MATH. SCAND. 52 (1983), 279–311

CONTRACTIVE PROJECTIONS ON OPERATOR TRIPLE SYSTEMS

YAAKOV FRIEDMAN and BERNARD RUSSO

A J*-algebra is a norm closed *complex* linear subspace M of $\mathscr{L}(H,K)$, the bounded linear operators from a Hilbert space H to a Hilbert space K, which is closed under the operation $a \to aa^*a$. J*-algebras were defined and studied as a generalization of C*-algebras by Harris [17], [18], [19] in connection with function theory on infinite dimensional bounded symmetric homogeneous domains. The principal examples of J*-algebras, besides (concrete) C*-algebras are JC*-algebras [33], [34], [36] and ternary rings of operators [21], [38]. By virtue of various extensions of the Gelfand Naimark theorem we may regard JB*-algebras [1], [37] (with the usual exclusion of the exceptional Jordan algebra $(M_3^8)^{\mathrm{C}}$) and C*-ternary rings [38], as well as abstract C*-algebras [10] as examples of J*-algebras.

A current objective of the author's is to show that the category of C*-triple systems (an abstract version of J*-algebras) is stable under the action of norm one projections (see [14], [15]). The main result of the present paper is the following.

THEOREM 3. *Let M be a J*-algebra and let P: $M \to M$ be a linear projection of norm one*: $P^2=P$, $\|P\|=1$. *Suppose the range $P(M)$ of P is finite dimensional. Then $P(M)$ is linearly isometric to a J*-algebra and is therefore a C*-triple system.*

Simple examples show that Theorem 3 is best possible in the sense that, in the absence of special assumptions on P, $P(M)$ is not in general a ternary ring of operators or a JC*-algebra even if M is a C*-algebra.

In the other direction, every finite dimensional J*-algebra (and many infinite dimensional ones — the Cartan factors, [17]) can be embedded in a C*-algebra A as the range of a contractive projection on A. For finite dimensions this follows from the classification [17], [26], and for spin factors it is proved in [3] and [12]. Moreover, if the J*-algebra M does not have a Hilbert space direct summand, then for every isometric embedding T of M into a C*-algebra A, there is a contractive projection P on A such that $\mathrm{T}(M)=P(A)$, [2], [3].

Received December 21, 1981.

A consequence of Theorem 3 is that the category of finite dimensional J*-algebras is stable under contractive projections.

A C*-triple system is a Jordan triple system [24], [30] of arbitrary dimension in which the underlying linear space is a Banach space whose norm satisfies a "C*-condition" (cf. [22]). For example, any linear space which is isometric to a J*-algebra is a C*-triple system with the induced triple product. The triple product which defines the Jordan triple system structure of $P(M)$ in Theorem 3 is given by

$$\{abc\} = \tfrac{1}{2}P(ab^*c+cb^*a), \qquad a,b,c \in P(M)\,. \tag{0.1}$$

The significance of Theorem 3 is that the connections between Jordan triple systems and complex analysis (via bounded symmetric domains [25]) and mathematical physics (via the quadratic Jordan formulation of quantum mechanics [16], Lie superalgebras [4], and quantization of finite [5], [6] or infinite dimensional [31] symmetric domains) can be further studied by use of a new tool, namely, contractive projections. The state space of any algebraic system is important in mathematical physics. Contractive projections are related to state spaces by virtue of the induced action on the unit ball of the system and of its dual.

The proof of Theorem 3 is long, requiring over twenty supporting lemmas and propositions, several of which (e.g., Proposition 3.3, Proposition 3.5, Proposition 3.7, Proposition 4.3) have independent interest. Two remarks about the proof of Theorem 3 are in order.

First, despite the strong interaction of Theorem 3 with other fields (operator theory, Jordan algebras, geometry, mathematical physics) the proof given in the present paper is reasonably self contained. The only deep results used are the Hahn Banach extension theorem, the universal representation of a C*-algebra, and the following two special results which we state here as Lemma 0.1 and Lemma 0.2.

LEMMA 0.1. (Effros, [11, Lemma 3.1], [10, 12.2.3]). *Let B be a von Neumann algebra, f an element of the predual B_* of B, e a projection in B. Let $e.f \in B_*$ be the functional $x \to f(ex)$. Then $f=e.f$ if and only if $\|f\| = \|e.f\|$.*

LEMMA 0.2. *Let N be an ultraweakly closed Jordan *-subalgebra of $\mathscr{L}(H)$ and let φ be a positive faithful ultraweakly continuous functional on N. Let S be the face in $(N_*)^+$ generated by φ, that is,*

$$S = \{\tau \in (N_*)^+ : \exists\,\lambda>0 \text{ with } 0\leqq\tau\leqq\lambda\varphi\}\,.$$

Then S is norm dense in $(N_*)^+$.

Lemma 0.2 is a special case of the following recent result of King [23], cf. [13, § 2]. Let A be JBW-algebra with predual E (cf. [32]). For a normal state f, let

$$V_f = \{g \in E : \exists \lambda \in \mathrm{R} \text{ with } -\lambda f \leqq g \leqq \lambda f\} .$$

Then f is faithful if and only if V_f is norm dense in E. This is also proved in [39, p. 200].

The second remark is that the assumption of finite dimensionality of the range $P(M)$ is not needed for most of the steps in the proof of Theorem 3. We believe that the finiteness assumption can be dropped in Theorem 3. Indeed in Theorem 2 we prove a version of Theorem 3 without this assumption. Moreover, versions of Theorem 3 without a dimensionality restriction on the range are known if additional assumptions are made on the space M and/or on the projection P.

Choi–Effros prove in [8] that if M is a C*-algebra and if P is completely positive and unital, then $P(M)$ (of arbitrary dimension) is a C*-algebra in a product given by

$$a * b = P(ab), \quad a, b \in P(M) . \tag{0.2}$$

Arazy–Friedman in [3] completely classified the range $P(M)$ of an arbitrary contractive projection in case M is the C*-algebra C_∞ of compact operators on a separable complex Hilbert space. Using their classification, Theorem 3 (with $M = C_\infty$ and $P(M)$ of arbitrary dimension) can be verified on a case by case basis. Effros–Størmer in [12] prove that if M is a JC-algebra and P is positive and unital, then $P(M)$ (of arbitrary dimension) is a JC-algebra in the product

$$a \circ b = P(\tfrac{1}{2}(ab + ba)), \quad a, b \in P(M) . \tag{0.3}$$

Finally, the authors proved Theorem 3 with $P(M)$ of arbitrary dimension in case M is a commutative C*-algebra and they gave a complete description of all contractive projections in this case in [14].

Note that Theorem 3 gives new proofs of each of these last three versions in the case of a finite dimensional range.

An interesting and important related problem is to show that a bicontractive projection P on a J*-algebra M, i.e., P and $\mathrm{id}_M - P$ are both contractive projections, has the property that $2P - \mathrm{id}_M$ is an involutive isometry of M onto M. This problem was solved for $M =$ the C*-algebra of compact operators by Arazy–Friedman [3] and is known for commutative C*-algebras (Bernau–Lacey [7]; see [14] for another proof). The particular case of this problem in which M is a C*-algebra and P is positive and unital has just recently been solved by Størmer [35]. In each of these known cases the structure of bicontractive projections was determined from the result corresponding to

Theorem 3. For more partial results in the context of JB-algebras see [29]. In connection with this problem we note that one of the nice properties of J*-algebras (and C*-triple systems) is that the surjective linear isometries coincide with the J*-isomorphisms, [17], [22].

The following remark is due to L. Harris and was communicated to the authors by E. Effros: if P is an Hermitian projection on a J*-algebra M, then $P(M)$ is a J*-subalgebra of M. This follows easily from Example 5 in [20] (see also [22]) which states that the Hermitian operators δ on a J*-algebra M are characterized by the formula

$$(0.4) \qquad \delta(ab^*a) = (\delta a)b^*a - a(\delta b)^*a + ab^*(\delta a), \qquad a, b \in M .$$

This paper is organized as follows. In Section 1 we discuss Jordan triple systems and their concrete analogs, the J*-algebras. The principal result of Section 2 is a polar decomposition for ultraweakly continuous linear functionals on a weakly closed J*-algebra (Theorem 1). By using the fact that the bidual of a J*-algebra is a J*-algebra (Proposition 2.1), an infinite dimensional version of Theorem 3 is proved (Theorem 2) in which it is assumed that the partial isometry occuring in the enveloping polar decomposition of some functional in the dual M' of the J*-algebra M covers the range of P' in an appropriate sense. The proof of Theorem 3 is given in Section 3 and the proof of the key lemma (Lemma 3.8) of Section 3 is given in section 4.

The following is some notation which will be used without further explanation. $\mathscr{L}(H)$ is $\mathscr{L}(H,H)$, and is a Jordan *-algebra with the usual involution and Jordan product $a \circ b = \frac{1}{2}(ab+ba)$. S^+ is the positive part of set S, B' the dual of a normed space B, $l(a)$ and $r(a)$ denote the range projection and support projection of an operator a. If ξ, η are Hilbert space vectors, $\omega(\xi, \eta)$ denotes the restriction of the linear functional $x \to (x\xi, \eta)$ to an appropriate subspace of $\mathscr{L}(H)$ which will be clear from the context.

The authors wish to express thanks to E. Effros for his advice and encouragement in this project.

1. Jordan triple systems — generalities.

Let V be a complex vector space and let $Q\colon V \to \operatorname{End}(V)$ be a quadratic map. Define maps $\{\cdot\cdot\cdot\}\colon V \times V \times V \to V$ and $D\colon V \times V \to \operatorname{End}(V)$ by the formula

$$(1.1) \qquad \{xyz\} = D(x,y)z = Q(x+z)y - Q(x)y - Q(z)y .$$

By a *Jordan triple system* (over the complex field) is meant a pair (V, Q) in which the function $\{xyz\}$ is complex linear and symmetric in x and z, complex antilinear in y, and satisfies

$$(1.2) \qquad \{xy\{uvz\}\} - \{uv\{xyz\}\} = \{\{xyu\}vz\} - \{u\{yxv\}z\} .$$

This identity can be expressed as

$$(1.3)\qquad [D(u,v),D(x,y)] = D(\{xyu\},v)-D(u,\{yxv\})$$

which shows that the linear span of $\{D(x,y)\colon x,y \in V\}$ is a Lie subalgebra of $gl(V)$. For each fixed y, V becomes a Jordan algebra under $(x,z) \to \{xyz\}$. In the other direction, any Jordan algebra becomes a Jordan triple system under the Jordan triple product $\{xyz\}=x(yz)-y(zx)+z(xy)$. Jordan triple systems were extensively studied by Loos [24], [25], under the name Jordan pair. A principal fact is that (in finite dimensions) there is a one-to-one correspondence between circled bounded symmetric domains and Jordan pairs with involution in which irreducible domains correspond to simple Jordan pairs. The classification of simple finite dimensional Jordan triple systems over the complex field (and hence of the corresponding domains [25]) given in [24] is the following (for the real case see [28]). There are two exceptional ones of dimensions 16 and 27. The non-exceptional ones fall into four classes:

$I_{p,q}$, $1 \leqq p \leqq q$,	$V = M_{p,q}(\mathbf{C})$	(p by q matrices)
II_n, $n \geqq 5$,	$V = A_n(\mathbf{C})$	(skew symmetric)
III_n, $n \geqq 2$,	$V = S_n(\mathbf{C})$	(symmetric)
IV_n, $n \geqq 4$,	$V = \mathbf{C}^n$	(spin factors) .

