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MODELS OF FINITE GROUP ACTIONS

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Introduction.

In this paper we continue work begun in [7]. There we dealt with actions of Z_2 on UHF C*-algebra and showed that a certain model action was characterized by its behaviour on central sequences. Here we call this property $\mathscr{R}_\infty$ (see below). The more general finite group situation is what we address here and in doing so the role of $\mathscr{R}_\infty$ becomes clearer. Indeed it appears more indispensable in this paper than it did in [7] and is crucial for our vanishing cohomology results. As in [2, 7] much of the work is done in the algebra A^∞ $(=l^\infty(Z,A)/c_0(Z,A))$ and A_∞ $(=\{(x_n : \forall x \in A, \lim_{n\to\infty} \|[x_n,x]\|=0\})$. We then transfer the results back to the C*-algebra A via various technical lemmas. What we show is that "property $\mathscr{R}_\infty$ characterizes an action of the finite group G as the tensor product of Ad of the left regular representation with it itself an infinite number of times" (call it s_G).

To reach this result we combine the techniques of [7] and [8] together with results of Ocneanu [10] and Sutherland [14]. Matrix units which behave well under the action of G are found in Section I. For this we need to pass to cocycle twisted actions of G. The main result appears in Section III, Theorem 3.6. As a corollary we obtain that s_G absorbs, by tensoring, any approximately inner action of the group G on a simple unital C*-algebra. Finally in Section IV we show that, for A a UHF C*-algebra, if G acts on A in such a manner that it remains outer in the II_1 representation, then the fixed point subalgebra, A^G, is the closed linear span of its projections.

Most of the results hold true only under the *standing assumption* that A is a simple unital C*-algebra which is isomorphic to its tensor product with a UHF C*-algebra of type $|G|^\infty$ and that the automorphisms in question are approximately inner. Indeed one of the final results requires A to be AF. Some of the lemmas are true otherwise and we have pointed out where the standing assumption enters. All C*-algebras considered here are assumed to be separable.

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I. 2-Cohomology.

The main goal of this section is to find equivariant matrix units for a cocycle perturbation of our original action. That is we shall find a cocycle $g \to v_g$ and $|G| \times |G|$ matrix units $e_{k,l}$ such that

$$\operatorname{Ad} v_g \alpha_g(e_{k,l}) = e_{gk,gl}.$$

To do this it will be necessary to consider cocycle twisted actions (see [14]). We say that (γ, u) is a *cocycle twisted action* if γ is a map from $G \to \operatorname{Aut}(A)$ and $u_{g,h}$ is a family of unitaries indexed by G such that

$$\gamma_g \gamma_h = \operatorname{Ad} u_{g,h} \gamma_{gh}$$

and

$$(*) \qquad u_{g,h} u_{gh,k} = \gamma_g(u_{h,k}) u_{g,hk}$$

with the normalization $u_{g,h} = 1$ if either g or $h = 1$.

There is an easy way to perturb cocycle twisted actions to others. We choose any family of unitaries $t_g, t_e = 1$ and form $\tilde{\gamma}_g = \operatorname{Ad} t_g \gamma_g$ with corresponding cocycle $\tilde{u}_{g,h} = t_g \gamma_g(t_h) u_{g,h} t_{gh}^*$.

REMARK I.1. Note that if α is an action, then perturbing $(\alpha, 1)$ by a cochain t_g, one obtains an action $(\tilde{\alpha}, 1)$ exactly when t_g is a 1-cocycle.

We begin by noting the result of perturbing twice. A chain rule exists.

LEMMA I.2. *If $(\tilde{\gamma}, \tilde{u})$ is the perturbation of the cocycle twisted action (γ, u) by the cochain v_g and w_g is another cochain, then the result $(\tilde{\tilde{\gamma}}, \tilde{\tilde{u}})$ of perturbing $(\tilde{\gamma}, \tilde{u})$ by $\{w_g\}$ is the same as the result of perturbing (γ, u) by the cochain $\{w_g v_g\}$.*

PROOF. This is just a simple calculation.

