MATH. SCAND. 52 (1983), 269-278

PRELIMINARY ALGEBRAS ARISING FROM LOCAL HOMEOMORPHISMS

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1. Introduction.

A preliminary algebra is defined to be a C*-algebra isomorphic to a hereditary subalgebra of $\mathscr{K} \otimes A$ for some abelian C*-algebra A (where $\mathscr{K}$ denotes the algebra of compact operators on a separable Hilbert space).

Given a local homeomorphism $\psi\colon X \in T$, it is possible to endow $C_c(X)$ with a $C_c(T)$-valued inner-product such that its completion (with respect to a suitable norm), denoted $l^2(\psi)$, may be viewed as the continuous sections vanishing at infinity of a Hilbert bundle over T. Then $l^2(\psi)$ is an imprimitivity bimodule [7] and the imprimitivity algebra, $C^*(\psi)$, is to be the object of our study. Thus $C^*(\psi)$ is strong Morita equivalent to $C_0(T)$ and hence preliminary (since $C^*(\psi)$ and $C_0(T)$ are stably isomorphic [1]). This algebra is shown to be the enveloping C*-algebra of the convolution algebra of the equivalence relation on X induced by ψ, $R(\psi)$. Consequently there is a diagonal projection, $P\colon C^*(\psi) \to C_0(X)$, where $C_0(X)$ is the natural diagonal masa (cf. [3], [4], [6]). There is a countable family of abelian elements in $C_c(X)$, so that finite sums constitute an approximate identity for $C^*(\psi)$.

Finally, the collection of generating *-derivations annihilating the diagonal (i.e. $\{\delta\colon \delta \circ P = 0\}$) is identified with $Z^1(R(\psi), \mathsf{R})$, the abelian group of continuous real-valued 1-cocycles on $R(\psi)$ ([3], [6]). Further, if ψ admits a section, then each 1-parameter automorphism group fixing the diagonal pointwise is implemented by a strictly continuous group of unitaries in the multiplier algebra of the diagonal (or more simply $H^1(R(\psi), \mathsf{R}) = 0$).

All topological spaces are implicitly locally compact, Hausdorff, and second countable (consequently paracompact). This work was partially supported by a grant from Danmark Amerika Fondet. The author wishes to thank George Elliott for his help in disambiguating certain delicate passages.

2. The algebra $C^*(\psi)$.

Let $\psi\colon X \to T$ be a continuous open surjection and let $\mathscr{U}(\psi)$ denote the collection of open subsets of X to which restrictions of ψ are injective. Then ψ is said to be a local homeomorphism if $\mathscr{U}(\psi)$ covers.

Received December 22, 1981; in revised form November 29, 1982.

Given a local homeomorphism $\psi: X \to T$, we define a linear map $\psi_*: C_c(X) \to C_c(T)$ by:

$$\psi_*(f)(t) = \sum_{\psi^{-1}(t)} f(x) \qquad t \in T$$

REMARKS. i) Regarding $C_c(X)$ as a right $C_c(T)$-module according to

$$(fg)(x) = f(x)g(\psi(x)) \qquad f \in C_c(X),\ g \in C_c(T)\,,$$

it is easily seen that ψ_* respects this action:

$$\psi_*(fg) = \psi_*(f)g\,.$$

ii) Borrowing notation from [8] we write $f \prec U$, for $f \in C_c(X)$ and U open, when $\operatorname{supp} f \subseteqq U$. If $f \prec U \in \mathscr{U}(\psi)$, then

$$f(x) = \psi_*(f)(\psi(x)) \qquad (\text{for } x \in U)$$

whence ψ_* is injective on $C_c(U)$.

