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SOME LEMMAS ABOUT DYNAMICAL SYSTEMS

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0. Introduction.

The following is an attempt to show how the following theorem (Ekeland [2]) can be used in the theory of dynamical systems.

THEOREM 0. *Let X be a complete metric space, $F\colon X \to \mathbf{R} \cup \{+\infty\}$ a lower semicontinuous function, bounded from below. Let there be given some $\varepsilon > 0$ and some point $x_0 \in X$ with $F(x_0) \leqq \operatorname{Inf} F + \varepsilon$. Then, for every $\alpha > 0$, some point $\bar{x} \in X$ can be found such that*

$$F(\bar{x}) \leqq F(x_0)$$

$$d(\bar{x}, x) \leqq \alpha$$

$$\forall x \in x, \quad F(x) \geqq F(\bar{x}) - \frac{\varepsilon}{\alpha}\|x - \bar{x}\| .$$

One would take $\alpha = \sqrt{\varepsilon}$ for instance.

We refer to [2] and [3] for details.

Brezis and Browder ([1]; see also [3]) have used a related result to study the existence of a flow on a closed subset and prove global estimates. Here we use theorem 0 directly, and we prove two classical results on the existence of closed orbits.

Section 1 provides a relatively new proof of the shadowing lemma.

Section 2 proves that if some neighbourhood of $\Omega(f)$ is hyperbolic, then periodic solutions are dense in $\Omega(f)$. The argument does not rely on compactness, and the result may be new in that generality.

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I. Shadowing lemma.

We begin by providing a new proof of Theorem 3.5 p. 29 in Newhouse's lectures. Notations and definitions as in Newhouse [4].

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Recall that a hyperbolic set Λ has a local product structure if there is an $\varepsilon>0$ such that $W_\varepsilon^s(x)\cap W_\varepsilon^u(y)\subset\Lambda$ for all x and y both in Λ. Here

$$W_\varepsilon^s(x) = \{x \in M \mid d(f^n(x), f^n(x'))\leqq\varepsilon, \quad \text{all } n\geqq 0\}$$
$$W_\varepsilon^u(y) = \{y' \in M \mid d(f^n(y), f^n(y'))\leqq\varepsilon, \quad \text{all } n\leqq 0\} .$$

A sequence $\{x_n\}$, $n \in \mathbf{Z}$, is a δ-pseudo-orbit if $d(x_n, f(x_{n-1}))\leqq\delta$ for all n; it is ε-shadowed by an orbit $\{\bar{x}_n\}$, $n \in \mathbf{Z}$, if $f(\bar{x}_n, x_n)\leqq\varepsilon$ for all n.

THEOREM 1. *Suppose M is a manifold, and Λ is a compact hyperbolic set for $f\colon M \to M$ with a local product structure. For every $\varepsilon>0$ there is a $\delta>0$ such that every δ-pseudo-orbit in Λ can be ε-shadowed by an orbit in Λ.*

PROOF. As pointed out in Newhouse, it is enough to shadow f^N pseudo-orbits, for some $N>1$. The hyperbolicity constant for f^N will be λ^N.

Let $S=\Lambda^{\mathbf{Z}}$ be the space of all sequences in Λ, with the l^∞ metric

$$d(\{x_n\},\{y_n\}) = \operatorname{Sup}\{d(x_n, y_n) \mid n \in \mathbf{Z}\} .$$

It is a complete metric space. We define a function $F\colon S \to \mathbf{R}$ as follows

$$F\{x_n\} = \operatorname{Sup}\{d(x_n, f(x_{n-1})) \mid n \in \mathbf{Z}\} .$$

Let $\{y_n\}$ be a δ-pseudo-orbit. We have

$$F(\{y_n\}) \leqq \delta .$$

Let $\varepsilon>0$ be given. By Theorem 0, there will be some point $\{\bar{x}_n\} \in S$ such that

$$(\alpha) \qquad F(\{\bar{x}_n\}) \leqq \delta ,$$

$$(\beta) \qquad d(\{\bar{x}_n\},\{y_n\}) \leqq \varepsilon ,$$

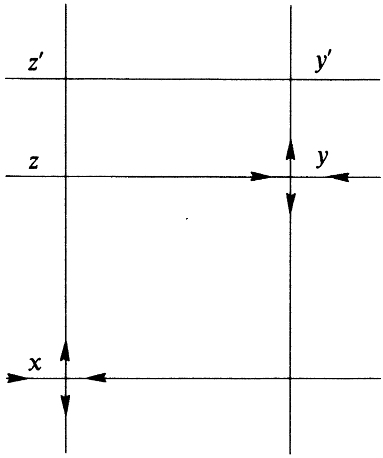
$$(\gamma) \qquad F(\{x_n\}) \geqq F\{\bar{x}_n\})-\delta\varepsilon^{-1}d(\{x_n\},\{\bar{x}_n\}), \quad \text{all } \{x_n) \in S .$$

Since $\{\bar{x}_n\}$ belongs to S, $\bar{x}_n$ belongs to Λ for every n. By condition (β), we have $d(\bar{x}_n, y_n)\leqq\varepsilon$ for every n. All that remains to show is that $\bar{x}_n$ is an orbit, i.e. $F\{\bar{x}_n\}=0$.

