(x) = k!\,\mathrm{Res}\,(P_{v,r}(s+\omega')x^s s^{-k-1})_{s=0}+O_k(\sigma_{-\frac{1}{2}}(v)) ,$$

since, for $s \in C$, we have

$$\begin{aligned} \zeta(s+\omega')^{-1} &\ll \log(|t|+2) , \\ G(s+\omega') &\ll 1 , \\ H_{v,r}(s+\omega') &\ll \sigma_{-\frac{1}{2}}(v) . \end{aligned}$$

The first of these three estimates and the fact that the origin is the only singularity of the integrand on the right hand side of C are well-known classical results.

To calculate the residue, we note that near $s=0$

$$\zeta(s+\omega')^{-1} = O(\omega'-1)+\sum_{j=1}^{\infty} O_j(1)s^j ,$$

$$G(s+\omega') = \sum_{j=0}^{\infty} O_j(1)s^j ,$$

$$H_{v,r}(s+\omega') = O(\sigma_{-\frac{1}{2}}(v)) \sum_{j=0}^{\infty} O_j(1)s^j , \tag{3.3}$$

$$x^s = \sum_{j=0}^{\infty} O_j(\log^j x)s^j .$$

We prove (3.3). Since $H_{v,r}(s+\omega')$ is holomorphic at the origin, it has a Maclaurin expansion, the coefficient of the jth term of which is

$$\begin{aligned}\frac{1}{2\pi i}\int_{|s|=\frac{1}{4}} H_{v,r}(s+\omega')s^{-j-1}\,ds &\ll_j \prod_{p|v} (1-p^{\frac{1}{4}-\omega'})^{-1} \\ &\ll \prod_{p|v} (1+p^{-\frac{1}{2}}) \ll \sigma_{-\frac{1}{2}}(v) .\end{aligned}$$

This yields (3.3).

Multiplying the above power series, we obtain

$$P_{v,r}(s+\omega')x^s = \sigma_{-\frac{1}{2}}(v)\left(O(\omega'-1)+\sum_{j=1}^{\infty} O_j(\log^{j-1} x)(1+(\omega'-1)\log x)s^j\right) .$$

Thus, we have

$$R_{v,r}(x) \ll_k \sigma_{-\frac{1}{2}}(v)(\log^{k-1} x)(1+(\omega'-1)\log x) .$$

Substituting this in the expression for E_r and using (3.1) and (3.2) we see that

$$\begin{aligned}E_r &\ll_k \sum_{m<z} m^{\omega-2\omega'} \sum_{a|m} a^{-\omega}(\sigma_{-\frac{1}{2}}(m)\log^{k-1} z)^2 \\ &\ll_k \log^{2k-2} z \sum_{m<z} m^{\omega-2\omega'}\sigma^3_{-\frac{1}{2}}(m) \\ &\ll \log^{2k-1} z .\end{aligned}$$

To complete the proof, we note finally that

$$\zeta(\omega) \ll (\omega-1)^{-1} \ll \log z .$$

The next two lemmas are corollaries of lemma 3.

LEMMA 4. *If a_n is defined by* (1.8), $\sigma \geqq \frac{1}{2} - y$ *and* $y \ll (\log z_1)^{-1}$, *then*

$$\sum_{z_1 < n \leqq M} a_n^2 n^{-2\sigma} \ll M^{O(1/\log z_1)} z_1^{1-2\sigma} (\log (z_2/z_1))^{-2} (\log z_2)^2 .$$

Graham [1] has given an asymptotic formula for $\sum_{1 \leqq n \leqq M} a_n^2$, for $M > z_1$.

PROOF OF LEMMA 4. Writing

$$L_d(z) = \begin{cases} \mu(d) \log (z/d), & \text{for } d \leqq z , \\ 0, & \text{for } d > z , \end{cases}$$

we have

$$a_n = (\log (z_2/z_1))^{-1} \sum_{d|n} (L_d(z_2) - L_d(z_1)) .$$

We note also that, for $z_1 < n \leqq M$,

$$n^{-2\sigma} \ll n^{-1-1/\log z_1} M^{O(1/\log z_1)} z_1^{1-2\sigma} .$$

Hence it remains to prove that, for $i = 1, 2$,

$$\sum_{z_1 < n \leqq M} \Big(\sum_{d|n} L_d(z_i) \Big)^2 n^{-1-1/\log z_1} \ll (\log z_2)^2 .$$

This follows from lemma 3 with $k = r = 1$, $\omega = \omega'$.