2.1. PROPOSITION. *ψ_* is surjective.*

PROOF. Claim: Given a compact set $K \subseteqq T$, there is $f \in C_c(X)$ so that $\psi_*(f)(t) = 1$ for all $t \in K$. Since K is compact there are a finite number of open sets $U_1, \ldots, U_n \subseteqq X$ so that

$$K \subseteqq \bigcup_{i=1}^{n} \psi(U_i)\,.$$

Choose $f_i \prec U_i$ so that $\psi_*(f_i)$ is a partition of unity for K. Set $f = \sum f_i$ and the claim follows by linearity.

Let $g \in C_c(T)$, put $K = \operatorname{supp} g$ and let f be as in the claim. Then $fg \in C_c(X)$ and

$$\psi_*(fg) = \psi_*(f)g = g\,.$$

DEFINITION. Let $(\cdot|\cdot)$ be the sesquilinear $C_c(T)$-valued form defined on $C_c(X)$ by the formula:

$$(f|g) = \psi_*(\bar{f}g) \qquad \text{for } f, g \in C_c(X)\,.$$

2.2. PROPOSITION. *The inner-product $(\cdot|\cdot)$ has the following properties:*

i) $(f|g) = \overline{(g|f)}$ $\forall f, g \in C_c(X)$.
ii) $(f|f) \geqq 0$ *and* $(f|f) = 0$ *iff* $f = 0$.
iii) $(f|gh) = (f|g)h = (f\bar{h}|g)$ $\forall f, g \in C_c(X),\ h \in C_c(T)$.
iv) $\forall h \in C_c(T),\ \exists f, g \in C_c(X)$ *so that* $h = (f|g)$.

PROOF. These routine verifications are left to the interested reader.

DEFINITION. For $f \in C_c(X)$ put $|f|_\psi^2 = |(f \mid f)|_\infty$. Let $l^2(\psi)$ denote the completion of $C_c(X)$ with this norm.

REMARKS. i) Since $|f|_\infty \leqq |f|_\psi$, $l^2(\psi)$ may be viewed as a dense subspace of $C_0(X)$.

ii) The Schwarz inequality obtains in this setting, mutatis mutandis

$$(f \mid g)(g \mid f) \leqq (f \mid f)(g \mid g) .$$

NOTATION. Let End $(l^2(\psi))$ denote the collection of bounded linear operators on $l^2(\psi)$ commuting with the right action of $C_0(T)$.

For $a \in$ End $(l^2(\psi))$, let a^* denote the unique operator for which

$$(a^* f \mid g) = (f \mid ag) \quad \forall f, g \in l^2(\psi) .$$

Let $\|a\|$ denote the usual operator norm for $a \in$ End $(l^2(\psi))$.

2.3. DEUS EX MACHINA. The right $C_0(T)$-module $l^2(\psi)$ together with its $C_0(T)$-valued inner-product fits Rieffel's definition of an imprimitivity bimodule [7]. Let $C^*(\psi)$ denote the imprimitivity algebra viz. the closed $*$-subalgebra of End $(l^2(\psi))$ generated by operators of the form $\langle f, g\rangle$ for $f, g \in l^2(\psi)$, where

$$\langle f, g\rangle h = f(g \mid h) \quad \text{for each } h \in l^2(\psi) .$$

Note that $\langle f, g\rangle^* = \langle g, f\rangle$ and

$$\|\langle f, g\rangle\| \leqq \|f\|_\psi \|g\|_\psi$$

(since $\|\langle f, g\rangle h\|_\psi = \|f(g \mid h)\|_\psi \leqq \|f\|_\psi \|(g \mid h)\|_\infty \leqq \|f\|_\psi \|g\|_\psi \|h\|_\psi$).

Finite linear combinations of operators of the form $\langle f, g\rangle$ for $f, g \in C_c(X)$ constitute a dense $*$-subalgebra of $C^*(\psi)$.