Our method for proving vanishing cohomology will be the following adaptation of Shapiro's lemma.

LEMMA I.3. *Suppose (γ, u) is a cocycle twisted action and that there are projections $\{e_g\}$ in A, $\sum_{g \in G} e_g = 1$ with $[u_{g,h}, e_k] = 0$ for all g, h, k, and $\gamma_g(e_h) = e_{gh}$. Then if $v_g = \sum_{h \in G} u_{g,h}^* e_{gh}$, the v_g's are a unitary cochain, and if $(\tilde{\gamma}, \tilde{u})$ is the result of perturbing (γ, u) by v_g, $\tilde{u}_{g,h} = 1$.*

PROOF. That the v_g's are unitary is clear. By definition

$$\tilde{u}_{g,h} = \left(\sum_{a\in G} u^*_{g,a}e_{ga}\right)\gamma_g\left(\sum_{b\in G} u^*_{h,b}e_{hb}\right)u_{g,h}\left(\sum_{c\in G} e_{ghc}u_{gh,c}\right).$$

Since the e_g's commute with the $u_{g,h}$'s, this becomes

$$\tilde{u}_{g,h} = \sum_{a,b,c} u^*_{g,a}\gamma_g(u^*_{h,b})u_{g,h}u_{gh,c}e_{ga}e_{ghb}e_{ghc}.$$

Now since the e_g's are orthogonal, the only contributions in the sum occur when $b=h^{-1}a$ and $c=h^{-1}a$ and then the cocycle relation for $u_{g,h}$ gives

$$u^*_{g,a}\gamma_g(u^*_{h,h^{-1}a})u_{g,h}u_{gh,h^{-1}a} = 1.$$

Hence $\tilde{u}_{g,h}=\sum_a e_{ga}=1$.

We now give a C* algebraic version of a result of C. Sutherland [14].

THEOREM I.4. *Let G be a finite group, A a unital C*-algebra and let (γ,u) be a cocycle twisted action of G on A. Suppose A possesses a partition of unity $\{e_g\}$ such that $\gamma_g(e_h)\sim e_{gh}$ for every $g,h\in G$. Then γ admits a perturbation of the form $(\tilde{\gamma},1)$.*

PROOF. For each $g\in G$ there is a unitary v_g such that $v_g\gamma_g(e_h)v_g^*=e_{gh}$, $v_1=1$ [5, Lemma 1.8]. Let $(\tilde{\gamma},\tilde{u})$ be the result of perturbing γ by the cochain $\{v_g\}$. Then $\tilde{\gamma}_g(e_h)=e_{gh}$ so that

$$\tilde{\gamma}_g\tilde{\gamma}_h(e_k) = e_{ghk} = \operatorname{Ad}\tilde{u}_{g,h}\tilde{\gamma}_{gh}(e_k) = \operatorname{Ad}\tilde{u}_{g,h}(e_{ghk}).$$

Hence $[\tilde{u}_{g,h},e_k]=0$ for any g,h,k. Hence by lemma I.3, $(\tilde{\gamma},\tilde{u})$ has a perturbation of the form $(\tilde{\tilde{\gamma}},1)$. Finally by Lemma I.2, (γ,u) has a perturbation $(\tilde{\gamma},1)$.

We shall use Theorem I.4 to prove the following.

THEOREM I.5. *Let A be a unital C*-algebra with $|G|^2\times|G|^2$ matrix units $\{f_{ij}\}$ and suppose $\alpha\colon G\to \operatorname{Aut} A$ is an action such that $\alpha_g(f_{11})\sim f_{11}$ for all $g\in G$. Then if $\{e_{g,h}\}$ are matrix units indexed by G in $\{f_{ij}\}''$, α may be perturbed by a 1-cocycle v_g so that $\operatorname{Ad} v_g\alpha_g(e_{h,k})=e_{gh,gh}$.*