LEMMA 5. *If $b_n(x)$ is defined by* (1.9), $x \ll (\log z_1)^{-1}$ *and*

$$\begin{cases} \log z_2 \ll \log v_1 , \\ \log v_2 \ll \log z_1 , \end{cases} \tag{3.4}$$

then

$$\sum_{n \leqq v_2 z_2} b_n^2(x) n^{-1} \ll (\log (z_2/z_1) \log (v_2/v_1))^{-2} (\log z_2)^4 .$$

PROOF. Let

$$K_d(z) = \begin{cases} \log (z/d), & \text{for } d \leqq z , \\ 0, & \text{for } d > z , \end{cases}$$

$$L_d(z) = \mu(d) K_d(z) .$$

Then

$$b_n(x) = (\log(z_2/z_1)\log(v_2/v_1))^{-1} \times$$
$$\times \sum_{d|n} (L_d(z_2)-L_d(z_1))(K_{n/d}(v_2)-K_{n/d}(v_1))\, d^x .$$

Thus, we have to show that

$$\sum_{n \leqq v_i z_j} \left(\sum_{d|n} L_d(z_j) K_{n/d}(v_i) d^x \right)^2 n^{-1} \ll (\log z_2)^4 , \tag{3.5}$$

for $i, j = 1, 2$.

For $n \leqq vz$, we have

$$\begin{aligned}
\sum_{d|n} L_d(z) K_{n|d}(v) d^x &= \sum_{\substack{d|n \\ n/v < d < z}} \mu(d) \log(z/d) \log(vd/n) d^x \\
&= \sum_{\substack{d|n \\ d<z}} \mu(d) \log(z/d) \log(vd/n) d^x \\
&\quad - \sum_{d|n} \mu(d) \log(z/d) \log(vd/n) d^x + \\
&\quad + \sum_{\substack{d|n \\ d<v}} \mu(n/d) \log(zd/n) \log(v/d)(n/d)^x \\
&= \sum_{r=1}^{7} c_r(n) ,
\end{aligned}$$

$$c_1(n) = \sum_{\substack{d|n \\ d<z}} \mu(d) \log(z/d) \log(vz/n) d^x ,$$

$$c_2(n) = -\sum_{\substack{d|n \\ d<z}} \mu(d) \log^2(z/d) d^x ,$$

$$c_3(n) = -\sum_{d|n} \mu(d) \log(vz/d) \log(v^3 z/n) d^x ,$$

$$c_4(n) = \sum_{d|n} \mu(d) \log^2(vz/d) d^x,$$

$$c_5(n) = \sum_{d|n} \mu(d) \log v \log(v^2 z/n) d^x ,$$

$$c_6(n) = n^x \sum_{\substack{d|n \\ d<v}} \mu(n/d) \log(v/d) \log(vz/n) d^{-x},$$

$$c_7(n) = -n^x \sum_{\substack{d|n \\ d<v}} \mu(n/d) \log^2(v/d) d^{-x}.$$

Hence

$$\sum_{n\leqq vz}\left(\sum_{d|n}L_d(z)K_{n/d}(v)d^x\right)^2 n^{-1} \ll \sum_{r=1}^{7} S_r\,,$$

where

$$S_r = \sum_{n\leqq vz} c_r^2(n)n^{-1}\,.$$

Consider the sum S_5. For simplicity, suppose that $x>0$. Denote

$$f_n(x) = n^{-\frac{1}{2}}\sum_{d|n}\mu(d)(d/vz)^x\,.$$

By Schwarz's inequality, we have

$$f_n^2(x) = \left(\int_0^x f_n'(y)\,dy\right)^2 \leqq x\int_0^x (f_n'(y))^2\,dy\,,$$

for $n>1$. Hence, there exists a number ξ such that $0<\xi<x$ and

$$\sum_{1<n\leqq vz} f_n^2(x) \leqq x^2 \sum_{n\leqq vz}(f_n'(\xi))^2\,.$$

Therefore,

$$S_5 \leqq (\log v \log(v^2z))^2(1+(vz)^{2x}x^2S_5')\,,$$

where

$$S_5' = \sum_{n\leqq vz}\left(\sum_{d|n}\mu(d)\log(d/vz)d^{\xi}\right)^2 n^{-1}\,.$$