REMARK. The linking algebra characterization of strong Morita Equivalence [1] provides a useful point of view in this situation. Let $Y = X \vee T$ (disjoint union) and put

$$\varphi : Y \to T \text{ by } \varphi(y) = \begin{cases} \psi(x) & \text{if } y = x \in X \\ t & \text{if } y = t \in T . \end{cases}$$

Then $C_0(T)$ and $C^*(\psi)$ are embedded in $C^*(\varphi)$ as complementary full corners while:

$$l^2(\psi) \cong \{a \in C^*(\varphi) : a^* a \in C_0(T) \text{ and } a a^* \in C^*(\psi)\} .$$

Suppose T is compact; then there is $f \in l^2(\psi)$ so that $(f \mid f)=1$. A routine calculation shows that $p=\langle f \mid f\rangle$ is a projection in $C^*(\psi)$. In fact every projection equivalent to p must be of this form. Moreover the corner determined by this projection is isomorphic to $C(T)$.

REMARK. For $f, g, h \in C_c(X)$ with $f, g \prec U \in \mathscr{U}(\psi)$, we have:

$$(\langle f, g\rangle h)(x) = f(x)\bar{g}(x)h(x) .$$

Evidently all such operators commute and the closure of the subalgebra generated is isomorphic to $C_0(X)$. This subalgebra is called the diagonal and henceforth is tacitly identified with $C_0(X)$. Note that if $f \prec U \in \mathscr{U}(\psi)$ then $f \in C^*(\psi)$ is an abelian element (cf. [5]).

Choose a partition of unity subordinate to a locally finite refinement of $\mathscr{U}(\psi)$, $\{(f_i, U_i) \mid f_i \prec U_i\}$. Put $g_n=\sum_1^n f_i$ and note that if $K \subseteqq X$ is compact, there is $m \geqq 1$ such that:

$$g_n(x) = 1 \quad \text{for all } x \in K,\ n \geqq m .$$

2.4. PROPOSITION. *The sequence* $\{g_n\}$ *constitutes an approximate identity for* $C^*(\psi)$.

PROOF. It suffices to check that $\lim_{n\to\infty} g_n\langle f, h\rangle=\langle f, h\rangle$ for f, $h \in C_c(X)$. Choose $m \geqq 1$ so that $g_n(x)=1$ for all $x \in \operatorname{supp}(f)$ and $n \geqq m$. Then

$$g_n\langle f, h\rangle = \langle g_n f, h\rangle = \langle f, h\rangle \quad \text{for all } n \geqq m .$$

3. Some examples.

i) If X is a countable set with the discrete topology and $T=\{t_0\}$ then we define ψ: $X \to T$ by $\psi(x)=t_0$. Then $l^2(\psi)$ is a Hilbert space of dimension equal the cardinality of X. Then $C^*(\psi) \cong \mathscr{K}(l^2(\psi))$.

ii) Suppose T is totally disconnected and $T=U_1 \supseteqq U_2 \supseteqq U_3 \supseteqq \ldots$ is a decreasing sequence of open subsets. Put

$$X = \{(t, n) \in T \times \mathbf{N} : t \in U_n\}$$

and define $\psi: X \to T$ by $\psi(t, n)=t$. Then ψ is evidently a local homeomorphism. $C^*(\psi)$ is then an AF algebra with dimension group

$$K_0(C^*(\psi)) \cong \{f: T \to \mathbf{Z} : f \text{ continuous}, \lim_{t\to\infty} f(t)=0\}$$

and dimension range $=\{f: t \in U_{f(t)}\}$. Every continuous trace AF algebra is of this form (this example arose in a discussion with G. Elliott).

iii) Let $X \cong T \cong \mathrm{T}$, the group of complex numbers of unit modulus. Let ψ be the n-fold covering i.e. $\psi(z)=z^n$. Consider $g_0, g_1, \ldots, g_{n-1} \in l^2(\psi)$ defined by $g_j(z)=z^j/\sqrt{n}$. Then $(g_i \mid g_i)=1$, while $(g_i \mid g_j)=0$ if $i \neq j$. It follows that $C^*(\psi) \cong M_n(C(T))$ because

$$\sum_{i=0}^{n-1} \langle g_j \mid g_j \rangle = 1 .$$

iv) Suppose $\psi: X \to T$ is a general covering map and let G denote the associated group of covering transformations. It is not difficult to see that $C^*(\psi)$ is precisely the associated transformation group C*-algebra, $C_0(X) \times G$. (Provided that X is connected).