In the first three cases one has $Q(x)y=x\bar{y}^{\mathrm{T}}x$ so that $\{xyz\}=x\bar{y}^{\mathrm{T}}z+z\bar{y}^{\mathrm{T}}x$. In case IV_n, one has

$$2\{xyz\} = \langle x,\bar{y}\rangle z+\langle z,\bar{y}\rangle x-\langle z,x\rangle\bar{y}\ ,$$

where

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i\ .$$

An important example of an infinite dimensional Jordan triple system is a J*-algebra which is defined as a norm closed linear subspace A of $\mathscr{L}(H,K)$ (=the bounded operators from Hilbert space H to Hilbert space K) which together with each element a contains aa^*a. By use of a polarization identity, the map $(a,b,c) \to ab^*c+cb^*a$ can be shown to define a Jordan triple system structure on A. J*-algebras were defined and studied by L. Harris [17], [18] in connection with function theory on infinite dimensional bounded symmetric homogeneous domains. Algebraic properties of J*-algebras are developed in [19]. In particular an analogue of the spectral theorem is proved in which spectral projections are replaced by partial isometries.

A bounded linear map L between two J*-algebras is called a J*-

homomorphism, if $L(aa^*a)=L(a)L(a)^*L(a)$ holds for all a. A J*-algebra M is said to be *embedded* in a C*-algebra A, if there is a J*-isomorphism of M into A (A considered as a J*-algebra).

We shall need three properties of J*-algebras, namely, (1) the weak closure of a J*-algebra is a J*-algebra [19]; (2) a Kaplansky density theorem [19]; (3) the characterization of the extreme points of the unit ball of a J*-algebra, [17].

Let V be a Jordan triple system. An element $e \in V$ is called a *tripotent* if $\{eee\} = e$. Each tripotent e gives rise to operators $E(e)$, $F(e)$, $G(e)$ on V as follows:

$$E(e) = Q(e)^2, \quad F(e) = \mathrm{id}_V - D(e,e) + E(e) \tag{1.4}$$

$$G(e) = \mathrm{id}_V - E(e) - F(e)\,. \tag{1.5}$$

It follows from [25; page A1 formula JP3, JP23, JP25] that $E(e)$, $G(e)$, $F(e)$ are idempotent and satisfy $E(e)+G(e)+F(e)=\mathrm{id}_V$. For $x \in V$, $x=E(e)x+G(e)x+F(e)x$ is the Peirce decomposition of x with respect to e, where $E(e)x=x_1$, $G(e)x=x_{\frac{1}{2}}$, $F(e)x=x_0$ (cf. Loos [24], McCrimmon [27]).

Formulas (1.4), (1.5) simplify considerably in the case of a J*-algebra M, which is the only Jordan triple system considered in this paper. In this case, the tripotents correspond to partial isometries in M. If v is a partial isometry in a J*-algebra M, then setting $l=vv^*$ and $r=v^*v$ the projections $E(v)$, $F(v)$, $G(v)$ on M are given by

$$\begin{aligned} E(v)x &= lxr = v(vx^*v)^*v \\ F(v)x &= (1-l)x(1-r) = x-(vv^*x+xv^*v)+v(vx^*v)^*v \\ G(v)x &= lx(1-r)+(1-l)xr = (vv^*x+xv^*v)-2v(vx^*v)^*v\,, \end{aligned}$$

and yield the familiar matrix representation of an element x in M:

$$x \sim \begin{bmatrix} E(v)x & lx(1-r) \\ (1-l)xr & F(v)x \end{bmatrix}$$

Throughout this paper, for convenience $E(v)$, $F(v)$, and $G(v)$ will also denote the projections induced on the dual or pre-dual of M, e.g., if g is a linear functional on M, $E(v)g$ denotes $g \circ E(v)$. The precise meaning of the symbols $E(v)$, etc. will be clear from the context.

The following Lemma is an easy consequence of the matrix representation.

LEMMA 1.1. *Let v be a partial isometry in a J*-algebra M. Then*

(a) $\|(E(v)+F(v))x\| = \max\{\|E(v)x\|, \|F(v)x\|\} \leqq \|x\|, \quad x \in M;$

(b) $\|E(v)g\| + \|F(v)g\| = \|(E(v)+F(v))g\| \leqq \|g\|, \quad g \in M'.$

2. Polar decomposition for functionals on a J*-algebra.

In this section we show that the second dual of a J*-algebra is itself a J*-algebra, we establish a polar decomposition for an ultraweakly continuous linear functional on a J*-algebra and we prove an infinite dimensional version of Theorem 3.

Five topologies of interest on $\mathscr{L}(H)$, namely the norm, strong, weak, ultrastrong, and ultraweak can be defined in the same way for $\mathscr{L}(H,K)$. The properties of these topologies (cf. Dixmier [9]) for $\mathscr{L}(H)$ are also valid for $\mathscr{L}(H,K)$. This follows easily, for example, from the fact that the map $\sigma\colon \mathscr{L}(H,K) \to \mathscr{L}(H \oplus K)$ defined by

$$\sigma(a) = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \quad \text{for } a \in \mathscr{L}(H,K)$$

is a homeomorphism (in all five topologies) of $\mathscr{L}(H,K)$ onto a closed subset of $\mathscr{L}(H \oplus K)$. Since σ is a linear isometry of $\mathscr{L}(H,K)$ into $\mathscr{L}(H \oplus K)$ satisfying

$$\sigma(ab^*c) = \sigma(a)\sigma(b)^*\sigma(c)\,,$$

every J*-algebra may be considered as a J*-subalgebra of some C*-algebra. We shall use this fact in the proofs of many of our results.

Proposition 2.1. *Let M be a J*-algebra. Then there is a J*-isomorphism π of M onto a J*-algebra $\pi(M)$ with the property that (identifying $\pi(M)$ with its canonical image in $\pi(M)''$) the identity map on $\pi(M)$ extends to an isometry of $\pi(M)''$ onto the weak closure N of $\pi(M)$. This isometry is a homeomorphism in the weak*-topology of $\pi(M)''$ and the weak topology of N.*

Proof. Let A be any C*-algebra which contains M as a J*-algebra and let π_1 be the universal representation of A. Then $\pi = \pi_1 | M$ satisfies the requirements of the proposition. The verification of this is a standard argument which we include for completeness. For $f \in \pi(M)'$, let $f_1 \in \pi_1(A)'$ be a Hahn Banach extension of f. Then f_1 and hence f is ultraweakly continuous so there is an ultraweakly continuous extension $\bar{f}$ of f to N with $\|f\| = \|\bar{f}\|$. By Harris' version of the Kaplansky density theorem [19], the map $f \to \bar{f}$ is a linear bijection of $\pi(M)'$ onto N_*, where N_* is the space of ultraweakly continuous linear functionals on N. The adjoint of $f \to \bar{f}$ is then a linear isometry of $N \cong (N_*)'$ onto $\pi(M)''$ which carries $\pi(M)$ onto the canonical image of $\pi(M)$ in $\pi(M)''$.

As pointed out by Harris [19], N is a J*-algebra. Therefore we may regard $M''(\cong \pi(M)'')$ as a J*-algebra which contains M as a J*-subalgebra.

Let us define a *von Neumann* J*-*algebra* to be an ultraweakly closed J*-algebra M. Its pre-dual, consisting of the normal functionals on M, will be denoted as usual by M_*. By Proposition 2.1, the bidual M'' of a J*-algebra is a von Neumann J*-algebra with pre-dual M'.

The next lemma plays a key role in the existence of a polar decomposition for a linear functional on a J*-algebra.

LEMMA 2.2. *Let M be a von Neumann J*-algebra, let $f \in M_*$ and let v be a partial isometry in M which satisfies $f(v) = \|f\|$. Then $f \circ E(v) = f$,* that is, $f(x) = f(E(v)x)$ *for* $x \in M$.

PROOF. Let A be any C*-algebra which contains M as a J*-sub-algebra and let $\tilde{f} \in A'$ be a Hahn Banach extension of f to A. Then

$$\|\tilde{f}\|_{A'} \geqq \|\tilde{f} \circ E(v)\|_{A'} \geqq |\tilde{f}(E(v)v)| = |\tilde{f}(v)| = |f(v)| = \|f\| = \|\tilde{f}\|_{A'} .$$

By Lemma 0.1, $\tilde{f} \circ E(v) = \tilde{f}$ and by restriction $f \circ E(v) = f$. Note that since $M \subset A$, $\tilde{f} \circ E(v)$ is defined on A.

The next lemma essentially gives the form of the polar decomposition.

LEMMA 2.3. *Let M be a von Neumann J*-algebra in $\mathscr{L}(H, K)$ and let v be a partial isometry in M. Then*

(a) *$N_v \equiv v^*Mr$ (with $r = v^*v$) is an ultraweakly closed Jordan *-subalgebra of $\mathscr{L}(H)$, with unit r.*
(b) *If $f \in M_*$ and $f(v) = \|f\|$, then φ_v, defined by*

$$\varphi_v(a) = f(va), \qquad a \in N_v$$

is a positive normal functional on N_v which satisfies

$$f(x) = \varphi_v(v^*xr), \qquad x \in M .$$

PROOF. (a) If $a \in N_v$, say $a = v^*xr$ with $x \in M$, then

$$a^2 = v^*xrv^*xr = v^*(xv^*x)r \in N_v ,$$

$$a^* = rx^*v = v^*(vx^*v)r \in N_v. \quad \text{and}$$

$$a \circ r = \tfrac{1}{2}(ar + ra) = \tfrac{1}{2}(v^*xrr + rv^*xr) = a .$$

Thus N_v is a Jordan *-subalgebra of $\mathscr{L}(H)$, with unit r. Note that if $a \in N_v$, then

$$va = v(v^*xr) = v(vx^*v)^*v \in M .$$

Let (a_γ) be a net in N_v with $a_\gamma \to b$ (ultraweakly). Then $va_\gamma \to vb$ (ultraweakly), and since M is ultraweakly closed, $vb \in M$. Also $a_\gamma = ra_\gamma r$ so $b = rbr$. Thus $b = rbr = v^*(vb)r$ belongs to N_v, and N_v is ultraweakly closed.

(b) Since $va \in M$ for $a \in N_v$, φ_v, defined by $\varphi_v(a) = f(va)$ is a bounded linear functional on N_v with $\|\varphi_v\| \leqq \|f\|$. By Lemma 2.2, for $x \in M$,

$$f(x) = f(E(v)x) = f(v(v^*xr)) = \varphi_v(v^*xr),$$

so that in particular $\|f\| = \|\varphi_v\|$. Let $\tilde{\varphi}_v$ be a Hahn Banach extension of φ_v to $r\mathscr{L}(H)r$. Then

$$\tilde{\varphi}_v(r) = \varphi_v(r) = f(vr) = f(v) = \|f\| = \|\varphi_v\| = \|\tilde{\varphi}_v\|.$$

Therefore $\tilde{\varphi}_v$ is a positive functional on the C*-algebra $r\mathscr{L}(H)r$, and so φ_v is positive on N_v. Let (a_γ) be a net in N_v with $a_\gamma \to 0$ (ultraweakly). We must show that $\varphi_v(a_\gamma) \to 0$. Now $a_\gamma \to 0$ ultraweakly in N_v implies $va_\gamma \to 0$ ultraweakly in $\mathscr{L}(H,K)$. Therefore $\varphi_v(a_\gamma) = f(va_\gamma) \to 0$.

The next lemma will be used in the uniqueness of the polar decomposition.

LEMMA 2.4. *Let N be an ultraweakly closed Jordan *-subalgebra of $\mathscr{L}(H)$, and let φ be a positive ultraweakly continuous linear functional on N with support projection $e \in N$. If $x \in N$ satisfies $\varphi(x) = \|\varphi\|\,\|x\|$, then $exe = \|x\|e$.*

PROOF. The restriction of φ to the Jordan algebra eNe is faithful and assumes its norm on $y = exe$. We may assume that $\|y\| = 1$ and $\|\varphi\| = 1$. Then we must show $y = e$.

CASE 1. If $y \in N^+$, then $e - y \in (eNe)^+$, $\varphi(e-y) = 1 - 1 = 0$, so $e - y = 0$.