Estimating the sums S_r, $r\neq 5$, and S_5' by Lemma 3, we obtain (3.5). Note that the trivial restriction $d<vz$ may be imposed on those sums, where d runs through all the divisors of n.

4. Mean value estimates.

In order to be able to treat simultaneously two cases we define the symbol $m_k=m_{k,t_1,t_2}$, for $t_1<t_2$, as follows:

$$f(s) = m_0(f(s)),\qquad \text{for any } t\in[t_1,t_2]\,,$$

$$\int_{t_1}^{t_2} f(s)\,dt = m_1(f(s))\,.$$

The following general lemma is proved in chapter 6 of [4].

LEMMA 6. *Let*

$$K(s,\chi) = \sum_{n=M+1}^{M+N} c_n\chi(n)n^{-s},$$

where χ is a character $\bmod q$. *Then, for $k=0,1$, we have*

$$\sum_{\chi \bmod q} m_k(|K(it,\chi)|^2) \ll (q(t_2-t_1)^k+N)\sum_{n=M+1}^{M+N}|c_n|^2 .$$

LEMMA 7. *If $S(s,X,\chi)$ is defined by* (2.2), $y \ll 1/\log z_1$, $\sigma \geqq \frac{1}{2}-y$, $X>1$, $h>1$, *and* $z_1 \leqq 2X$, *then*

$$\sum_{\chi \bmod q} m_k(|S(s,X,\chi)|^2) \tag{4.1}$$
$$\ll z_1^{1-2\sigma}(q(t_2-t_1)^k+X)X^{O(1/\log z_1)}(\log(z_2/z_2))^{-2}(\log z_2)^2 ,$$

for $k=0,1$.

PROOF. Denote by S_r the part of the sum $S(s,X,\chi)$ corresponding to the indices $2^rX<n\leqq 2^{r+1}X$, for $r\geqq 1$, and by S_0 the part corresponding to the indices $z_1<n\leqq 2X$. Thus,

$$S(s,X,\chi) = \sum_{r=0}^{\infty} S_r .$$

By Schwarz's inequality the left hand side of (4.1) is

$$\ll \sum_{r=0}^{\infty}(r+1)^2\sum_{\chi} m_k(|S_r|^2) .$$

Here the inner sum is, by Lemma 6,

$$\ll (q(t_2-t_1)^k+2^rX)\sum_{z_1<n\leqq 2^{r+1}X} a_n^2 n^{-2\sigma}\exp(-2^{rh}) .$$

Hence, using Lemma 4 and the fact that $h>1$, we obtain the desired result.

LEMMA 8. *If $I_0(s,X,\chi)$ is defined by* (2.3), $y\ll 1/\log z_1$, $\sigma>\frac{1}{2}-y$, $h>2\sigma$, $1<X_1 < X_2$ *and* (3.4) *is valid, then*

$$\sum_{\chi \bmod q} m_k\left(\left|(\log(X_2/X_1))^{-1}\int_{X_1}^{X_2} X^{-1}I_0(s,X,\chi)\,dX\right|^2\right) \tag{4.2}$$
$$\ll X_1^{1-2\sigma}(q(t_2-t_1)^k+v_2z_2)(y\log(X_2/X_1)\log(z_2/z_1)\log(v_2/v_1))^{-2}\times$$
$$\times\log z_2)^4(q(\max(|t_1|,|t_2|)+1))^{4y}X_2^{2y} ,$$

for $k=0,1$.