PROOF. Begin by noting that $\alpha_g(f_{11})\sim f_{11}$ implies the existence of unitaries, t_g, such that $\operatorname{Ad} t_g\alpha_g(f_{ij})=f_{ij}$. In particular $\operatorname{Ad} t_g\alpha_g(e_{h,k})=e_{h,k}$. If $r_g=\sum_{h\in G}e_{gh,h}$, then $\operatorname{Ad} r_gt_g\alpha_g(e_{h,k})=e_{gh,gk}$. If (γ,u) is the perturbation of α by the cochain $\{r_gt_g\}$, then, as in the proof of I.4, we have $[u_{g,h},e_{a,b}]=0$ for all $g,h,a,b\in G$. Thus (γ,u) may be viewed as a cocycle twisted action of G on the commutant $\{e_{g,h}\}'$. But the hypotheses of Theorem I.5 allow us to apply I.4 to conclude that (γ,u) may be perturbed by a cochain x_g *in the commutant* $\{e_{g,h}\}'$ to $(\tilde{\gamma},1)$. In particular

$\tilde{\gamma}(e_{h,k})=e_{gh,gk}$. Now by I.2, $(\tilde{\gamma}, 1)$ is a perturbation by some cochain $\{v_g\}$ of $(\alpha, 1)$. Finally it follows from Remark I.1 that $\{v_g\}$ is a cocycle.

REMARK. One might be tempted to conclude that Theorem I.5 is valid with only $|G|\times|G|$ matrix units in A. This is not true as seen by the following counterexample. Let $G=\mathbf{Z}_2\times\mathbf{Z}_2$ and $A=M_4(\mathbf{C})$. The Pauli spin matrices σ_g form a nontrivial projective representation of G in $M_2(\mathbf{C})$ and hence G acts on $M_4(\mathbf{C})=M_2(\mathbf{C})\otimes M_2(\mathbf{C})$ by $\alpha_g=\mathrm{Ad}\,(\sigma_g\otimes 1)$. Here A has a basis of $|G|\times|G|$ matrix units. But we claim that α cannot be perturbed by a cocycle so that $\mathrm{Ad}\, v_g\alpha_g(e_{h,k})=e_{gh,gk}$. For the crossed product $A\times_\alpha G$ is a factor ([8, 7.4.4 and 2.2.2]) so that any cocycle is a coboundary. This would mean that the action α would be conjugate to the action coming from Ad of the regular representation which is not the case.

II. Equivariant matrix units.

In this section we show how to find equivariant matrix units assuming the existence of approximately equivariant matrix units. The result will be used later on in obtaining our final structure theorem.

First we need a simple lemma.

LEMMA II.1. *Let $g\to y_g$ be a unitary one-cocycle for the action $g\to\alpha_g$ on A. If $\|y_g-I\|<\delta<\sqrt{2}$, then $y_g=z\alpha_g(z^*)$ for a unitary operator z with $\|z-I\|<\delta$.*

PROOF. If we let $x=1/|G|\sum_{g\in G}y_g$, the x is seen to be invertible (its numerical range is a proper subset of the right half-plane). A simple calculation shows that $\alpha_g(x)=y_g^*x$. If we take the unitary z in polar decomposition of x, then we get $y_g=z\alpha_g(z^*)$. Moreover, $\|z-I\|<\delta$. (For the estimates see [1, Lemma 2.3 and the remarks on the top of p. 172].)

DEFINITION. An action α of the finite group G will be said to have property $\mathscr{R}_\infty$, if there is a central sequence of partitions of unity $\{e_g^n\}$ indexed by G such that $\lim_{n\to\infty}\|\alpha_g(e_h^n)-e_{gh}^n\|=0$.

If E_g are the projections in A_∞ represented by $\{e_g^n\}$, then $\alpha_g(E_h)=E_{gh}$. If $G=\mathbf{Z}_2$ this is the same as the property $\mathscr{U}_\infty$ of [4] and [9].