CASE 2. If $y = y^*$, then $y = y^+ - y^-$ implies

$$1 = \varphi(y) = \varphi(y^+) - \varphi(y^-) \leqq \varphi(y^+) \leqq 1.$$

By case 1, $y^+ = e$. Therefore $y^- = 0$ and $y = y^+ - y^- = e$.

CASE 3. For arbitrary y, $\varphi(\frac{1}{2}(y+y^*)) = \frac{1}{2}\varphi(y) + \frac{1}{2}\overline{\varphi(y)} = 1$, so $\frac{1}{2}(y+y^*) = e$ by case 2. Since e is an extreme point of the unit ball of eNe, $e = y$.

THEOREM 1. *Let M be a von Neumann J*-algebra in $\mathscr{L}(H,K)$ and let $f \in M_*$. Then there is a unique partial isometry v in M with these properties*:

(i) *$N_v = v^*Mr$ is an ultraweakly closed Jordan *-subalgebra of $\mathscr{L}(H)$, with unit $r = v^*v$;*

(ii) φ_v, *defined by* $\varphi_v(a)=f(va)$, $a \in N_v$, *is a faithful normal positive functional on* N_v;

(iii) $f(x)=\varphi_v(v^*xr)$, $x \in M$ and $f(v)=\|\varphi_v\|=\|f\|$.

PROOF. For simplicity assume $\|f\|=1$. Let v be an extreme point of the ultraweakly compact and convex non-empty set $\{x \in M: f(x)=1=\|x\|\}$. Then it follows that v is an extreme point of the unit ball of the J*-algebra M, so by Harris [17], v is a partial isometry with $f(v)=1=\|f\|$. By Lemma 2.3, v satisfies (i), (ii), (iii), except possibly for faithfulness. Let $e=\operatorname{supp}\varphi_v \in N_v$, so $e \leqq r$. Set $u=ve$ so that

$$u^*u = ev^*ve = ere = e, v^*u = e$$

and

$$f(u) = \varphi_v(v^*ur) = \varphi_v(er) = \varphi_v(e) = 1 .$$

By Lemma 2.3 again u also satisfies (i), (ii), (iii) except possibly for faithfulness. However, since $N_u=eN_ve\subset N_v$ and $\varphi_u=\varphi_v \mid N_u$, φ_u is faithful.

To prove uniqueness suppose we have two partial isometries v_1 and v_2 with $v_1^*v_1=r_1$ and $v_2^*v_2=r_2$, say, in M, each satisfying conditions (i), (ii), (iii). For $a \in N_{v_2}$,

$$\varphi_{v_2}(a) = f(v_2a) = \varphi_{v_1}(v_1^*v_2ar_1) .$$

Thus

$$\varphi_{v_1}(v_1^*v_2r_1) = \varphi_{v_1}(v_1^*v_2r_2r_1) = \varphi_{v_2}(r_2) = 1 .$$

By Lemma 2.4, $v_1^*v_2r_1=r_1=v_1^*v_1$, so that $v_1^*v_2v_1^*=v_1^*$ or $v_1v_2^*v_1=v_1$. If we now interchange the roles of v_1 and v_2, we get $v_2v_1^*v_2=v_2$. The two conditions $v_1v_2^*v_1=v_1$ and $v_2v_1^*v_2=v_2$ force $v_1=v_2$ as follows. Let $\xi \in H$. Then

$$\|v_1\xi\| = \|v_1v_2^*v_1\xi\| \leqq \|v_2^*v_1\xi\| \leqq \|v_1\xi\| .$$

Thus $\|v_2^*v_1\xi\|=\|v_1\xi\|$, and by definition of $r(v_2^*)$,

$$v_1\xi = r(v_2^*)v_1\xi = l(v_2)v_1\xi .$$

Hence $l(v_1)\leqq l(v_2)$ and by symmetry $l(v_1)=l(v_2)$. Therefore $r(v_2)=l(v_2^*)=l(v_1^*)=r(v_1)$. Finally

$$v_2 = v_2v_2^*v_2 = v_2(v_2^*v_1v_2^*)v_2 = l(v_2)v_1r(v_2) = l(v_1)v_1r(v_1) = v_1 .$$

We shall say (φ_v, N_v) *is the polar decomposition* of f, if (i), (ii), (iii) hold, and we shall write $v=v(f)$. In case M is a von Neumann algebra, and $f=u.|f|$ is the usual polar decomposition of f, then $N_v\subset M$, $u=v^*$, and $\varphi_v=$the restriction of $|f|$ to N_v. This follows easily from uniqueness.

Thus the partial isometry occurring in the polar decomposition for a functional on a von Neumann algebra (considered as a J*-algebra), is the adjoint of the natural one. Since von Neumann algebras are self-adjoint, this phenomenon cannot be observed in the category of von Neumann algebras.

Let M be a J*-algebra and let $f \in M'$. The polar decomposition of f, considered as a normal functional on the von Neumann J*-algebra M'' will be called the *enveloping polar decomposition* of f.

We shall now use the polar decomposition to define the support projections (left and right) of any subset of the pre-dual M_* of a von Neumann J*-algebra M. For each $g \in M_*$ with polar decomposition (φ_v, N_v), write $l(g) = vv^*$, $r(g) = v^*v$. Then $l(g)$ and $r(g)$ are projections in the von Neumann algebra A'', where A is any C*-algebra containing M as a J*-subalgebra. For any subset $S \subset M_*$ let

$$l(S) = l(S,A) = \sup\{l(g) : g \in S\}, \quad r(S) = r(S,A) = \sup\{r(g) : g \in S\},$$

so that $l(S)$ and $r(S)$ are projections in A''. These projections define contractive projections $\mathscr{E} = \mathscr{E}(S,A)$ and $\mathscr{T} = \mathscr{T}(S,A)$ on A'' by $\mathscr{E}z = l(S)zr(S)$ and $\mathscr{T}z = (1-l(S))z(1-r(S))$, $z \in A''$.

REMARK 2.5. Let M be a J*-algebra and let A be any C*-algebra containing M as a J*-subalgebra. For $g \in M'$, let $\tilde{g} \in A'$ denote any Hahn Banach extension of g. Then

(a) $$\tilde{g}(z) = \tilde{g}(l(g)zr(g)) \quad \text{for } z \in A'';$$

(b) $$\text{for } S \subset M' = (M'')_* \quad \text{and } h \in S$$
$$h(x) = \tilde{h}(l(S)xr(S)) \quad \text{for } x \in M''.$$

PROOF. (a)

$$\|\tilde{g}\|_{A'} \geqq \|l(g)\cdot\tilde{g}\cdot r(g)\|_{A'} \geqq \langle l(g)\cdot\tilde{g}\cdot r(g), v(g)\rangle$$
$$= \langle \tilde{g}, v(g)\rangle = \langle g, v(g)\rangle = \|g\|_{M'} = \|\tilde{g}\|_{A'}.$$

By Lemma 0.1, $\tilde{g} = l(g)\cdot\tilde{g}\cdot r(g)$.

(b) follows from (a) using $z = l(S)xr(S)$.

The following Lemma with $Q = P'$ yields the isometries which are in the statements of Theorems 2 and 3.

LEMMA 2.6. *Let Q be a contractive projection on the dual M' of a J*-algebra M. Let A be any C*-algebra containing M as a J*-subalgebra. Then the map*

$$\mathscr{E} = \mathscr{E}(Q(M'),A) : A'' \to A''$$

is isometric on $Q'(M'') \subset A''$, i.e.,

$$\|Q'a\|_{M''} = \|\mathscr{E}Q'a\|_{A''}, \qquad a \in M'' .$$

PROOF. It suffices to prove $\|Q'a\|_{M''} \leqq \|\mathscr{E}Q'a\|_{A''}$. For any $g \in M'$, let $\tilde{h} \in A'$ denote any Hahn Banach extension of $h = Qg$. Then $\langle Q'a, g\rangle = \langle Q'a, Qg\rangle = \langle Q'a, h\rangle = \langle \mathscr{E}Q'a, \tilde{h}\rangle$ (by Remark 2.5.(b)). Thus

$$|\langle Q'a, g\rangle| \leqq \|\mathscr{E}Q'a\|_{A''}\|\tilde{h}\|_{A'} = \|\mathscr{E}Q'a\|_{A''}\|Qg\|_{M'} \leqq \|\mathscr{E}Q'a\|_{A''}\|g\|_{M'} .$$

Since g is arbitrary, $\|Q'a\|_{M''} \leqq \|\mathscr{E}Q'a\|_{A''}$.

The following is a version of Theorem 3 which does not assume that the range of P is finite dimensional. Its proof involves a reduction to the case of a positive projection on a JC-algebra and uses ideas from Effros–Størmer [12].

THEOREM 2. *Let M be a* J**-algebra and let P: $M \to M$ be a contractive projection. Suppose there is an element $f = P'f$ such that $E(f)P' = P'$, where $E(f) = E(v(f))$. Then $E(f)P''(M'')$ is a* J**-subalgebra of M'' which is linearly isometric to $P''(M'')$.*

The assumption in Theorem 2 is satisfied in the following cases: $Px = \frac{1}{2}(x + x^{\mathrm{T}})$, $P(M)$ = symmetric matrices; $Px = \frac{1}{2}(x - x^{\mathrm{T}})$, $P(M)$ = anti symmetric n by n matrices, n even or $n = \infty$; $P(M)$ = spin factors, cf. Effros–Størmer [12; Lemma 2.3]; $Pf(x) = \frac{1}{2}(f(x) - f(-x))$, $P(M)$ = continuous odd functions on R [14, Example 2].

The Corollary of the following Lemma is needed for the proof of Theorem 2.

LEMMA 2.7. *Let M be a von Neumann* J**-algebra and let $f \in M_*$ have polar decomposition (φ_v, N_v). If $x \in M$ satisfies $f(x) = 1 = \|x\| = \|f\|$, then $E(v)x = v$, $G(v)x = 0$, and therefore $x = v + F(v)x$.*

PROOF. $1 = f(x) = \varphi_v(v^*xr)$. By Lemma 2.4, $v^*xr = r$ and $E(v)x = vv^*xr = vr = v$. We show next that $G(v)x = 0$. As noted above $r = v^*xr$. Therefore

$$\begin{aligned} 1 \geqq \|v^*x\|^2 &= \|v^*xr + v^*x(1-r)\|^2 = \|r + v^*x(1-r)\|^2 \\ &= \|(r + v^*x(1-r))(r + v^*x(1-r))^*\| \\ &= \|r + v^*x(1-r)(v^*x(1-r))^*\| . \end{aligned}$$

Since $v^*x(1-r)(v^*x(1-r))^*$ belongs to $(r\mathscr{L}(H)r)^+$, $v^*x(1-r) = 0$. Thus $lx(1-r) = vv^*x(1-r) = 0$. By a similar argument $(1-l)xr = 0$.

COROLLARY 2.8. *Let Q be a contractive projection on the dual M' of a* J*-*algebra M and let (φ_v, N_v) be the enveloping polar decomposition of f. Then $E(v)Q'v=v$ and $Q'v=v+F(v)Q'v$.*

PROOF. $\langle Q'v, f\rangle = \langle v, Qf\rangle = \langle v, f\rangle = \|f\|$ by Theorem 1. Now apply Lemma 2.7.

PROOF OF THEOREM 2. Let (φ_v, N_v) be the enveloping polar decomposition of f. By Lemma 2.3 (a), $N_v = v^*M''r$ is an ultraweakly closed Jordan *-subalgebra of $\mathscr{L}(H)$. Define $\tilde{P}$: $N_v \to N_v$ by

$$\tilde{P}a \;=\; v^*E(f)P''(va) \;=\; v^*P''(va)r, \qquad (a \in N_v)\,.$$

Then $\tilde{P}$ is contractive, idempotent (since $P''E(f)=P''$), unital ($\tilde{P}r=r$ by Corollary 2.8 with $Q=P'$) and ultraweakly continuous. In particular $\tilde{P}$ is positive.