Because of the compactness of M and the hyperbolic structure of Λ, for any constants $c>1$ and $\eta>0$, some $\delta_0>0$ can be found such that, whenever $d(x,y)\leqq\delta_0$ with x and y in Λ, we have the situation in the next picture.

There is a single point z such that

$$z \in W_\eta^u(x) \cap W_\eta^s(y)$$
$$d(x,z) \leqq cd(x,y)$$
$$d(y,z) \leqq cd(x,y) .$$



For any $y' \in W^u(y)$ with $d(y,y') \leqq \delta_0/5$, there is a single point z' such that

$$z' \in W^s_\eta(y') \cap W^u_\eta(x)$$
$$d(z',z) \leqq cd(y',y)$$
$$d(z',y') \leqq cd(z,y)\,.$$

Assume $\delta \leqq \delta_0$. For each $n \in \mathbf{Z}$, set $x = f^N(\bar{x}_{n-1})$ and $y = \bar{x}_n$; then define $z_n = z$. Set $y' = f^{-N}(z_{n+1})$; since $z_{n+1} \in W^u_\eta(f^N(\bar{x}_n))$, we have $y' \in W^u_\eta(y)$ and

$$d(y,y') = d(\bar{x}_n, f^N(z_{n+1})) \leqq \lambda^{-N}\eta\,.$$

Assume N is so large that $\lambda^{-N}\eta \leqq \delta_0/5$, so that we can define $z'_n = z'$ as above.

We know $\bar{x}_n$ belongs to Λ for every n. Then so does $f^N(\bar{x}_{n-1})$, since Λ is f^N-invariant, so does z_n by the local product structure, so does $f^{-N}(z_{n+1})$ since Λ is f^N-invariant, and finally so does z'_n by the local product structure.

For each $n \in \mathbf{Z}$, we have:

$$\begin{aligned} d(z'_n, \bar{x}_n) &\leqq d(z'_n, f^{-N}(z_{n+1})) + d(f^{-N}(z_{n+1}), \bar{x}_n) \\ &\leqq cd(z_n, \bar{x}_n) + \lambda^{-N}\eta \\ &\leqq c\delta + \lambda^{-N}\eta \end{aligned}$$
$$d(z'_{n+1}, f^N(z'_n)) \leqq \lambda^{-N}\eta \quad \text{since } z'_n \in W^s_\eta(f^{-N}(z'_{n+1}))\,.$$

The sequence $\{z'_n\}$ belongs to S, and we can test condition (γ), which is supposed to hold for all $\{x_n\}$ in S, at $\{z'_n\}$. We get

$$\lambda^{-N}\eta \geqq F(\{\bar{x}_n\}) - \delta\varepsilon^{-1}(c\delta + \lambda^{-N}\eta)$$

$$F(\{\bar{x}_n\}) \leqq \delta(\varepsilon^{-1}c\delta + \lambda^{-N}\eta\varepsilon^{-1}) + \lambda^{-N}\eta = D(\delta).$$

Once the constants $\varepsilon > 0$, $c > 1$, and $\eta > 0$ have been chosen, we can associate with them $\delta_0 > 0$ and N such that whenever $\delta \leqq \delta_0$, we can find a $D(\delta)$-pseudo-orbit $\bar{x}_n$, $n \in \mathbf{Z}$, in Λ ε-shadowing $\{y_n\}$. Letting $\delta \to 0$, we have $D(\delta) \to 0$, and taking cluster points of the $\bar{x}_n$, we get an actual orbit in Λ ε-shadowing $\{y_n\}$.

II. Periodic orbits are dense in $\Omega(f)$.

Now M is a complete Riemannian manifold, possibly infinite-dimensional, and f: $M \to M$ a C^1 map. We do not require it to be invertible. We endow M with the geodesic distance d.