4. The equivalence relation $R(\psi)$.

Much of the structure of $C^*(\psi)$ becomes transparent when a dense *-subalgebra is identified with $C_c(R(\psi))$, where $R(\psi)$ is the topological equivalence relation on X induced by ψ.

DEFINITION. Put $R(\psi)=\{(x,y) \in X \times X: \psi(x)=\psi(y)\}$ and endow $R(\psi)$ with the relative topology from $X \times X$.

REMARKS. i) Let $r: R(\psi) \to X$ and $s: R(\psi) \to X$ be the projections onto the first and second coordinates respectively (both are local homeomorphisms). Evidently, $R(\psi)$ is a principal discrete groupoid in the terminology of [6].

ii) Put $\Omega(\psi)=\mathscr{U}(s) \cap \mathscr{U}(r)$, i.e. $\Omega(\psi)=\{\omega \subseteq R(\psi) : r|_\omega, s|_\omega \text{ inj.}\}$. Note that $\Omega(\psi)$ covers $R(\psi)$.

We view $f \in C_c(R(\psi))$ as an element of End $(l^2(\psi))$ according to the formula

$$(fg)(x) = \sum_y f(x,y)g(y) \quad \text{where } g \in l^2(\psi) .$$

The reader is left to verify that this formula respects the usual groupoid convolution and involution. Henceforth, the distinction between X and the diagonal in $R(\psi)$, $\{(x,x): x \in X\}$, is blurred; Thus, $C_c(X)$ is tacitly identified with the corresponding subalgebra of $C_c(R(\psi))$.

4.1. PROPOSITION. *Let $h \prec \omega \in \Omega(\psi)$; there is $f, g \in C_c(X)$ and $k \prec \omega$ so that*

i) $h=fkg$
ii) $h(x,y)=f(x)g(y)$.

PROOF. Choose $k \prec \omega$ so that $h(x,y) \neq 0 \Rightarrow k(x,y)=1$. Set

$$f(x) = h(x,y)|h(x,y)|^{-\frac{1}{2}}, \quad g(y) = |h(x,y)|^{\frac{1}{2}}.$$

We note that $C^*(\psi)$ is the norm closure of $C_c(R(\psi))$; given $n \geqq 1$ and $g_i, h_i \in C_c(X)$ for $1 \leqq i \leqq n$, there is $f \in C_c(R(\psi))$ such that:

$$f = \sum_i \langle g_i, h_i \rangle \quad \left(\text{put } f(x,y) = \sum_i g_i(x)\overline{h_i(y)}\right).$$

The above proposition combined with the fact that $\Omega(\psi)$ covers implies that every element of $C_c(R(\psi))$ can be so expressed.

DEFINITION. Let $P\colon C_c(R(\psi)) \to C_c(X)$ denote the restriction to the diagonal.

4.2. PROPOSITION. *Let $\{(f_i, U_i)\}$ be a partition of unity for X as in 2.4. Let $f \in C_c(R(\psi))$; then*

$$P(f) = \sum f_i^{\frac{1}{2}} f f_i^{\frac{1}{2}} \quad (\textit{the sum is finite}).$$

PROOF. There is a compact subset $K \subseteq X$ such that $\operatorname{supp} f \subseteq K \times K$. There is $n_0 \geqq 1$ so that $\forall n \geqq n_0$. $\operatorname{supp} f_n \cap K = \varnothing$; whence

$$f = \sum_{i=1}^{n} f_i f = \sum_{i=1}^{n} f f_i .$$

Since $f_i^{\frac{1}{2}} f f_i^{\frac{1}{2}} \in C_c(X)$, one checks directly that

$$(f_i^{\frac{1}{2}} f f_i^{\frac{1}{2}})(x) = f_i(x) f(x,x)$$

and the result follows by linearity.