We show that $\tilde{P}$ is faithful. Let $a \in N_v^+$ and suppose $\tilde{P}a=0$. Since $v\tilde{P}a \in M''$, we have

$$0 \;=\; \langle v\tilde{P}a, f\rangle \;=\; \langle lP''(va)r, f\rangle \;=\; \langle va, f\rangle \;=\; \varphi_v(a)\,.$$

Since φ_v is faithful, $a=0$ so $\tilde{P}$ is faithful.

By [12, Corollary 1.5], $\tilde{P}(N_v)$ is a Jordan *-subalgebra of N_v. We show next that $E(f)P''(M'')$ is closed under the operation $b \to bb^*b$. If $b \in E(f)P''(M'')$, then $v^*b \in N_v$ and

$$\tilde{P}(v^*b) \;=\; v^*E(f)P''(vv^*b) \;=\; v^*E(f)P''b \;=\; v^*b\,.$$

Since $\tilde{P}(N_v)$ is in particular a J*-subalgebra of N_v, we have

$$v^*bb^*b \;=\; (v^*b)(v^*b)^*(v^*b) \in \tilde{P}(N_v)\,.$$

Therefore $v^*bb^*b = \tilde{P}(v^*bb^*b) = v^*P''(bb^*b)r$. This implies

$$bb^*b \;=\; vv^*bb^*b \;=\; vv^*P''(bb^*b)r \;=\; E(f)P''(bb^*b)\,.$$

Thus $E(f)P''(M'')$ is a J*-subalgebra of M''. Since $E(f)P'=P'$ implies that $E(f) = \mathscr{E}(P'(M'), A)\,|\,M$ for any C*-algebra A containing M as a J*-subalgebra, it follows from Lemma 2.6 that $E(f)P''(M'')$ is isometric to $P''(M'')$.

The following gives some important commutativity relations.

LEMMA 2.9. *Let M be a* J*-*algebra and let $f, g \in M'$. Then*

(i) $f=E(g)f \Leftrightarrow l(f)\leqq l(g)$ and $r(f)\leqq r(g)$;
(ii) $f=F(g)f \Leftrightarrow l(f)l(g)=0=r(g)r(f)$ (*f and g are then said to be orthogonal*);
(iii) $f=G(g)f \Rightarrow l(f)l(g)=l(g)l(f)$ *and* $r(f)r(g)=r(g)r(f)$.

In particular if f and g satisfy one of the three (mutually exclusive) relations $f=E(g)f$, $f=F(g)f$, $f=G(g)f$, then $l(f)l(g)=l(g)l(f)$ and $r(f)r(g)=r(g)r(f)$, and therefore $\{E(f),F(f),G(f),E(g),F(g),G(g)\}$ is a commutative family of operators.

PROOF. (i) Assume $f=E(g)f$ and let $v=v(f)$. Then

$$\|f\| = f(v) = f(l(g)vr(g)),$$

so by Lemma 2.7, $l(g)vr(g)=v+F(f)(l(g)vr(g))$. Hence

$$v^*l(g)vr(g) = v^*v = r(f), \quad l(g)vr(g)v^* = vv^* = l(f)$$

and therefore $l(f)\leqq l(g)$ and $r(f)\leqq r(g)$. The converse is trivial.

(ii) By similar argument $f=F(g)f$ implies $l(f)\leqq 1-l(g)$ and $r(f)\leqq 1-r(g)$.

(iii) Assume $f=G(g)f$ and let $v=v(f)$, $u=v(g)$. Then

$$\|f\| = f(v) = f(G(g)v),$$

so by Lemma 2.7, $G(g)v=v+F(f)G(g)v$. Hence

$$v^*(G(g)v) = v^*v \quad \text{and} \quad (G(g)v)v^* = vv^*.$$

Thus

$$v^*v = v^*((1-uu^*)vu^*u+uu^*v(1-u^*u)) \tag{2.1}$$

and

$$v^*vu^*u = v^*vu^*u-v^*uu^*vu^*u,$$

so that $v^*uu^*vu^*u = 0$. Therefore (2.1) becomes $v^*v=v^*vu^*u+v^*uu^*v$ and

$$\begin{aligned} r(g)r(f) &= u^*uv^*v = u^*u(v^*vu^*u+v^*uu^*v) \\ &= r(g)r(f)r(g)+(v^*uu^*vu^*u)^* = r(g)r(f)r(g), \end{aligned}$$

which by taking adjoints implies $r(g)r(f)=r(f)r(g)$. A similar proof using the equation $(G(g)v)v^*=vv^*$ shows that $l(g)l(f)=l(f)l(g)$.

REMARK 2.10. In case M is a C*-algebra, for any projections $l,r\in M''$ we have

(iv) $f=l.f.r \Leftrightarrow l(f)\leqq l$ and $r(f)\leqq r$.

The proof is the same as in Lemma 2.9 (i).

3. The main theorem.

Throughout this section M will be a J*-algebra and Q will denote a contractive projection on the dual M' of M. In the proof of Theorem 3 we shall put $Q = P'$, where P is a contractive projection on M, so that $P'' = Q'$ will be an ultraweakly continuous contractive projection on the von Neumann J*-algebra M''. It will be easily seen that the entire section can be phrased in terms of ultraweakly continuous projections on von Neumann J*-algebras.

For a partial isometry v in M'' the projections $E(v)$, $G(v)$, $F(v)$ will be considered as defined either on M' or on M'', and if $f \in M'$ has enveloping polar decomposition $(\varphi_v, \mathrm{N}_v)$, we shall write $E(f)$ for $E(v)$, etc. If M is embedded in a C*-algebra A, then since $M'' \subset A''$ each of the projections $E(v)$, $G(v)$, $F(v)$ with $v \in M''$ also acts on A'' and A'.

The first lemma in this section is a simple reformulation of Lemma 0.1 in a more general setting.

LEMMA 3.1. *Let M be a J*-algebra, let $g \in M'$ and let v be a partial isometry in M''. Then*

(i) $\|E(v)g\| = \|g\|$ *implies* $E(v)g = g$
(ii) $\|F(v)g\| = \|g\|$ *implies* $F(v)g = g$.

PROOF. (i) Let $\tilde{g} \in A'$ be a Hahn Banach extension of g. Then

$$\|\tilde{g}\|_{A'} = \|g\|_{M'} = \|E(v)g\|_{M'} \leqq \|E(v)\tilde{g}\|_{A'} \leqq \|\tilde{g}\|_{A'}\,.$$

By Lemma 0.1, $E(v)\tilde{g} = \tilde{g}$ and by restriction $E(v)g = g$. The proof of (ii) is similar.

The following remark is an easy consequence of the construction of the polar decomposition.

REMARK 3.2. Let v be a partial isometry in the second dual M'' of a J*-algebra M. Then

(a) $E(v)(M'') \ni x \to v^*x \in N_v$ is an isometric isomorphism onto with inverse $a \to va$.
(b) The map $\Phi\colon E(v)(M') \to N'_v$ defined by $\langle \Phi(g), a\rangle = \langle g, va\rangle$ for $g \in E(v)(M')$, $a \in N_v$, is an isometric linear isomorphism onto a subspace B of N'_v.
(c) The dual B' of B is isometrically isomorphic with $E(v)(M'')$ and therefore by (a), $B' \cong N_v$.

The following is the first of several commutativity formulas.

PROPOSITION 3.3. *Let Q be a contractive projection on the dual M' of a* J*-*algebra M and let $f=Qf$. Then with $E=E(f)$ we have*

(i) *QE is a contractive projection on M';*
(ii) *$QE=EQE$;*
(iii) *$EQ'=EQ'E$.*

PROOF. (i) and (iii) follow from (ii). To prove (ii) we shall first use Remark 3.2 (b) with $v=v(f)$. Let $T\colon B\to E(v)(M')$ be the inverse of Φ and let $\tilde{Q}=QT$. Then $\tilde{Q}(B)=QE(v)(M')$, so to prove (ii) it suffices to show that $\tilde{Q}(B)\subset E(v)(M')$. By Remark 3.2 (c), B is the space of ultraweakly continuous linear functionals on N_v. By Theorem 1, $\varphi_v\in B$ is faithful. Since B is linearly spanned by B^+, it suffices, by Lemma 0.2, to prove that $\tilde{Q}(S)\subset E(v)(M')$, where S is the face in B^+ generated by φ_v i.e.,

$$S=\{\tau\in B^+ : \tau\leqq \text{const.}\,\varphi_v\}.$$

For $\tau\in S$ we can write $\varphi_v=\alpha\tau+\sigma$, with $\tau,\sigma\in B^+$ and $\alpha>0$. Then

$$f=\tilde{Q}\varphi_v=\alpha\tilde{Q}\tau+\tilde{Q}\sigma \quad\text{and}\quad f=E(v)f=\alpha E(v)\tilde{Q}\tau+E(v)\tilde{Q}\sigma.$$

Now

$$\|f\|=\|\varphi_v\|=\alpha\|\tau\|+\|\sigma\|\geqq\alpha\|\tilde{Q}\tau\|+\|\tilde{Q}\sigma\|$$
$$\geqq\alpha\|E(v)\tilde{Q}\tau\|+\|E(v)\tilde{Q}\sigma\|\geqq\|f\|.$$

Thus $\|\tilde{Q}\tau\|=\|E(v)\tilde{Q}\tau\|$, and by Lemma 3.1, $\tilde{Q}\tau=E(v)\tilde{Q}\tau$.

LEMMA 3.4. *Let Q be a contractive projection on the dual M' of a* J*-*algebra M, and let $f=Qf$. Then $F(f)QG(f)=0$.*

PROOF. Denote $E(f)$, $G(f)$, $F(f)$ by E, G, F respectively. Let A be a C*-algebra which contains M as a J*-subalgebra and identify A with its universal representation. Let $h\in M'$ and set $g=Gh$. We shall show $FQg=0$. To this end let $\tilde{h}\in A'$ be a Hahn Banach extension of h, and let

$$g_1=l.\tilde{h}.(1-r)\,|\,M,\qquad g_2=(1-l).\tilde{h}.r\,|\,M$$

(where $l=l(f)$, $r=r(f)$). Then $g=g_1+g_2$ and we shall prove $FQg_1=0=FQg_2$. We can find vectors ξ,η such that $g_1=\omega(\xi,\eta)$, $l\eta=\eta$, $r\xi=0$, and $\|\eta\|=1$. Therefore

$$(v^*\eta,\xi)=(v^*l\eta,\xi)=(rv^*\eta,\xi)=0.$$

For scalar t, let $\omega_t=g_3+tg_1$, where $g_3=\omega(v^*\eta,\eta)=Eg_3$.

By Proposition 3.3, $FQg_3 = FQEg_3 = FEQEg_3 = 0$. Thus to prove $FQg_1 = 0$, it suffices to prove $FQ\omega_t = 0$ for all t. To this end we first estimate the norm of ω_t:

$$\|\omega_t\| \leqq \|v^*\eta + t\xi\| = (\|v^*\eta\|^2 + \|t\xi\|^2)^{\frac{1}{2}} \leqq (1 + \|t\xi\|^2)^{\frac{1}{2}} .$$

We next show that $\|EQ\omega_t\| \geqq 1$. By Corollary 2.8, $EQ'v = v$. Thus

$$\langle Qg_3, v\rangle = \langle g_3, EQ'v\rangle = \langle g_3, v\rangle = \|\eta\|^2 = 1 ,$$

so $\langle Q\omega_t, v\rangle = 1 + t\langle Qg_1, v\rangle$, and therefore

$$|1 + t\langle Qg_1, v\rangle| \leqq \|Q\omega_t\| \leqq \|\omega_t\| \leqq (1 + |t|^2\|\xi\|^2)^{\frac{1}{2}} .$$

Since t is arbitrary, $\langle Qg_1, v\rangle = 0$ so that $\langle EQ\omega_t, v\rangle = \langle Q\omega_t, v\rangle = 1$, and $\|EQ\omega_t\| \geqq 1$.