PROOF. We have

$$\int_{X_1}^{X_2} X^{-1} I_0(s, X, \chi)\, dX = I_0^*(s, X_2, \chi) - I_0^*(s, X_1, \chi), \tag{4.3}$$

where the asterisque means that w is replaced by w^2 in the denominator of the integrand. For $w \in C_0$, we have

$$\psi(s+w, \chi) \ll (q(|t|+1))^{2y}, \tag{4.4}$$

$$\Gamma(1+w/h) \ll 1. \tag{4.5}$$

The estimate (4.4) follows from Lemma 1. By (4.3)–(4.5) and Schwarz's inequality the square of the left hand side of (4.3) is

$$\ll (q(|t|+1))^{4y} X_1^{1-2\sigma} X_2^{2y} y^{-1} \int_{C_0} |M(s+w, \chi) F(1-s-w, \bar{\chi})|^2 |w^{-2}| |dw| .$$

Hence, in order to obtain (4.2), it remains to prove that, for $w \in C_0$,

$$\sum_{\chi} m_k(|M(s+w, \chi) F(1-\overline{(s+w)}, \chi)|^2) \tag{4.6}$$
$$\ll (q(t_2-t_1)^k + v_2 z_2)(\log(z_2/z_1)\log(v_2/v_1))^{-2}(\log z_2)^4 .$$

For any complex z, we have

$$M(z, \chi) F(1-\bar{z}, \chi) = \sum_{n < v_2 z_2} \chi(n) n^{\bar{z}-1} b_n(1 - 2\,\mathrm{Re}\, z),$$

where $b_n(\cdot)$ is defined by (1.9). Hence, by Lemma 6, the left hand side of (4.6) is

$$\ll (q(t_2-t_1)^k + v_2 z_2) \sum_{n < v_2 z_2} n^{2\sigma + 2\,\mathrm{Re}\, w - 2} b_n^2(1 - 2\sigma - 2\,\mathrm{Re}\, w) .$$

Since $w \in C_0$, $2\sigma + 2\,\mathrm{Re}\, w - 2 \leqq -1$ and $1 - 2\sigma - 2\,\mathrm{Re}\, w \ll y \ll 1/\log z_1$, so that an application of Lemma 5 yields (4.6).

LEMMA 9. *If* $I_1(s, X, \chi)$ *is defined by* (2.3), $0 < y < \frac{1}{4}$, $\frac{1}{2} - y \leqq \sigma \leqq 2$, $4 < h < A$ *and* $X > 1$, *then*

$$I_1(s, X, \chi) \ll_A (q(|t|+1) z_2 / X v_1)^{h/2 + \frac{1}{2} - \sigma} (v_1 z_2)^{\frac{1}{2}} X^{\frac{1}{2} - \sigma} .$$

PROOF. If $w \in C_1$, then $\mathrm{Re}\,(s+w) = \sigma - h/2$. Hence, for $w \in C_1$,

$$M(s+w, \chi) \ll z_2^{h/2 + 1 - \sigma},$$

$$L(1-s-w, \bar{\chi}) - F(1-s-w, \bar{\chi}) \ll v_1^{\sigma - h/2} .$$

We have, by Stirling's formula and Lemma 1,

$$\int_{C_1} |\Psi(s+w,\chi)\Gamma(1+w/h)X^w w^{-1}|\,|dw|$$
$$\ll X^{-h/2}\int_{-\infty}^{\infty} |\Psi(\sigma-h/2+i(t+h\omega),\chi)|e^{-\pi|\omega|/2}\,d\omega$$
$$\ll_A X^{-h/2}(q(|t|+1))^{\frac{1}{2}-\sigma+h/2}\,.$$

Combining the previous estimates completes the proof.

5. Proof of the theorem.

We state without proof the following slightly modified version of a lemma of Selberg ([8, Lemma 14]).