THEOREM II.2. *Let α be an approximately inner action of the finite group G on the C^*-algebra A having property $\mathscr{R}_\infty$. Then for any $\varepsilon>0$ there is a $\delta>0$ such* *that if* $\{f_{g,h}\}$ *are matrix units in* A *with* $\|\alpha_g(f_{h,k})-f_{gh,gk}\|<\delta$, *then there are matrix units* $\{e_{g,h}\}$ *with*

$$\alpha_g(e_{h,k}) = e_{gh,gk} \quad \text{and} \quad \|e_{g,h}-f_{g,h}\| < \varepsilon .$$

PROOF. By a result of Glimm [5], we may find unitaries w_g such that $w_g\alpha_g(f_{h,k})w_g^*=f_{gh,gk}$ and $\|w_g-1\|$ controlled by δ. This means that if $\tilde{\alpha}_g=\operatorname{Ad} w_g\alpha_g$ and $x_{g,h}=w_g\alpha_g(w_h)w_{gh}^*$, then $(\tilde{\alpha}, x_{g,h})$ is a cocycle twisted action on the commutant B of the $f_{h,k}$'s. Also $\|x_{g,h}-1\|$ is controlled by δ. By $\mathscr{R}_\infty$ and a little approximation, we may choose a partition of unity $\{e_g\}$ in B with $\|\tilde{\alpha}_g(e_h)-e_{gh}\|$ as small as we like. Moreover by [Ibid] we may then find unitaries t_g with $t_g\tilde{\alpha}_g(e_h)t_g^*=e_{gh}$ and $\|t_g-1\|$ small. But then if $(\beta, u_{g,h})$ is the result of perturbing $(\tilde{\alpha}, x_{g,h})$ by $t_g, [u_{g,h}, e_k]=0$ and $\|u_{g,h}-1\|$ is controlled by δ. Thus if $v_g=\sum_{h\in g}u_{g,h}^*e_{gh}$, then $\|v_g-1\|$ is controlled by δ and if $(\tilde{\beta}, \tilde{u}_{g,h})$ is the result of perturbing β by v_g, then as in Lemma I.3, $\tilde{u}_{g,h}=1$. But by Remark I.1, this means that $y_g=v_gt_gw_g$ is an α cocycle and $\|y_g-1\|$ is controlled by δ. Also since t_g and v_g are in B, $\operatorname{Ad} y_g\alpha_g(f_{h,k})=f_{gh,gk}$. Now use Lemma II.1 to choose a unitary z with $\|z-1\|$ controlled by δ and $z\alpha_g(z^*)=y_g$. Then put $e_{g,h}=z^*f_{g,h}z$.

III. Conjugacy of $\mathscr{R}_\infty$ actions.

The importance of property $\mathscr{R}_\infty$ is that it implies stability of the action α on A_∞ as shown in the next lemma.

LEMMA III.1. *If* α *is an action on* A *with property* $\mathscr{R}_\infty$ *and* $\{V_g\}$ *is a unitary cocycle for the action of* G *on* A_∞, *then there is a* $U \in A_\infty$ *such that* $U\alpha_g(U^*)=V_g$.

PROOF. By property $\mathscr{R}_\infty$ we may choose a partition of I in A_∞ such that $\alpha_g(E_h)=E_{gh}$. By choosing a subsequence [2] we may further suppose $[V_h, E_g]=0$ for all $h, g \in G$. Let $U=\sum_{g\in G}V_gE_g$.

LEMMA III.2. *Let* A *and* α *satisfy the standing assumptions. In addition assume that it is an* AF-*algebra. There exist* $|G|^2\times|G|^2$ *matrix units* $\{F_{i,j}\}$ *in* A_∞ *with* $\alpha_g(F_{11})\sim F_{11}$.