Finally by Lemma 1.1,

$$\begin{aligned} 1 + |t|\|FQg_1\| &= 1 + \|FQ\omega_t\| \leqq \|EQ\omega_t\| + \|FQ\omega_t\| \\ &= \|(E+F)Q\omega_t\| \leqq \|\omega_t\| \leqq (1 + |t|^2\|\xi\|^2)^{\frac{1}{2}} \end{aligned}$$

and this forces $\|FQg_1\| = 0$. This proves $FQg_1 = 0$, and a similar proof shows $FQg_2 = 0$.

We now have the following additional commutativity formulas.

PROPOSITION 3.5. *Let Q be a contractive projection on the dual M' of a* J*-*algebra M and let $f = Qf$. Then, with $F = F(f)$ and $G = G(f)$,*

(i) FQ and GQ *are contractive projections*;
(ii) $FQ = FQF = QFQ$;
(iii) $Q'F = FQ'F = Q'FQ'$.

PROOF. $FQ = FQ(E+F+G) = FQE + FQF + FQG = FQF$ by Proposition 3.3 and Lemma 3.4. Hence FQ is a contractive projection. To complete the proof of (ii) let $g \in M'$. Then

$$\|FQg\| = \|FQFQg\| \leqq \|QFQg\| \leqq \|FQg\| .$$

By Lemma 3.1, $QFQg = FQFQg = FQg$. Thus (ii) is proved and (iii) follows immediately from (ii). Finally

$$\begin{aligned} GQ = GQ(E+F+G)Q &= G(QE)Q + G(QFQ) + GQGQ \\ &= GEQEQ + GFQ + GQGQ = GQGQ , \end{aligned}$$

since $GE = GF = 0$.

DEFINITION 3.6. Let Q be a contractive projection on the dual M' of a J*-algebra M. By an atom of Q, we mean any extreme point of the unit sphere $Q(M')_1$ of $Q(M')$.

The main property of atoms is contained in Proposition 3.7.

PROPOSITION 3.7. *Let Q be a contractive projection on the dual M' of a J*-algebra M and let f be an atom of Q with enveloping polar decomposition (φ_v, N_v). Then with $E = E(f)$,*

(i) *QE is one dimensional;*
(ii) *$QE = \langle \cdot, v \rangle f$*
(iii) *$EQ' = \langle f, \cdot \rangle v$.*

PROOF. Let B be the space defined in Remark 3.2 (b) with $v = v(f)$, let T: $B \to E(M')$ be the inverse of Φ, and let S be the face in B^+ generated by φ_v. For $\tau \in S$, we can write

$$\varphi_v = \alpha\|\tau\|^{-1}\tau + (1-\alpha)\|\sigma\|^{-1}\sigma$$

with $\tau, \sigma \in S$ and $\alpha \in [0,1]$. Then

$$f = Qf = QT\varphi_v = \alpha QT(\|\tau\|^{-1}\tau) + (1-\alpha)QT(\|\sigma\|^{-1}\sigma)$$

and since f is an atom, $f = QT(\|\tau\|^{-1}\tau)$ that is,

$$QT(\tau) = \|\tau\| f = \langle \tau, r \rangle f, \quad \tau \in S. \tag{3.1}$$

By Theorem 1, φ_v is faithful on N_v; by Lemma 0.2, S is norm dense in B^+. Therefore (3.1) holds for all $\tau \in B^+$. Since B is linearly spanned by B^+, (3.1) holds for each $\tau \in B$. Finally if $g \in M'$

$$\begin{aligned} QEg &= QT\Phi(Eg) = \langle \Phi(Eg), r \rangle f = \langle Eg, vr \rangle f \\ &= \langle g, v \rangle f. \end{aligned}$$

The next lemma is the key step in moving from a local result to a global result. Its proof is rather lengthy and is deferred to section 4.

LEMMA 3.8. *Let Q be a contractive projection on the dual M' of a J*-algebra M with $Q(M')$ finite dimensional, and let f be an atom of Q with enveloping polar decomposition (φ_v, N_v). Then for every $g \in Q(M')$, $F(g)F(f)Q'v = F(f)Q'v$.*

Let P be a contractive projection on a J*-algebra M, and let $S = P'(M')$. For any C*-algebra A which contains M as a J*-subalgebra recall that $\mathscr{E}$

$=\mathscr{E}(P'(M'),A)$ and $\mathscr{T}=\mathscr{T}(P'(M'),A)$ are contractive projections on A'' defined by

$$\mathscr{E}z = l(S)zr(S), \quad \mathscr{T}z = (1-l(S))z(1-r(S)), \quad z \in A'' ,$$

where

$$l(S) = \sup\{l(g) : g \in P'(M')\} \in A''$$

and

$$r(S) = \sup\{r(g) : g \in P'(M')\} \in A'' .$$

COROLLARY 3.9. *Let P be a contractive projection on a* J*-*algebra M and let f be an atom of P′ with enveloping polar decomposition* (φ_v, N_v). *For any* C*-*algebra A containing M as a* J*-*algebra,*

$$F(f)P''v = \mathscr{T}P''v , \tag{3.2}$$

where $\mathscr{T}=\mathscr{T}(P'(M''),A)$; *and*

$$\mathscr{E}P''v = v , \tag{3.3}$$

where $\mathscr{E}=\mathscr{E}(P'(M'),A)$.

PROOF. Set $a=F(f)P''v$, $L=l(P'(M'))$, $R=r(P'(M'))$. By Lemma 3.8 with $Q=P'$, we have $F(g)a=a$ for every $g \in P'(M')$, that is, $(1-l(g))a(1-r(g))=a$ or $l(g)a=0=ar(g)$ for all $g \in P'(M')$. This entails $La=0=aR$, that is, $\mathscr{T}a=a$. Hence

$$\mathscr{T}P''v = (\mathscr{T}F(f))P''v = \mathscr{T}a = a$$

and (3.2) is proved. Corollary 2.8 (with $Q=P'$) and (3.2) imply (3.3).

THEOREM 3. *Let M be a* J*-*algebra and let* $P\colon M \to M$ *be a linear projection of norm one*: $P^2=P$, $\|P\|=1$. *Suppose the range* $P(M)$ *of P is finite dimensional. Then* $P(M)$ *is a* C*-*triple system with the triple product* $\{abc\}=\frac{1}{2}P(ab^*c+cb^*a)$, $a,b,c \in P(M)$.

More precisely, with $\mathscr{E}=\mathscr{E}(P'(M'),A)$ *where A is any* C*-*algebra containing M as a* J*-*algebra,* $\mathscr{E}P(M)$ *is a* J*-*subalgebra of* A'' *and* $\mathscr{E}Px \to Px$ *is a* C*-*triple system isomorphism of* $\mathscr{E}P(M)$ *onto* $P(M)$.

PROOF. We shall show that for any $a \in M$ we have

$$(\mathscr{E}Pa)(\mathscr{E}Pa)^*(\mathscr{E}Pa) = \mathscr{E}P(Pa(Pa)^*Pa) . \tag{3.4}$$

Then (3.4) shows that $\mathscr{E}P(M)$ is a J*-subalgebra of A'' and Lemma 2.6 says that the map T: $\mathscr{E}P(M) \to P(M)$ defined by $\mathrm{T}\mathscr{E}Pa = Pa$ is a linear isometry. If we apply T to (3.4) we obtain

$$\mathrm{T}((\mathscr{E}Pa)(\mathscr{E}Pa)^*(\mathscr{E}Pa)) = P(Pa(Pa)^*Pa) \tag{3.5}$$

which shows (via polarization) that the triple product $\{abc\} = \frac{1}{2}P(ab^*c + cb^*a)$ for $a, b, c \in P(M)$ is the transport of the Jordan triple system structure on $\mathscr{E}P(M)$ given by its J*-algebra structure. It remains to prove (3.4). To this end we show first that if $a \in P(M)$, then there exist mutually orthogonal partial isometries $v_1, v_2, \ldots, v_n$ in M'' and scalars $\alpha_1, \ldots, \alpha_n \in [0, \infty)$ such that

$$\mathscr{E}a = \sum_{i=1}^{n} \alpha_i v_i . \tag{3.6}$$

To prove (3.6) first choose an atom f_1 of P' such that $\langle f_1, a\rangle = \|a\|$. The functional f_1 can be obtained as an extreme point of the weak*-compact convex set

$$\{f \in P'(M') : \langle f, a\rangle = \|a\| \quad \text{and} \quad \|f\| = 1\} .$$

Then letting (φ_{v_1}, N_{v_1}) be the enveloping polar decomposition of f_1 we have

$$\mathscr{E}P''(v_1) = v_1 \quad \text{(by Corollary 3.9) and} \tag{3.7}$$

$$a = \alpha_1 v_1 + F(f_1)a \quad \text{(by Lemma 2.7)} . \tag{3.8}$$

We now apply $\mathscr{E}P''$ to (3.8) obtaining

$$\mathscr{E}a = \alpha_1 v_1 + \mathscr{E}P''F(f_1)a . \tag{3.9}$$

Since $P(M)$ is finite dimensional, we have $P''(M'') = P(M)$. Let $a_2 = P''F(f_1)a \in P(M)$. Then by Proposition 3.5 (iii), $a_2 = F(f_1)a_2$. This implies

$$l(a_2) \leqq 1 - l(f_1) \quad \text{and} \quad r(a_2) \leqq 1 - r(f_1) .$$

If $a_2 \neq 0$, then since $a_2 \in P(M)$, we can choose an atom f_2 of P' with enveloping polar decomposition (φ_{v_2}, N_{v_2}) such that $a_2 = \alpha_2 v_2 + F(f_2)a_2$. This implies that $l(f_2) \leqq l(a_2)$ and $r(f_2) \leqq r(a_2)$, so that v_1 and v_2 are orthogonal and

$$\mathscr{E}a_2 = \mathscr{E}P''a_2 = \alpha_2 v_2 + \mathscr{E}P''F(f_2)a_2 .$$

Now

$$P''F(f_2)a_2 = P''F(f_2)P''F(f_1)a = P''F(f_2)F(f_1)a = P''F(f_1 + f_2)a .$$

Therefore

$$\mathscr{E}a = \alpha_1 v_1 + \mathscr{E}P''F(f_1)a = \alpha_1 v_1 + \mathscr{E}a_2 = \alpha_1 v_1 + \alpha_2 v_2 + \mathscr{E}P''F(f_1 + f_2)a .$$

Set $a_3 = P''F(f_1 + f_2)a$, so that $a_3 \in P(M) \cap F(f_1 + f_2)(M)$. If $a_3 \neq 0$, we can choose an atom f_3 of P' with enveloping polar decomposition (φ_{v_3}, N_{v_3}) such that

$$\mathscr{E}a = \sum_{i=1}^{3} \alpha_i v_i + \mathscr{E}P''F(f_1+f_2+f_3)a ,$$

with v_1, v_2, v_3 mutually orthogonal. Continuing in this way, we arrive at (3.6) (with $v_1, v_2, \ldots, v_n$ mutually orthogonal), by the finite dimensionality of $P(M)$.