4.3. PROPOSITION. *The map $P\colon C_c(R(\psi)) \to C_c(X)$ extends to a faithful conditional expectation $P\colon C^*(\psi) \to C_0(X)$* (cf. [4], [6]).

PROOF. It is a direct consequence of 4.2 that $\|P(f)\| \leqq \|f\|$ for $f \in C_c(R(\psi))$ and thus P extends to a projection of unit norm onto $C_0(X)$. Suppose $a \in C^*(\psi)$ so that $P(a^*a)=0$. Then $f_i^{\frac{1}{2}} a^* a f_i^{\frac{1}{2}} = 0$ for each i. Whence $af_i = (af_i^{\frac{1}{2}}) f_i^{\frac{1}{2}} = 0$ for each i. Since $a = \lim_{n\to\infty} \sum_{i=1}^{n} af_i$, we conclude that $a=0$, and that P is faithful.

REMARK. Since $C^*(\psi)$ is preliminary and, hence, of continuous trace, there is a dense ideal on which the unique center-valued trace, τ, is defined. Clearly, $C_c(R(\psi))$ is contained in this ideal

$$\tau(f) = \psi_*(P(f)) \quad \forall f \in C_c(R(\psi)).$$

4.4. LEMMA. *Let* $a \in C^*(\psi)$; *then* $a \in C_c(R(\psi))$ *iff there is* $h \in C_c(X)$ *such that* $a = hah$.

PROOF. "Only if" is clear.

"if"—it will suffice to prove that the map extending the identity on $C_c(R(\psi))$ is an embedding of $C^*(\psi)$ in $C_0(R(\psi))$. We may then conclude from $\mathrm{supp}\,(a) \subseteqq \mathrm{supp}\,(h) \times \mathrm{supp}\,(h)$ that $a \in C_c(R(\psi))$. Let $f \cdot g$ denote the pointwise product of $f, g \in C_c(R(\psi))$. Recall that $s\colon R(\psi) \to X$ $(s(x,y) = y)$ is a local homeomorphism and note that $P(f^*g) = s_*(\bar{f} \cdot g)$. Since $\|P\| = 1$ it follows that:

$$\|f\| \geqq \|f\|_s \geqq \|f\|_\infty .$$

That P is faithful implies that $C^*(\psi)$ embeds in $l^2(s)$ which embeds in $C_0(R(\psi))$.

5. Automorphisms and *-derivations.

In [3] Feldman and Moore characterize the automorphisms preserving the diagonal in the von Neumann algebra of a Borel equivalence relation in terms of circle-valued 1-cocycles on the relation (II, Theorem 2).

DEFINITION. Let $Z^1(R(\psi), \mathrm{T})$ denote the collection of continuous functions

$$v\colon R(\psi) \to \mathrm{T} \quad \text{for which}$$

$$v(x,z) = v(x,y)v(y,z) \quad \text{all } x \sim y \sim z .$$

NB: This implies $v(x,x) = 1$ and $v(x,y) = \overline{v(y,x)}$ each $x \sim y$.

Renault shows that pointwise multiplication by $v \in Z^1(R(\psi), \mathrm{T})$ defines an automorphism of $C^*(\psi)$ which fixes the diagonal that is $\alpha \circ P = P$ ([6], II. 5.1). We offer the following converse:

5.1. THEOREM. *The group of automorphisms of* $C^*(\psi)$ *fixing the diagonal* (viz. $\alpha \circ P = P$) *is isomorphic to* $Z^1(R(\psi), \mathrm{T})$ *according as, given* $a \in \mathrm{Aut}\,(C^*(\psi))$ *with* $\alpha \circ P = P$ *there is* $v \in Z^1(R(\psi), \mathrm{T})$ *so that*

$$\alpha(f) = v \cdot f$$

for all $f \in C_c(R(\psi))$ (*where* $(v \cdot f)(x,y) = v(x,y) f(x,y)$).