THEOREM 2. *Assume $z \in M$ is non-wandering. Then, for all $\varepsilon > 0$, there is some $x \in M$ and $n \geqq 1$ such that:*

(1) $$d(x, z) \leqq 2\varepsilon$$

(2) $$d(f^n(x), z) \leqq 2\varepsilon$$

(3) $$\|(I - D_\xi f^n)^*(\eta - \xi)\| \leqq \varepsilon\|\eta - \xi\|,$$

where η and ξ in T_xM, and $D_\xi \in \mathscr{L}(T_xM)$, are defined by:

$$\exp \xi = x$$

$$\exp \eta = f^n(x)$$

$$D_\xi f^n = (T_\eta \exp)^{-1}(Tf^n)(T_\xi \exp)$$

and exp: $T_zM \to M$ *is the exponential map at z.*

PROOF. Let $\varepsilon > 0$ be given. Since z is non-wandering, there is some $y \in M$ and $n \geqq 1$ such that

$$d(y, z) \leqq \varepsilon^2 \quad \text{and} \quad d(f^n(y), z) \leqq \varepsilon^2.$$

Define a function F: $M \to \mathbf{R}_+$ by

$$F(x) = d(x, f^n(x)).$$

By Theorem 0, there will be some point x such that

$$F(x) \leqq F(y) \leqq \varepsilon^2$$

$$d(x, y) \leqq \varepsilon$$

$$F(x') \geqq F(x) - \varepsilon d(x', x), \quad \text{all } x' \in M.$$

The two first conditions obviously imply (1) and (2)

$$d(x,z) \leqq d(x,y)+d(y,z) \leqq \varepsilon+\varepsilon^2$$

$$d(f^n(x),z) \leqq F(x)+d(x,z) \leqq \varepsilon+2\varepsilon^2 .$$

We now use the exponential map at z as a local chart around z. We replace x by $\exp^{-1} x=\xi$, $f^n(x)$ by $\exp^{-1} f^n(x)=\eta$, and f^n by $\exp^{-1}\circ f^n\circ\exp=\varrho$. The Taylor expansion for ϱ near ξ now is

$$\varrho(\xi+\zeta) = \eta+D_\xi f^n\zeta+o(\zeta) ,$$

where $o(\zeta)$ denotes second-order terms.

Rewrite the third condition with $x'=\exp(\xi+\zeta)$. We get

$$d(\exp(\xi+\zeta),\exp\varrho(\xi+\zeta)) \geqq d(\exp\xi,\exp\varrho(\xi))-\varepsilon d(\exp\xi,\exp(\xi+\zeta)) .$$

Writing in first-order expansions, this becomes

$$\|\xi+\zeta-\eta-D_\xi f^n\zeta\| \geqq \|\xi-\eta\|-\varepsilon\|\zeta\|+o(\zeta) .$$

If $\xi=\eta$, condition (3) holds trivially. If $\xi\neq\eta$, the right-hand side is positive for small enough ζ, and we can square both sides. We get

$$2(\zeta-D_\xi f^n\zeta,\xi-\eta) \geqq -2\varepsilon\|\zeta\|\,\|\xi-\eta\|+o(\zeta)$$

$$((I-D_\xi f^n)^*(\eta-\xi),\zeta) \leqq \varepsilon\|\zeta\|\,\|\eta-\xi\|+o(\zeta) .$$

Letting $\zeta \to 0$, we get the desired result.

If $\xi=\eta$, x is a periodic point for f. If $\xi\neq\eta$, condition (3) means that the geodesics from x to $f^n(x)$ is almost an eigenray of $(I-D_x f^n)^*$ associated with the eigenvalue one.

We shall now assume that f is hyperbolic in a neighbourhood of z. This means that, for all x near z, there is a splitting of T_xM into $E_x^s\oplus E_x^u$, depending continuously on y, and f-invariant, such that

$$\forall\xi\in E_x^s,\quad \|(T_xf^n)\xi\| \leqq \lambda^{-n}\|\xi\|,\quad \text{all } n\geqq 1 ,$$

$$\forall\zeta\in E_x^u,\quad \|(T_xf^n)\zeta\| \geqq \lambda^{-n}\|\zeta\|,\quad \text{all } n\geqq 1 ,$$

for some constant $\lambda>1$. No generality is lost in assuming E_x^s and E_x^u to be orthogonal (change the Riemannian structure accordingly).

COROLLARY. *Assume f is hyperbolic in a neighbourhood of z. Then there is a sequence of periodic points x_p converging to z.*

PROOF. Setting $\varepsilon=p^{-1}$, we get sequences $x_p \to z$ and $n_p\geqq 1$ such that

$$d(x_p, f^{n_p}(x_p)) \to 0$$

$$(*) \qquad \|(I - D_{x_p} f^{n_p})^*(\eta_p - \xi_p)\| \leqq \|\eta_p - \xi_p\| p^{-1}$$

with $x_p = \exp \xi_p$ and $f^{n_p}(x_p) = \exp \eta_p$.