Note that (3.6) shows that $\mathscr{E}P''(M'') = \mathscr{E}P(M) \subset M''$. We show next that

$$a = \mathscr{E}a + \mathscr{T}a \quad \text{for } a \in P(M) . \tag{3.11}$$

To see this set

$$b = P''(\sum \alpha_i v_i) = \sum \alpha_i v_i + \mathscr{T}P''(\sum \alpha_i v_i) = \mathscr{E}a + \mathscr{T}b \text{ (By Corollary 3.9)} .$$

Thus $\mathscr{E}(b-a)=0$, and since $b-a \in P(M)$, Lemma 2.6 implies $b-a=0$. Thus $a=b=\mathscr{E}a+\mathscr{T}b=\mathscr{E}a+\mathscr{T}a$, so (3.11) is proved. We can now prove (3.4). If $a \in P(M)$, then $a=\mathscr{E}a+\mathscr{T}a$ so that

$$aa^*a = \mathscr{E}a(\mathscr{E}a)^*\mathscr{E}a + \mathscr{T}a(\mathscr{T}a)^*\mathscr{T}a . \tag{3.12}$$

Note that since $aa^*a \in M$ and $\mathscr{E}a(\mathscr{E}a)^*\mathscr{E}a = \sum \alpha_i^3 v_i \in M''$, we have $\mathscr{T}a(\mathscr{T}a)^*\mathscr{T}a \in M''$, so we can apply $\mathscr{E}P''$ to (3.12) to get

$$\mathscr{E}P(aa^*a) = \mathscr{E}a(\mathscr{E}a)^*\mathscr{E}a + \mathscr{E}P''(\mathscr{T}a(\mathscr{T}a)^*\mathscr{T}a) .$$

It remains only to prove $P''(\mathscr{T}a(\mathscr{T}a)^*\mathscr{T}a)=0$. To see this let $x=\mathscr{T}a(\mathscr{T}a)^*\mathscr{T}a$. For arbitrary $f \in M'$, let $\tilde{f} \in A'$ be a Hahn Banach extension of $P'f$. Then by Remark 2.5 (b),

$$\langle P''x, f\rangle = \langle x, P'f\rangle = \langle x, \tilde{f}\rangle = \langle \mathscr{E}x, \tilde{f}\rangle = 0 ,$$

since $\mathscr{E}x = \mathscr{E}\mathscr{T}x = 0$.

4. Proof of the main lemma (Lemma 3.8).

This section is organized as follows. We first reduce the proof of Lemma 3.8 to three cases in Remark 4.4 (b). The first two cases are easy and are dealt with in Remark 4.5 and Lemma 4.6 respectively. The remaining case is more complicated and necessitates a further reduction to two cases (Lemma 4.7). These final two cases are proved under the additional assumption that g is an atom of $G(f)Q$ in Lemma 4.9 and Lemma 4.10. This suffices for the present paper since in Theorem 3 we assume that $P(M)$ is finite dimensional.

The first lemma in this section is a significant generalization of Lemma 0.1 and it uses Lemma 0.1 in its proof. Lemma 4.1 will be used in the sequel only when there is a partial isometry v with $v^*v=r$ and $vv^*=l$. However this assumption is not needed for its proof.

LEMMA 4.1. *Let A be a C*-algebra, let l and r be projections in A'' and suppose $\varphi, \psi \in A'$ satisfy*

$$\|l.\psi.r+l.\varphi.(1-r)+(1-l).\varphi.r\| = \|l.\varphi.(1-r)+(1-l).\varphi.r\| .$$

Then $l.\psi.r=0$.

If we write $E\psi=l.\psi.r$ amd $G\varphi=l.\varphi.(1-r)+(1-l).\varphi.r$, then since $\|E\psi+G\varphi\| \geqq \|G\varphi\|$, it suffices to prove that $\|E\psi+G\varphi\| \leqq \|G\varphi\|$ implies $E\psi=0$.

PROOF. Set $g=l.\varphi.(1-r)$, $h=(1-l).\varphi.r$, so that $G\varphi=g+h$ and consider a decomposition

$$E\psi = k_{11}+k_{12}+k_{21}+k_{22}$$

given by the block matrix support Diagram 1.

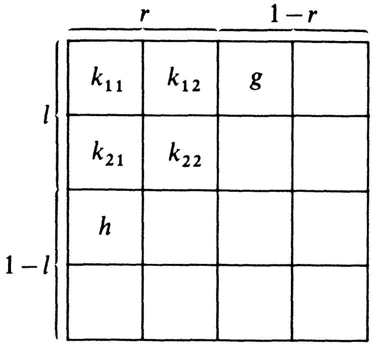

Diagram 1.

Explicitly:

$$k_{11} = l(g).\psi.r(h), \qquad k_{12} = l(g).\psi(r-r(h)),$$
$$k_{21} = (l-l(g)).\psi.r(h), \quad k_{22} = (l-l(g)).\psi.(r-r(h)).$$

We shall prove in steps that $k_{22}=0$, $k_{21}=0=k_{12}$, and finally $k_{11}=0$. The first two steps depend on the easily verified facts that the projections P_1 and P_2 on A' defined by

$$P_1 f = l(g).f.r(g)+(l-l(g)).f.(r-r(h))+l(h).f.r(h)$$

and

$$P_2 f = (l(h)+(l-l(g))).f.r(h)+l(g).f.(r(g)+(r-r(h)))$$

are contractive. These projections are represented schematically by Diagram 2:

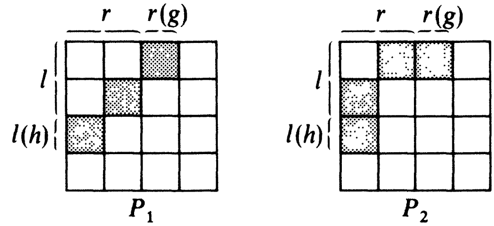

Diagram 2.

STEP 1.

$$\begin{aligned}\|g\|+\|h\| &= \|g+h\| = \|G\varphi\| = \|E\psi+G\varphi\| \\ &\geqq \|P_1(E\psi+G\varphi)\| = \|g+h+k_{22}\| = \|g\|+\|h\|+\|k_{22}\| .\end{aligned}$$

Therefore $k_{22}=0$.

STEP 2.

$$\begin{aligned}\|g\|+\|h\| &= \|g+h\| = \|G\varphi\| = \|E\psi+G\varphi\| \\ &\geqq \|P_2(E\psi+G\varphi)\| = \|k_{21}+k_{12}+g+h\| = \|g+k_{12}\|+\|h+k_{21}\| \\ &\geqq \|(g+k_{12}).r(g)\|+\|l(h).(h+k_{21})\| = \|g\|+\|h\| .\end{aligned}$$

Thus by Lemma 0.1,

$$h+k_{21} = l(h).(h+k_{21}) = h$$

and

$$g+k_{12} = (g+k_{12}).r(g) = g ,$$

so $k_{21}=0=k_{12}$.

STEP 3. We now prove that $k_{11}=0$. Let (φ_u, N_u) and (φ_w, N_w) be the enveloping polar decompositions of g and h respectively and set $v=u+w$. Since g and h are orthogonal,

$$\begin{aligned}\langle G\varphi, v\rangle &= \langle g+h, v\rangle = \langle g+h, u+w\rangle \\ &= \langle g,u\rangle+\langle h,w\rangle = \|g\|+\|h\| = \|g+h\| = \|G\varphi\| .\end{aligned}$$

Let the isometry Φ: $E(v)(A') \to (N_v)_*$ be as defined in Remark 3.2 (b), and let $r_1=v^*v$. Then

$$\langle k_{11}, v\rangle = \langle k_{11}, l(g)vr(h)\rangle = \langle k_{11}, l(g)(u+w)r(h)\rangle = 0 ,$$

since $ur(h)=l(g)w=0$. Also

$$\langle \Phi(G\varphi + k_{11}), r_1 \rangle = \langle G\varphi + k_{11}, vr_1 \rangle = \langle G\varphi, v \rangle = \|G\varphi\|$$
$$= \|G\varphi + k_{11}\| \quad \text{(by assumption)} .$$

Thus $\Phi(G\varphi + k_{11}) \in (N_v)_*^+$ and $\Phi G\varphi \in (N_v)_*^+$, since r_1 is the unit of N_v. Therefore $\Phi(k_{11})$ is a self-adjoint functional on N_v.

Choose $a \in A''$ such that $\|a\| = 1$ and $\langle k_{11}, a \rangle = \|k_{11}\|$. We may assume that $a = l(g)ar(h)$. Then $b = v^*a \in N_v$ and

$$\langle \Phi(k_{11}, b \rangle = \langle k_1, vb \rangle = \langle k_{11}, a \rangle = \|k_{11}\| .$$

But

$$\overline{\langle \Phi(k_{11}), b \rangle} = \langle \Phi(k_{11}), b^* \rangle = \langle k_{11}, va^*v \rangle$$
$$= \langle l(g) . k_{11} . r(h), vr(h)a^*l(g)v \rangle = 0 ,$$

since $l(g)vr(h) = uu^*(u+w)w^*w = 0$. Therefore $\|k_{11}\| = 0$.

COROLLARY 4.2. *Let M be a* J*-*algebra, and let $f, g, h \in M'$. Then*

(i) $\|E(f)g + G(f)h\| = \|G(f)h\|$ *implies* $E(f)g = 0$;
(ii) $\|F(f)g + G(f)h\| = \|G(f)h\|$ *implies* $F(f) = 0$.

PROOF. It suffices to prove (i). Let A be any C*-algebra which contains M as a J*-subalgebra. Let E, F, G denote $E(f)$, $F(f)$, $G(f)$, respectively. The functional $Eg + Gh \in M'$ vanishes on $F(M'')$, and therefore there is a Hahn Banach extension $\varphi \in A'$ of $Eg + Gh$ which vanishes on $F(A'')$. Note that $E\varphi + G\varphi = (1 - F)\varphi = \varphi$. Thus

$$\|E\varphi + G\varphi\| = \|\varphi\| = \|Eg + Gh\| = \|Gh\| \leqq \|G\varphi\|$$

the last inequality being true, since $G\varphi$ is an extension of Gh. By Lemma 4.1, $E\varphi = 0$. Thus for $x \in M$,

$$0 = E\varphi(x) = \varphi(Ex) = (Eg + Gh)(Ex) = Eg(x) .$$

Throughout the rest of this section Q denotes a contractive projection on the dual M' of a J*-algebra M and (φ_v, N_v) is the enveloping polar decomposition of an $f \in Q(M')$.

We can now state yet another collection of commutativity formulas.

PROPOSITION 4.3. (i) $G(f)Q = QG(f)Q$; (ii) $E(f)Q = QE(f)Q$; *in particular $E(f)Q$ is a contractive projection.*

PROOF. Let $E = E(f)$, $G = G(f)$, $F = F(f)$.

(i) $QGQ = EQGQ + FQGQ + GQGQ = EQGQ + GQ$ by Lemma 3.4 and Proposition 3.5. Thus for arbitrary $g \in M'$,

$$\|E(QGQg) + G(Qg)\| = \|QGQg\| \leqq \|G(Qg)\|$$

so by Corollary 4.2, $EQGQg = 0$. This proves (i).

(ii) $EQ = (1 - G - F)Q = Q - GQ - FQ = Q - QGQ - QFQ = Q(1 - G - F)Q = QEQ$.

REMARK 4.4. (a) Let $f, g \in Q(M')$ and suppose that $g = g_1 + g_2 + \ldots + g_n$ with $g_i \in Q(M')$ and that Lemma 3.8 is true for the pairs (f, g_i), $i = 1, 2, \ldots, n$, that is, $F(g_i)F(f)Q'v = F(f)Q'v$. Then Lemma 3.8 is true for the pair (f, g). Indeed, set $a = F(f)Q'v$. For any $h \in M'$,

$$F(h)a = a \Leftrightarrow l(h)a = 0 = ar(h)\,.$$

Set $\tilde{l} = \sup\{l(g_i)\}$, $\tilde{r} = \sup\{r(g_i)\}$. Then $\tilde{l} \geqq l(g)$, $\tilde{r} \geqq r(g)$, and therefore

$$F(g_i)a = a,\ \forall i \Rightarrow l(g_i)a = 0 = ar(g_i)\ \forall i$$

$$\Rightarrow \tilde{l}a = 0 = a\tilde{r} \Rightarrow l(g)a = 0 = ar(g) \Rightarrow F(g)a = a\,.$$

(b) Let $f, g \in Q(M')$ and write $g = E(f)g + F(f)g + G(f)g$ where by Proposition 4.3 and Proposition 3.5 (ii), each summand belongs to $Q(M')$. Then by part (a) it suffices to prove Lemma 3.8 in the three cases $g \in E(f)Q(M')$, $g \in F(f)Q(M')$, and $g \in G(f)Q(M')$. This will be done in Remark 4.5, Lemma 4.6, and Remark 4.11 (b), respectively.