PROOF. We have already made use of this idea in [7] although it was not made explicit there. That $|G|^2\times|G|^2$ matrix units exist in A_∞ is clear from the assumptions on A. Given a central sequence of matrix units $\{e_{ij}^{(n)}\}$, clearly $\{\alpha_g(e_{ij}^{(n)})\}$ is also central. Since α_g is approximately inner clearly $\{\alpha_g(e_{ij}^{(n)})\}$ is equivalent to $\{e_{ij}^{(n)}\}$. Using the fact that A is AF we pick first a finite dimensional subalgebra M_1 and then choose n so large (say $n=n_1$) so that

$\{e_{ij}^{(n)}\}$ and $\{\alpha_g(e_{ij}^{(n)})\}$ are nearly in M_1^c. There are then two sets of matrix units in M_1^c, near to $\{e_{ij}^{(n)}\}$ and $\{\alpha_g(e_{ij}^{(n)})\}$ respectively, which are equivalent in M_1^c. This last equivalence provides an equivalence for $\{e_{ij}^{(n)}\}$ and $\{\alpha_g(e_{ij}^{(n)})\}$ by an element nearly in M_1^c. In particular $e_{11}^{(n)} \sim \alpha_g(e_{11}^{(n)})$ by an element near to M_1^c. Picking an increasing nest $\{M_k\}$ of finite dimensional subalgebras and a subsequence of the $\{e_{ij}^{(n)}\}$ with increasingly better commutation properties, we obtain our result.

THEOREM III.3. *Let A be an* AF *algebra with A and α satisfying the standing assumptions. Suppose that α is an action of G on A with property $\mathscr{R}_\infty$. Then there is a unitary representation V_g of G on A^∞ such that $\alpha_g(V_h) = V_{ghg^{-1}}$ and* Ad $V_g = \alpha_g$ *on $A \subseteqq A^\infty$.*

PROOF. This proof follows exactly the calculation of [8, Theorem 4.3.1]. One forms the semi-direct product $G \rtimes G$ and makes use of the results of Section I, which are available to us as A_∞ contains $|G|^2 \times |G|^2$ matrix units. Suppose that W_h are chosen (by approximate innerness) in A^∞ such that Ad $W_h = \alpha_h$ on $A \subseteqq A^\infty$, then

$$\begin{aligned}\alpha_g(W_k)X\alpha_g(W_k^*) &= \alpha_g(W_k\alpha_{g^{-1}}(X)W_k^*)\\ &= \alpha_g(\alpha_{kg^{-1}}(X))\\ &= W_{gkg^{-1}}XW_{gkg^{-1}}^*\end{aligned}$$

so that $\alpha_g(W_k) \in W_{gkg^{-1}}A_\infty$. We may then use III.2 and I.4 to replace the use of [8, Corollary 4.1.7] and II.1 to replace the use of [8, 3.1.3].

COROLLARY III.4. *There are matrix units $\{E_{g,h}\}$ in A^∞ with $E_{g,g} \in A_\infty$, $\alpha_g(E_{h,k}) = E_{gh,gk}$* and Ad $(\sum_{h\in G} E_{gh,h}) = \alpha_g$ *on $A \subseteqq A^\infty$.*

PROOF. We have the existence of F_g in A_∞ with $\alpha_g(F_h) = F_{gh}$ (Property $\mathscr{R}_\infty$). Passing to a subsequence we may suppose $V_gF_hV_g^* = F_{gh}$ with $\alpha_g(V_h) = V_{ghg^{-1}}$ and Ad $V_g = \alpha_g$ on $A \subseteqq A^\infty$. Let $E_{g,h} \equiv V_{gh^{-1}}F_h$.

COROLLARY III.5. *If $a_1, a_2, \ldots, a_n$ are elements of A and $\varepsilon > 0$ is given, then there are matrix units $e_{g,h}$ in A with $\|[e_{g,g}, a_1]\| < \varepsilon$ and*

$$\|\text{Ad}\, r_g(a_1) - \alpha_g(a_1)\| < \varepsilon \quad \text{for } i = 1, 2, \ldots, n$$

where $r_g = \sum_{h\in G} e_{gh,h}$.

PROOF. Combine III.4 and II.2.