If the sequence n_p is bounded, the first condition will imply that z itself is a periodic point. If $\eta_p = \xi_p$ for an infinite number of p, the corresponding x_p are periodic, and the result is proved.

Assume then $n_p \to \infty$ and $\eta_p \neq \xi_p$ for all p. Define $w_p = (\eta_p - \xi_p)\|\eta_p - \xi_p\|^{-1}$, so that w_p is unitary. Set $y_p = f^{n_p}(x_p)$, and define

$$E_p^s = (T\exp_\xi)^{-1} E_{x_p}^s \quad \text{and} \quad E_p^u = (T\exp_\xi^{-1}) E_{x_p}^u$$

$$F_p^s = (T\exp_\eta)^{-1} E_{y_p}^s \quad \text{and} \quad F_p^u = (T\exp_\eta^{-1}) E_{y_p}^u .$$

Since the splitting $E_x^s \oplus E_x^u$ of $T_x M$ arising from the hyperbolic structure depends continuously on x, the subspaces E_p^s and F_p^s converge to the subspace $(T\exp_0)^{-1} E_z^s$ when $p \to \infty$. Similarly E_p^u and F_p^u have a common limit.

Analysing the hyperbolic structure in the local chart provided by exp, we see that $D_{x_p} f^{n_p}$ sends E_p^s onto F_p^s and E_p^u onto F_p^u. Moreover, for some constant c, with $c \geqq 1$, we have

$$\zeta \in E_p^u \Rightarrow D_{x_p} f^{n_p} \zeta \in F_p^u \quad \text{and} \quad \|D_{x_p} f^{n_p} \zeta\| \geqq c^{-1} \lambda^{n_p} \|\zeta\|$$

$$\zeta \in E_p^s \Rightarrow D_{x_p} f^{n_p} \zeta \in F_p^s \quad \text{and} \quad \|D_{x_p} f^{n_p} \zeta\| \leqq c \lambda^{-n_p} \|\zeta\| .$$

Now $(D_x f^{n_p})^*$ is going to have the same properties as $(D_x f^{n_p})$, with E_p^u, E_p^s, F_p^u, and F_p^s replaced by isomorphic subspaces $(E_p^u)^*$, $(E_p^s)^*$, $(F_p^u)^*$, and $(F_p^s)^*$. If now we split w_p into $w_p^u + w_p^s$, with $w_p^u \in (E_p^u)^*$ and $w_p^s \in (E_p^s)^*$, we have

$$v_p^u = (D_x f^{n_p})^* w_p^u \in (F_p^u)^* \quad \text{and} \quad \|v_p^u\| \geqq c^{-1} \lambda^{n_p} \|w_p^u\|$$

$$v_p^s = (D_x f^{n_p})^* w_p^s \in (F_p^s)^* \quad \text{and} \quad \|v_p^s\| \leqq c \lambda^{-n_p} \|w_p^s\| \leqq c \lambda^{-n_p} .$$

But remember condition (*), we have

$$\|v^p - w^p\| \leqq p^{-1} .$$

Hence

$$\|v_p^u\| \leqq \|v_p^s\| + \|v_p - w_p\| + \|w_p\|$$

$$c^{-1} \lambda^{n_p} \|w_p^u\| \leqq c \lambda^{-n_p} + p^{-1} + 1 .$$

It follows that $\|w_p^u\| \to 0$ when $p \to \infty$. On the other hand, since $\|v_p^s\| \leqq c^2 \mu^{-n_p}$, we will also have $\|v_p^s\| \to 0$. This will enable us to study w_p^s.

We have $w_p = v_p + \zeta$ with $\|\zeta\| \leqq p^{-1}$. So w_p^s is the component in $(E_p^s)^*$ of the vector $v_p + \zeta$. Since the $(F_p^u)^*$ component of this vector is known, namely $v_p^s + \zeta_p^s$ with $\zeta_p^s \to 0$, and the subspaces $(E_p^u)^*$ and $(F_p^u)^*$, as well as the subspaces $(E_p^s)^*$ and $(E_p^u)^*$, have a common limit, we get

$$\|w_p^s\| \leqq 2(\|v_p^s\| + \|\zeta_p^s\|) \to 0 \quad \text{when } p \to \infty$$

so both components of w_p converge to zero, while $\|w_p\| = 1$ for all p. This is a contradiction, and our assumption that $n_p \to \infty$ and $\eta_p \neq \xi_p$ for all p must be false. Hence the result.

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