LEMMA 10. *Let $f(s)$ be holomorphic in the rectangle $\sigma_1 \leqq \sigma \leqq \sigma_2$, $t_1 \leqq t \leqq t_2$, where $\sigma_2 > \operatorname{Re}\varrho$ whenever ϱ is a zero of $f(s)$ such that $t_1 \leqq \operatorname{Im}\varrho \leqq t_2$. Writing $f(s) = 1-g(s)$, suppose that $|g(\sigma_2+it)| \leqq \frac{1}{2}$, for $t_1 \leqq t \leqq t_2$. Then we have*

$$(t_2-t_1)\sum_{\substack{\varrho,\, f(\varrho)=0\\ \operatorname{Re}\varrho>\sigma_1\\ t_1<\operatorname{Im}\varrho<t_2}} \sin\left(\pi\frac{\operatorname{Im}\varrho-t_1}{t_2-t_1}\right)\sinh\left(\pi\frac{\operatorname{Re}\varrho-\sigma_1}{t_2-t_1}\right)$$
$$\ll \int_{t_1}^{t_2}|g(\sigma_1+it)|\,dt+\exp\left(\pi\frac{\sigma_2-\sigma_1}{t_2-t_1}\right)\int_{t_1}^{t_2}|g(\sigma_2+it)|\,dt$$
$$+\int_{\sigma_1}^{\sigma_2}\exp\left(\pi\frac{\sigma-\sigma_1}{t_2-t_1}\right)(|g(\sigma+it_1)|+|g(\sigma+it_2)|)\,d\sigma\,.$$

We may suppose that $\alpha \leqq 1$ and $T_2-T_1 > 10^{2/\varepsilon}/\log 2q$. Let

(5.1) $$t_1 = (3T_1-T_2)/2, \quad t_2 = (3T_2-T_1)/2\,,$$

whence

(5.2) $$t_2-t_1 = 2(T_2-T_1)\,.$$

In particular,

(5.3) $$t_2-t_1 > 2\pi/\varepsilon\log 2q\,.$$

Let $T = t_2-t_1+1$, whence

(5.4) $$qT > 10^{2/\varepsilon}\,.$$

Let

(5.5) $$\alpha_0 = \alpha-1/\log qT\,,$$
$$z_1 = q^{c/2-\varepsilon/2}T^{1/2-\varepsilon/2}, \quad z_2 = z_1(qT)^{\varepsilon/6}\,.$$

In particular, by (5.4),

(5.6) $$z_1 > 10\,.$$

Recalling (1.7), let

(5.7) $$g(s,\chi) = (M(s,\chi)L(s,\chi)-1)^2\,.$$

Then, for $k=0,1$ and $\alpha_0 \leqq \sigma \leqq 2$, we have

(5.8) $$\sum_{\chi \bmod q}{}' m_k(|g(s,\chi)|)$$
$$\ll_\varepsilon q^{1-(c-\varepsilon)(\sigma-\frac{1}{2})}(T-k)T^{-(1-\varepsilon)(\sigma-\frac{1}{2})}\,.$$

To see this, we first fix the parameters that occur in lemmas 2, 7, 8, and 9 as follows:

$$v_1 = q^{1-c/2}T^{1/2}, \qquad v_2 = v_1(qT)^{\varepsilon/6}\,,$$
$$X_1 = (qT)^{1-\varepsilon/6}, \qquad X_2 = qT,$$
$$y = 1/\log qT\,,$$
$$h = 4+6/\varepsilon\,.$$

By Lemma 2 the left hand side of (5.8) is

$$\ll B_1+B_2+B_3+B_4\,,$$

where

$$B_j = \sum_\chi{}' m_k\left(\left|(\log(X_2/X_1))^{-1}\int_{X_1}^{X_2} X^{-1}A_j\,dX\right|^2\right)$$

and

$$A_1 = \exp(-X^{-h})-1\,,$$
$$A_2 = S(s,X,\chi)\,,$$
$$A_j = I_{j-3}(s,X,\chi); \quad j=3,4\,.$$

Obviously,

$$B_1 \ll q(t_2-t_1)^k(qT)^{-6+4\varepsilon/3-12/\varepsilon}\,.$$

By Schwarz's inequality,

$$B_2 \ll (\log(X_2/X_1))^{-1}\sum_\chi{}' m_k\left(\int_{X_1}^{X_2} X^{-1}|A_2|^2\,dX\right).$$

Hence, by Lemma 7,

$$B_2 \ll_\varepsilon q^{-(c-\varepsilon)(\sigma-\frac{1}{2})}T^{-(1-\varepsilon)(\sigma-\frac{1}{2})}(q(t_2-t_1)^k+qT/\log qT)\,.$$