PROOF. By 4.4 we know that $C_c(R(\psi))$ is left invariant by α. Given $h \prec \omega \in \Omega(\psi)$, we may select $f, g \in C_c(X)$ and $k \prec \omega$ as in 4.1, so $h = fkg$. Then,

$$\alpha(h) = \alpha(f)\alpha(k)\alpha(g) = f\alpha(k)g$$

and so:

$$h(x,y) = 0 \Rightarrow f(x)g(y) = 0 \Rightarrow (\alpha(h))(x,y) = 0 .$$

Since $R(\psi)$ is paracompact and $\Omega(\psi)$ covers we may choose a partition of unity subordinate to a countable locally finite refinement of $\Omega(\psi)$:

$$\{(h_i, \omega_i) : h_i \prec \omega_i\} .$$

Put $v = \sum \alpha(h_i)$; then v is a continuous function as, in a neighbourhood of a given point, there are only finitely many non-zero summands.

Choose h as above with $h(x,y) = f(x)g(y)$, then

$$h = \sum f h_i g \quad \text{and} \quad \alpha(h) = \sum f\alpha(h_i)g = \sum f(v \cdot h_i)g = v \cdot h;$$

this obtains generally as

$$C_c(R(\psi)) = \operatorname{sp}\{h : h \prec \omega \in \Omega(\psi)\} .$$

Assume $h(x,y) \neq 0$; since $\alpha(h^*) = v \cdot h^* = (v \cdot h)^* = \alpha(h)^*$, then clearly $v(x,y) = \overline{v(y,x)}$.

Since $h^*h \in C_c(X)$ we have $h^*h = \alpha(h^*h)$ and

$$|h(x,y)|^2 = h^*h(y) = (\alpha(h^*h))(y) = |h(x,y)|^2 |v(x,y)|^2 .$$

Thus $v(x,y) \in \mathsf{T}$; it is a routine matter to check the cocycle property.

As in [6] we may establish a correspondence between $Z^1(R(\psi), \mathsf{R})$ and $*$-derivations on $C_c(R(\psi))$. These derivations annihilate the diagonal and are pregenerators for $C^*(\psi)$, moreover $C_c(R(\psi))$ consists of analytic elements (cf. II. 5.2). Suppose $d \in Z^1(R(\psi), \mathsf{R})$ (so $d(x,z) = d(x,y) + d(y,z)$), then put

$$\delta f = \mathrm{id} \cdot f \quad \text{for } f \in C_c(R(\psi)) .$$

We have $\delta \circ P = 0$, since $d(x,x) = 0 \ \forall x \in X$.

5.2. THEOREM. *Let δ be a $*$-derivation on $C_c(R(\psi))$ so that $\delta \circ P = 0$. There is $d \in Z^1(R(\psi), \mathsf{R})$ such that:*

$$(*) \qquad \delta f = \mathrm{id} \cdot f .$$

Further, if ψ has a section, then $d \in B^1(R(\psi), \mathsf{R})$ (viz. $\exists p: X \to \mathsf{R}$ continuous so that $d(x,y) = p(x) - p(y)$) and the 1-parameter automorphism group is inner (cf. [6], II. 5.3).