We now have the main

THEOREM III.6. *If α is an approximately inner action of the finite group G on an* AF C*-*algebra A isomorphic to its tensor product with a* $|G|^\infty$ UHF *algebra B such that α has property $\mathscr{R}_\infty$, then α is conjugate to $s_G \otimes \mathrm{id}$ for some factorization $B \otimes A$, where s_G is the infinite tensor product action of the regular representation acting by conjugation on the $|G| \times |G|$ matrix algebra.*

PROOF. The argument of 5.3.3 of [8] goes through with the modifications used in of [7] to obtain that α is conjugate to $s_G \otimes \mathrm{id}$ on some splitting $B \otimes (B' \cap A)$ of A. But since the tensor product of the regular representation with itself is conjugate to the tensor product of the regular representation with a trivial representation of dimension $|G|$, we deduce that s_G is conjugate to $s_G \otimes \mathrm{id}$ on $B \otimes B$. Hence α is conjugate to $s_G \otimes \mathrm{id} \otimes \mathrm{id}$ on $B \otimes (B \otimes B' \cap A) = B \otimes A$.

COROLLARY III.7. *If β is an approximately inner automorphic action of G on a simple unital* C*-*algebra B and if A is the* $|G|^\infty$ UHF *algebra, form the action $\beta \otimes s_G$ on $B \otimes A$. We then have $\beta \otimes s_G \sim \mathrm{id} \otimes s_G$.*

PROOF. This is identical with the corresponding result in [7], an additional comment is, however, warranted. We are applying our Theorem III.6 to the algebra $B \otimes A$ and there is no assumption that $B \otimes A$ is AF. However this assumption was only used in one place, viz. Lemma III.2. It is apparent though that $\beta \otimes s_G$ has the required matrix units (as s_G does).

IV. The fixed point algebra.

A question which so far remains unanswered is whether or not the fixed point subalgebra of an outer action of a finite group on an AF-algebra is again AF. We show the following:

THEOREM IV.1. *Suppose that A is a* UHF C*-*algebra and α is an outer action of the finite group G on A. If α remains outer in the trace representation then the fixed point subalgebra A^G is the closed linear span of its projections.*

PROOF. That A^G is simple holds more generally [3, 13]. We claim that A^G has a unique trace. Since $G \times_\alpha A$ is simple A^G and $G \times_\alpha A$ are stably isomorphic [12, 13], so it will be enough to know that $G \times_\alpha A$ has a unique trace. If u_g denote the unitary of the crossed product giving α_g on A there is only one trace τ extending the trace on A so that $\tau(u_g) = 0$ for all $g \in G \setminus \{e\}$. If φ is another trace on $G \times_\alpha A$ it will suffice then to show that $\varphi(u_h)=0$, if $h \in G \setminus \{e\}$. We may thus restrict attention to the cyclic subgroup H of order n generated by a given $h \in G \setminus \{e\}$ and restrict φ to $H \times_\alpha A \subseteqq G \times_\alpha A$. Since H is abelian we consider the dual action $\hat{\alpha}$ of $\hat{H}$ in $H \times_\alpha A$. But then $(\varphi(x)+\varphi(\hat{\alpha}(x))+\ldots+\varphi(\hat{\alpha}^n(x)))/(n+1)$ is the trace $\tau | H \times_\alpha A$. Since α is outer in the tracial representation τ is a factor trace and so extremal [16]. Thus $\varphi(x)=\tau(x)$ for all $H \times_\alpha A$ and so $\varphi(u_h)=0$.

To complete the proof of the theorem it remains only to show [11] that A^G has a projection. For this we appeal to Elliot's extension [3] of a result of Connes [2]. Since α is outer, given $\varepsilon>0$ we can find, under any projection e, a projection f such that $\|f\alpha_h(f)\|<\varepsilon$. If we form $f+\alpha_h(f)+\ldots+\alpha_k(f)=x$, we see that x is a fixed element and x^2-x is small. That is, there is a gap in the spectrum of x. A continuous function of x results in a projection in A^G.

We remark that David Handelman [6] and Antony Wassermann [15] have analyzed in detail the situation with regard to traces on the fixed point algebra. We thank them for conversations about their work.

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