By (1.5) and (5.1),

$$\max(|t_1|,|t_2|) \ll q^{1-c}T\,, \tag{5.9}$$

whence, by Lemma 8,

$$B_3 \ll_\varepsilon (qT)^{-(2-\varepsilon/3)(\sigma-\frac{1}{2})}(q(t_2-t_1)^k+(qT)^{1-\varepsilon/6})\,.$$

By Lemma 9,

$$B_4 \ll_\varepsilon q(t_2-t_1)^k\left(\frac{q(\max(|t_1|,|t_2|)+1)z_2}{X_1 v_1}\right)^{h+1-2\sigma} v_1 z_2 X_1^{1-2\sigma}\,,$$

whence, by (5.9),

$$B_4 \ll_\varepsilon q(t_2-t_1)^k(qT)^{-(2-\varepsilon/3)(\sigma-\frac{1}{2})-\varepsilon/2}\,.$$

Combining these estimates gives (5.8).

Recalling (5.7) we define

$$\Phi(s,\chi) = 1-g(s,\chi)\,.$$

Obviously each zero of an L-function is a zero of the corresponding Φ-function, too. Thus, since $\alpha-\alpha_0=1/\log qT$ and $\frac{1}{4}\leqq(t-t_1)/(t_2-t_1)\leqq\frac{3}{4}$, for $T_1\leqq t\leqq T_2$, we have (see (1.1), (1.3))

$$N'(q) \ll (\log qT)(t_2-t_1)\sum_{\chi}{}' \sum_{\substack{\varrho,\,\Phi(\varrho,\chi)=0\\ \operatorname{Re}\varrho>\alpha_0\\ t_1<\operatorname{Im}\varrho<t_2}} \sin\left(\pi\frac{\operatorname{Im}\varrho-t_1}{t_2-t_1}\right)\sinh\left(\pi\frac{\operatorname{Re}\varrho-\alpha_0}{t_2-t_1}\right).$$

Here we estimate the right hand side by Lemma 10 with $\sigma_1=\alpha_0$ and $\sigma_2=2$. Note that by (5.6), $|g(s,\chi)|\leqq(\sum_{n>z_1}\tau(n)n^{-2})^2\leqq\frac{1}{2}$. Hence, we find

$$N'(q) \ll (U+V)\log qT\,,$$

where

$$U = \sum_{\chi}{}'\int_{t_1}^{t_2}|g(\alpha_0+it,\chi)|\,dt+q^{\varepsilon}\sum_{\chi}{}'\int_{t_1}^{t_2}|g(2+it,\chi)|\,dt$$

and

$$V = \sum_{j=1}^{2}\sum_{\chi}{}'\int_{\alpha_0}^{2}q^{\varepsilon(\sigma-\alpha_0)/2}|g(\sigma+it_j,\chi)|\,d\sigma\,.$$

Here we made use of (5.3).

Now we shall estimate U and V by (5.8). Writing $A(c,\sigma)=(c-\varepsilon)(\sigma-\frac{1}{2})$, we have, since $c\geqq 2\varepsilon$,

$$U \ll_\varepsilon q^{1-A(c,\alpha_0)}(t_2-t_1)T^{-A(1,\alpha_0)}$$

and

$$V \ll_\varepsilon \int_{\alpha_0}^{2} q^{\varepsilon(\sigma-\alpha_0)/2+1-A(c,\sigma)}T^{1-A(1,\sigma)}\,d\sigma$$
$$\ll_\varepsilon q^{1-A(c,\alpha_0)}T^{1-A(1,\alpha_0)}/\log qT.$$

Now the theorem has been proved in the case that χ_0 is omitted in the summation, for by (5.5) and (5.2), α_0 can be replaced by α and the numbers t_i by the T_i's.

Finally we note that $N_{\chi_0}(\alpha;\, T_1, T_2)$ is also absorbed in the right hand side of (1.6). In case $T_2-T_1>4$ this follows from (1.4) and (1.5). In case $4\geqq T_2-T_1 \gg_\varepsilon 1/\log 2q$ it follows from (1.5) and the well-known estimate

$$N_{\chi_0}(\tfrac{1}{2};\, Y, Y+1) \ll \log(|Y|+2).$$

Thus, the proof is complete.

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