REMARK 4.5. Let $g \in E(f)Q(M')$. Then $F(f)Q'v = F(g)F(f)Q'v$, i.e. Lemma 3.8 is true in this case.

Indeed since $g \in E(f)(M')$, $F(g)F(f) = F(f)$, by Lemma 2.9 (i).

LEMMA 4.6. *Let $g \in F(f)Q(M')$. Then $F(f)Q'v = F(g)F(f)Q'v$, that is Lemma 3.8 is true in this case.*

PROOF. Set $a = F(f)Q'v$. Then

$$F(g)a = a - l(g)a - ar(g) + l(g)ar(g)\,.$$

We shall show that $l(g)a = 0 = ar(g)$. To this end let (φ_u, N_u) be the enveloping polar decomposition of g. Then

$$Q'(u \pm v) = Q'u \pm Q'v = u + F(g)Q'u \pm (v + a)$$

(by Corollary 2.8) and

$$Q'(u \pm v)r(g) = u + 0 + 0 \pm ar(g).$$

Therefore

$$\|u \pm ar(g)\| = \|Q'(u+v)r(g)\| \leqq \|u \pm v\| \leqq 1,$$

the last inequality being true, since $g = F(f)g$. Now $A''r(g)$ is a J*-algebra, where A is any C*-algebra containing M as a J*-subalgebra, and u is a partial isometry in $A''r(g)$ with $(1-uu^*)A''r(g)(1-u^*u)=0$. Therefore by Harris [17], u is an extreme point of the unit ball of $A''r(g)$. Hence $ar(g)=0$ and similarly

$$l(g)Q'(u \pm v) = u \pm l(g)a, \quad \|u \pm l(g)a\| \leqq 1$$

and $l(g)a = 0$.

The remainder of this section is devoted to the proof of Lemma 3.8 in the (remaining) case that $g \in G(f)Q(M')$. The first step is a subdivision of this case into two other cases.

LEMMA 4.7. *Suppose f is an atom of Q and $g \in G(f)Q(M')$, $g \neq 0$. Then either* (i) $f = E(g)f$ *(and* $G(f)F(g) = 0$) *or* (ii) $f = G(g)f$.

PROOF. By Lemma 2.9 (iii) the projections $F(g)$, $E(f)$, $E(g)$, $G(g)$ commute in pairs. Therefore if we write $f = E(g)f + F(g)f + G(g)f$, then each term in the sum is a multiple of f. Indeed using Proposition 3.5 (ii) and Proposition 3.7 (ii) we have

$$\begin{aligned} F(g)f &= F(g)Qf = QF(g)Qf = QF(g)f \\ &= QF(g)E(f)f = QE(f)F(g)f = \langle F(g)f, v\rangle f \end{aligned}$$

and similarly $E(g)f = \langle E(g)f, v\rangle f$ and $G(g)f = \langle G(g)f, v\rangle f$. Thus

$$0 = E(g)G(g)f = \langle E(g)f, v\rangle \langle G(g)f, v\rangle f,$$

so that either $E(g)f = 0$ or $G(g)f = 0$. It remains to prove that $F(g)f = 0$. Since $F(g)f$ is a multiple of f, if $F(g)f \neq 0$, we must have $F(g)f = f$. Thus by Lemma 2.9 (ii),

$$r(f)r(g) = 0 = l(f)l(g).$$

Therefore

$$g = F(f)g = F(f)G(f)g = 0,$$

a contradiction.

LEMMA 4.8 *Suppose f is an atom of Q, g is an atom of $G(f)Q$ and that $f = G(g)f$. Then g is an atom of Q.*

PROOF. Let $g=\frac{1}{2}(h+k)$, with $h,k \in Q(M')_1$. We are to prove that $g=h$. Now

$$g = E(g)g = \tfrac{1}{2}(E(g)h+E(g)k),$$

so that $\|E(g)h\|=1=\|h\|$ and by Lemma 3.1, $h=E(g)h$. On the other hand

$$g = G(f)Qg = \tfrac{1}{2}(G(f)Qh+G(f)Qk),$$

and since g is an atom of $G(f)Q$, we have $g=G(f)Qh=G(f)h$. Since

$$\begin{aligned} h &= E(f)h+F(f)h+G(f)h \\ &= E(f)h+F(f)h+g, \end{aligned}$$

it remains to prove that $E(f)h=0=F(f)h$. Now by Proposition 4.3 (ii) and Proposition 3.7,

$$E(f)h = E(f)Qh = QE(f)Qh = \lambda f = \lambda G(g)f$$

for some scalar λ. Since $h=E(g)h$ we have by Lemma 2.9 (iii),

$$\lambda G(g)f = E(f)h = E(f)E(g)h = E(g)E(f)h.$$

By multiplying on the left by $G(g)$, we have $\lambda G(g)f=0=E(f)h$.
Finally

$$\|F(f)h+G(f)h\| = \|h-E(f)h\| = \|h\| = 1 = \|g\| = \|G(f)h\|.$$

So by Corollary 4.2, $F(f)h=0$.

LEMMA 4.9 *Suppose f is an atom of Q, g is an atom of $G(f)Q$ and that $f=G(g)f$. Then*

$$F(f)Q'v = F(g)F(f)Q'v.$$

PROOF. Let $a=F(f)Q'v$ and write $a=a_{11}+a_{12}+a_{21}+a_{22}$, where $a_{11}=E(g)a$, $a_{12}+a_{21}=G(g)a$, and $a_{22}=F(g)a$ according to Diagram 3. Here

$$a_{12} = l(g)(1-l(f))a(1-r(g)-r(f))$$

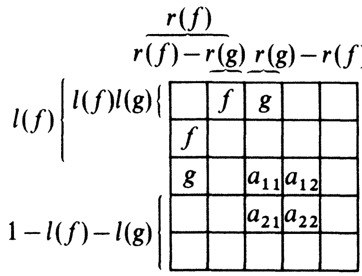

Diagram 3.

and

$$a_{21} = (1-l(f)-l(g))a(1-r(f))r(g)$$

lie in A'', where A is any C*-algebra containing M as a J*-subalgebra. Note that $l(g)l(f)=l(f)l(g)$, $r(g)r(f)=r(f)r(g)$ by Lemma 2.9.

We shall show that $a_{11}=0$ and $a_{21}=a_{12}=0$. This will imply $a=a_{22}=F(g)a$ completing the proof of the lemma.

In the first place, with $u=v(g)$ we have

$$a_{11} = E(g)a = E(g)(v+a) = E(g)Q'v = \langle v,g\rangle u$$

$$= \langle E(g)v,g\rangle u = 0$$

(by Proposition 3.7 (iii), since g is an atom of Q by Lemma 4.8).

We next show that $a_{21}=0$. To this end consider the contractive projection P_1 on A'' defined by

$$P_1(b) = (l_1+l_2)b(r_1+r_2),$$

where $l_1=l(f)l(g)$, $l_2=1-l(f)-l(g)$, $r_1=r(f)-r(g)$, $r_2=r(g)-r(f)$. The projection P_1 is represented schematically by the following (cf. Diagram 3):

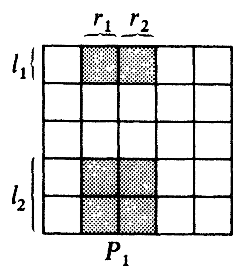

Diagram 4.

Let $v_1=l_1vr_1$ and $u_1=l_1ur_1$. Then

$$\|tv_1+u_1\|^2 = \|(tv_1+u_1)(tv_1+u_1)^*\| = \||t|^2l_1+l_1\| = 1+|t|^2 .$$

Similarly letting $v_2=(1-l_1-l_2)v(1-r_1-r_2)$ and $u_2=(1-l_1-l_2)u(1-r_1-r_2)$, we get

$$\|tv_2+u_2\|^2 = 1+|t|^2 ,$$

and therefore

$$\|tv+u\| = \max(\|tv_1+u_1\|, \|tv_2+u_2\|) = (1+|t|^2)^{\frac{1}{2}} .$$

For convenience we shall write

$$P_1(b) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

where $b_{11} = l_1 b r_1$, $b_{12} = l_1 b r_2$), etc.

Let $z_t = P_1 Q'(tv+u)$, where t is a scalar and $v = v(f)$. Since

$$P_1 Q'v = P_1(v+a) = v_1 + a_{21},$$

and

$$P_1 Q'u = P_1(u + F(g)Q'u) = u_1 + c_1,$$

where $c_1 = P_1 F(g) Q'u$, z_t has the form

$$z_t = \begin{bmatrix} tv_1 & u_1 \\ c_1 & ta_{21} \end{bmatrix}.$$

We have

$$\begin{aligned} \|z_t\| &= \|P_1 Q'(tv+u)\| \\ &\leqq \|tv+u\|_{M''} = (1+|t|^2)^{\frac{1}{2}}. \end{aligned}$$

Now

$$z_t z_t^* = \begin{bmatrix} (1+|t|^2) l_1 & tv_1 c_1^* + \bar{t} u_1 a_{21}^* \\ \# & \# \end{bmatrix}$$

and $\|z_t z_t^*\|^2 = \|z_t\|^2 \leqq 1 + |t|^2$. Therefore $tv_1 c_1^* + \bar{t} u_1 a_{21}^* = 0$ for all scalars t. Using the values $t = 1$ and $t = i$ yields $u_1 a_{21}^* = 0$ and thus $a_{21} = a_{21} u_1^* u_1 = 0$. A similar proof yields $a_{12} = 0$.

LEMMA 4.10. *Suppose f is an atom of Q, g is an atom of $G(f)Q$, and $f = E(g)f$. Then $F(f)Q'v = F(g)F(f)Q'v$.*

PROOF. Suppose there is a functional $h \in F(f)Q(M')$ such that $l(f+h) = l(g)$ and $r(f+h) = r(g)$. Then we would have $F(g) = F(f+h) = F(h)F(f)$, and by Lemma 4.6,

$$F(g)F(f)Q'v = F(h)F(f)Q'v = F(f)Q'v.$$

To prove the lemma it suffices to construct such a functional h.

The assumptions $g = G(f)g$ and $f = E(g)f$ enable us to employ the following support Diagram 5:

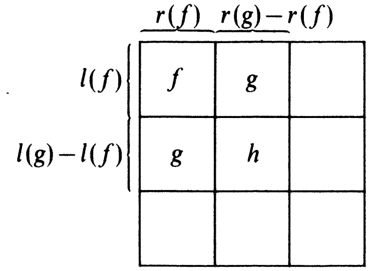

Diagram 5.

We shall construct functionals $\omega_{ij} \in M'$, of norm 1, with supports as indicated in Diagram 6 (the blocks as in Diagram 5):

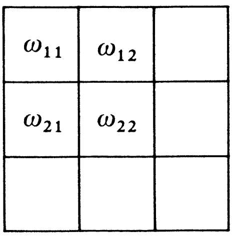

Diagram 6.

such that $Q\omega_{11}=f$, $Q\omega_{12}=Q\omega_{21}=g$ and $Q\omega_{22}=\lambda f+h$ for some scalar λ. We shall prove that the h so defined has the required properties.