PROOF. Choose $h \prec \omega \in \Omega(\psi)$ as in 4.1, then $h = fkg$ with $f, g \in C_c(X)$. By the certain derivation property we have

$$\delta(h) = \delta(f)kg + f\delta(k)g + fk\delta(g)$$
$$= f\delta(k)g \quad (\text{since } \delta \circ P = 0) .$$

Whence $h(x,y)=0 \Rightarrow f(x)g(y)=0 \Rightarrow \delta(h)(x,y)=0$. Choose a partition of unity $\{(h_j, \omega_j): h_j \prec \omega_j\}$ as in 5.1. Define $d: R(\psi) \to \mathbf{C}$ by

$$d(x,y) = -i \sum_j (\delta h_j)(x,y) .$$

Again reasoning as in 5.1, we see that d is continuous. While, if h is as above $h=\sum f h_j g$, then

$$\delta(h) = \sum_j f\delta(h_j)g = \mathrm{id}\cdot h;$$

then (*) follows by linearity.

Since $h^*h \in C_c(X)$ we have $0=\delta(h^*h)=\delta(h)^*h+h^*\delta(h)$. Whence $h^*\delta(h)$ is skew-adjoint. A simple calculation reveals that $h^*\delta(h) \in C_c(X)$; suppose $h(x,y) \neq 0$

$$(h^*\delta(h))(y) = |h(x,y)|^2(\mathrm{id}\,(x,y)) ,$$

whence $d(x,y) \in \mathbf{R}$.

The derivation property implies $d(x,z)=d(x,y)+d(y,z)$. Suppose now that ψ has a continuous section $\sigma: T \to X$. Put $p(x)=d(x,\sigma(\psi(x)))$. Note that

$$(x,y) \in R(\psi) \Rightarrow \sigma(\psi(x)) = \sigma(\psi(y)) .$$

Thus

$$\begin{aligned} d(x,y) &= d(x,\sigma(\psi(x)))+d(\sigma(\psi(y)),y) \\ &= d(x,\sigma(\psi(x)))-d(y,\sigma(\psi(y))) \\ &= p(x)-p(y) . \end{aligned}$$

5.3. Lemma. *Let S be a topological space. Let $C(S,\mathbf{T})$ denote the group of circle-valued continuous functions (under pointwise multiplication) equipped with the topology of uniform convergence on compact subsets of S. Let $\varphi: \mathbf{R} \to C(S,\mathbf{T})$ be a continuous homomorphism. There is a continuous function*

$$\theta: S \to \mathbf{R}$$

such that

$$\varphi_t(s) = e^{it\theta(s)} \qquad \forall t \in \mathbf{R} \text{ and } s \in S .$$

Proof. We may assume that S is compact. Observe that $\varphi_0(s)=1$ for all $s \in S$. There is $\varepsilon>0$ so that $\mathrm{Re}\,(\varphi_t(s))>0$ for all $s \in S$ and $t \in (-\varepsilon,\varepsilon)$. Put

$$\theta(s) = \frac{-i}{\varepsilon}\mathrm{Log}\,(\varphi_\varepsilon(s)) .$$

5.4. PROPOSITION. *Let* α: $\mathsf{R} \to \mathrm{Aut}\,(C^*(\psi))$, *so that* $\alpha_t \circ P = P$ *and* $\alpha_t(a)$ *is norm-continuous for each* $a \in C^*(\psi)$. *Then there is* $d \in Z^1(R(\psi), \mathsf{R})$ *so that*

$$\alpha_t(f) = e^{itd} \cdot f \quad \text{each } f \in C_c(R(\psi))\,.$$

PROOF. By 5.1 there is $v_t \in Z^1(R(\psi), T)$ so that

$$\alpha_t(f) = v_t \cdot f \quad \text{for each } t \in \mathsf{R}\,.$$

Clearly $v_0(x,y) = 1$ and

$$v_{s+t}(x,y) = v_s(x,y)v_t(x,y) \qquad \forall\,(x,y) \in R(\psi),\ s,t \in \mathsf{R}\,.$$

The norm continuity of $v_t \cdot h$ implies that v: $\mathsf{R} \to Z^1(R(\psi), \mathsf{T})$ is continuous in the topology of uniform convergence on compact subsets of $R(\psi)$. Application of the above lemma yields the desired result.

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