We define $\omega_{12}=\omega(\xi,\eta)$, where $\xi=(r(g)-r(f))\xi$, $\|\xi\|=1$, $\eta=u\xi$, and $u=v(g)$. Note that since $\eta=l(f)\eta$, $\omega_{12} \in G(f)(M')\cap E(g)(M')$ and by Proposition 3.7,

$$G(f)Q\omega_{12} = G(f)QE(g)\omega_{12} = \langle\omega_{12},u\rangle g = (u\xi,\eta)g = \|\eta\|^2 g = g\,.$$

Also $F(f)Q\omega_{12}=F(f)QG(f)\omega_{12}=0$ by Lemma 3.4. Therefore

$$Q\omega_{12} = E(f)Q\omega_{12}+G(f)Q\omega_{12}+F(f)Q\omega_{12} = E(f)Q\omega_{12}+g\,,$$

so

$$\|E(f)Q\omega_{12}+G(f)g\| = \|E(f)Q\omega_{12}+g\| = \|Q\omega_{12}\| \leqq 1=\|g\| = \|G(f)g\|\,.$$

By Corollary 4.2, $E(f)Q\omega_{12}=0$ and $Q\omega_{12}=g$.

Next we define $\omega_{11}=\omega(\tilde{\xi},\eta)$, where $\tilde{\xi}=v^*u\xi=r(f)\tilde{\xi}$, so that $\omega_{12} \in E(f)(M')$. Then

$$Q\omega_{11} = QE(f)\omega_{11} = \langle\omega_{11},v\rangle f = (v\tilde{\xi},\eta)f = (vv^*u\xi,\eta)f = (l(f)\eta,\eta)f = f\,.$$

Next we define $\omega_{21}=\omega(\tilde{\xi},\tilde{\eta})$, where $\tilde{\eta}=uv^*\eta=l(g)\tilde{\eta}$, so that

$\omega_{21} \in E(g)(M') \cap G(f)(M')$. By the same argument that established $Q\omega_{12}=g$, we find $Q\omega_{21}=g$.

Finally we define $\omega_{22}=\omega(\xi,\tilde{\eta})$, so that $\omega_{22} \in E(g)(M') \cap F(f)(M')$. Then

$$Q\omega_{22} = E(f)Q\omega_{22}+G(f)Q\omega_{22}+F(f)Q\omega_{22} .$$

For the middle term,

$$G(f)Q\omega_{22} = G(f)QE(g)\omega_{22} = \langle\omega_{22},u\rangle g = (u\xi,\tilde{\eta})g = (\eta,\tilde{\eta})g = 0 .$$

For the first term $E(f)Q\omega_{22}=QE(f)Q\omega_{22}$ (by Proposition 4.3 (ii)) $=\langle Q\omega_{22},v\rangle f=\lambda f$ say. If we now set $h=F(f)Q\omega_{22}$, then $Q\omega_{22}=\lambda f+h$.

For $|\alpha|=1$, set $\omega_\alpha=\alpha\omega_{11}+\omega_{12}+\omega_{21}+\bar{\alpha}\omega_{22}$. Because $(\xi,\tilde{\xi})$ and $(\eta,\tilde{\eta})$ are orthonormal pairs, $\|\omega_\alpha\|$ is not greater than the trace norm of the matrix $\begin{bmatrix}\alpha & 1\\ 1 & \bar{\alpha}\end{bmatrix}$, which is 2. Thus we have

$$Q\omega_\alpha = 2g+\alpha f+\bar{\alpha}(\lambda f+h) \tag{4.1}$$

and $\|Q\omega_\alpha\| \leqq \|\omega_\alpha\| \leqq 2$.

We show next that $h \neq 0$. Suppose $h=0$. Set $\alpha=1$ in (4.1) to obtain

$$\|2g+(1+\lambda)f\| \leqq 2 = \|2g\| .$$

Since $g \in G(f)(M')$ and $f \in E(f)(M')$, Corollary 4.2 implies $1+\lambda=0$. Now set $\alpha=i$ in (4.1) to obtain $\|2g+i(1-\lambda)f\| \leqq 2$. Again by Corollary 4.2, $1-\lambda=0$, contradiction.

We now have our $h \in F(f)Q(M')$ and $h \neq 0$. Note that also $h \in E(g)(M')$, since

$$h = Q\omega_{22}-\lambda f = QE(g)\omega_{22}-\lambda f = E(g)(QE(g)\omega_{22}-\lambda f) .$$

It remains to show that $l(h)=l(g)-l(f)$ and $r(h)=r(g)-r(f)$. This is a consequence of the fact that g is an atom of $G(f)Q$ as follows. Consider $G(h)g$. Clearly

$$G(h)g \in E(g)(M') \cap G(f)(M') \cap Q(M') ,$$

so $G(h)g=G(f)QE(g)G(h)g=\beta g$ for some scalar β. If $\beta=0$, then

$$g = E(h)g+G(h)g+F(h)g = F(h)g ,$$

since $G(h)g=0$ and $E(h)g=0$. This equivalent to $h=F(g)h$. Thus

$$h = F(g)h = F(g)E(g)h = 0 ,$$

contradiction. Therefore $\beta \neq 0$, and therefore $G(h)g=g$.

Now $l(f)l(h)=0=r(f)r(h)$ and $g=G(f)G(h)g$. This implies $E(f+h)g=g$, so by Lemma 2.9

$$l(g) \leqq l(f+h) = l(f)+l(h),$$
$$r(g) \leqq r(f+h) = r(f)+r(h).$$

REMARK 4.11.
(a) By Lemmas 4.7, 4.9, 4.10, the proof of Lemma 3.8 is complete in case g is an atom of $G(f)Q$.
(b) By (a) and Remark 4.4 (a), the proof of Lemma 3.8 in the case $g \in G(f)Q$ is complete, since $Q(M')$ is assumed finite dimensional.
(c) Lemma 3.8 is valid only for complex J*-algebras. For a counterexample in the real case, see [3, Example 7.10, p. 163].

ADDED IN PROOF MARCH 21, 1983. By using the tools developed in this paper, the authors have been able to drop the finite dimensionality assumption in Theorem 3. They have also been able to show that for a bicontractive projection P on a J*-algebra M, $2P-\mathrm{Id}_M$ is an isometry.

REFERENCES

1. E. Alfsen, F. W. Shultz, and E. Størmer, *A Gelfand-Naimark theorem for Jordan algebras*, Adv. in Math. 28 (1978), 11–56.
2. J. Arazy and Y. Friedman, *The isometries of $C_p^{m,n}$ into C_p*, Israel J. Math. 26 (1977), 151–165.
3. J. Arazy and Y. Friedman, *Contractive projections in C_1 and C_∞*, Mem. Amer. Math. Soc. 13 (1978), No. 200.
4. I. Bars and M. Gunaydin, *Construction of Lie algebras and Lie superalgebras from ternary algebras*, J. Math. Phys. 20 (1979), 1977–1993.
5. F. A. Berezin, *Quantization*, Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), No. 5. Translation: Math. USSR-Izv. 8 (1974), 1109–1165.
6. F. A. Berezin, *Quantization in complex symmetric spaces*, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), No. 2. Translation: Math. USSR-Izv. 9 (1975), 341–379.
7. S. J. Bernau and H. E. Lacey, *Bicontractive projections and reordering of L_p spaces*, Pacific J. Math. 69 (1977), 291–302.
8. M. D. Choi and E. Effros, *Injectivity and operator spaces*, J. Funct. Anal. 24 (1977), 156–209.
9. J. Dixmier, *Les algèbres d'operateurs dans l'espace Hilbertien*, (Cahiers Scientifiques 25), Gauthier Villars, 1957, Paris, second edition 1969.
10. J. Dixmier, *Les C*-algèbres et leur représentations* (Cahiers Scientifiques 24), Gauthier Villars, 1964, Paris, second edition 1969.
11. E. Effros, *Order ideals in a C*-algebra and its dual*, Duke Math. J. 30 (1963), 391–412.
12. E. Effros and E. Størmer, *Positive projections and Jordan structure in operator algebras*, Math. Scand. 45 (1979), 127–138.
13. G. G. Emch and W. P. C. King, *Faithful normal states on* JBW-*algebras*, Proc. Sympos. Pure Math. 38 Part 2 (1982), 305–307.
14. Y. Friedman and B. Russo, *Contractive projections on $C_0(K)$*, Trans Amer. Math. Soc. 273 (1982), 57–73.
15. Y. Friedman and B. Russo, *Contractive projections on* C*-*algebras*, Proc. Sympos. Pure Math. 38 Part 2 (1982), 615–618.

16. M. Gunaydin, *Quadratic Jordan formulation of quantum mechanics and construction of Lie (super) algebras from Jordan (super) algebras*, VIII International Colloquium on Group Theoretical Methods in Physics, Kiriat Anavim, Israel, 1979.
17. L. Harris, *Bounded symmetric homogeneous domains in infinite dimensional spaces* in *Proceedings on infinite dimensional holomorphy*, eds. T. L. Hayden and T. J. Suffridge, (Lecture Notes in Mathematics 364), pp. 13–40, Springer-Verlag, Berlin - Heidelberg - New York, 1974.
18. L. Harris, *Analytic Invariants and the Schwarz–Pick inequality*, Israel J. Math. 34 (1979), 177–197.
19. L. Harris, *A generalization of* C*-*algebras*, Proc. London Math. Soc. 42 (1981), 331–361.
20. L. Harris and W. Kaup, *Linear algebraic groups in infinite dimensions*, Illinois J. Math. 21 (1977), 666–674.
21. M. Hestenes, *A ternary algebra with applications to matrices and linear transformations*, Arch. Rational Mech. Anal. 11 (1962), 138–194.
22. W. Kaup, *Algebraic characterization of symmetric complex Banach manifolds*, Math. Ann. 228 (1977), 39–64.
23. W. P. C. King, Ph. D. Thesis, University of Rochester, 1980.
24. O. Loos, *Jordan Pairs* (Lecture Notes in Math. 460), Springer-Verlag, Berlin - Heidelberg - New York, 1975.
25. O. Loos, *Bounded symmetric domains and Jordan pairs*, Lecture Notes, University of California, Irvine, 1977.
26. O. Loos and K. McCrimmon, *Speciality of Jordan triple systems*, Comm. Alg. 5 (1977), 1057–1082.
27. K. McCrimmon, *Peirce ideals in Jordan triple systems*, Pacific J. Math. 83 (1979), 415–439.
28. E. Neher, *Cartan-Involutioner von halbeinfachen reellen Jordan-Tripelsystemen*, Math. Z. 169 (1979), 271–292.
29. A. G. Robertson and M. A. Youngson, *Positive projections with contractive complements on Jordan algebras*, J. London Math. Soc. (2) 25 (1982), 365–374.
30. I. Satake, *Algebraic structures of symmetric domains*, Publ. Math. Soc. Japan, 14, 1980.
31. I. A. Shereshevskii, *Quantization based on infinite-dimensional hermitian symmetric space*, Moscow Univ. Math. Bull. 32 (1977), 28–36.
32. F. W. Shultz, *On normed Jordan algebras which are Banach dual spaces*, J. Funct. Anal. 31 (1979), 360–376.
33. E. Størmer, *Jordan algebras of type* I, Acta Math. 115 (1966), 165–184.
34. E. Størmer, *Irreducible Jordan algebras of self adjoint operators*, Trans. Amer. Math. Soc. 130 (1968), 153–166.
35. E. Størmer, *Positive projections with contractive complements on* C*-*algebras*, J. London Math. Soc. (2) 26 (1982), 132–142.
36. D. Topping, *Jordan algebras of self-adjoint operators*, Mem. Amer. Math. Soc. 53 (1965), 48 pp.
37. J. D. M. Wright, *Jordan* C*-*algebras*, Michigan Math. J. 24 (1977), 291–302.
38. H. Zettl, *A characterization of ternary rings of operators*, to appear in Adv. in Math.
39. B. Iochum, *Cones autopolaires et algèbres de Jordan*, Thèse, Université de Provence, 1982.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
IRVINE, CALIFORNIA 92717
U.S.A.