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EIGENSPACE REPRESENTATIONS OF NILPOTENT LIE GROUPS II

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1. Introduction.

The present paper generalizes and completes the results of [2] on eigenspace representations and invariant differential operators on homogeneous spaces for nilpotent Lie groups.

Let G be a connected and simply connected, complex or real, nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let α: $\mathfrak{g} \to \mathbf{C}$ be a linear functional on $\mathfrak{g}$ and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ subordinate to α, i.e. with $\alpha([\mathfrak{k},\mathfrak{k}])=\{0\}$.

For G complex we consider the left regular representation $\zeta_{\alpha,\mathfrak{k}}$ of G on the joint eigenspace

$$\mathscr{H}_{\alpha,\mathfrak{k}}(G) := \{f \in \mathscr{H}(G) \mid Xf=\alpha(X)f \,\forall\, X \in \mathfrak{k}\}$$

of holomorphic functions on G, and for G real we consider the left regular representation $\lambda_{\alpha,\mathfrak{k}}$ of G on the joint eigenspace

$$\mathscr{E}_{\alpha,\mathfrak{k}}(G) := \{f \in \mathscr{E}(G) \mid Xf=\alpha(X)f \,\forall\, X \in \mathfrak{k}\}$$

of C^∞-functions on G.

We show that these representations may be realized on $\mathscr{H}(\mathbf{C}^n)$ (respectively $\mathscr{E}(\mathbf{R}^n)$), $n=\dim \mathfrak{g}/\mathfrak{k}$, in such a way that if $\mathfrak{k}$ satisfies a certain maximality condition relative to α, then the derived representation $d\zeta_{\alpha,\mathfrak{k}}$ (respectively $d\lambda_{\alpha,\mathfrak{k}}$) maps the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ onto the algebra of all differential operators on $\mathbf{C}^n$ (respectively $\mathbf{R}^n$) with polynomial coefficients (Theorems 3.1 and 3.2). From this, irreducibility of the group representations is derived (Corollary 3.3).

These results contain and extend Theorems 4.1 and 5.1 of [2], where only real groups were considered and where it was assumed that α is of the special form $\alpha=c\beta$ for some $c \in \mathbf{C}$ and $\beta \in \mathfrak{g}^*$.

We prove that the maximality condition on $\mathfrak{k}$ relative to α, imposed as a sufficient condition in the above irreducibility statement, is also a necessary one (Corollary 4.4). No result of this kind was obtained in [2].

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connected subgroup H of G ensuring that the algebra $D(G/H)$ of G-invariant differential operators on the homogeneous space G/H is generated by a single vector field (Theorem 4.5). Sufficiency of this condition was proved (for real G) in [2].

The results on the eigenspace representations $\zeta_{\alpha,\mathfrak{k}}$ and $\lambda_{\alpha,\mathfrak{k}}$ described above are analogous to classical results on the unitary representations of nilpotent Lie groups, Theorems 7.1 and 5.2(1) of Kirillov [3].

2. Definitions and preliminaries.

Throughout this paper G denotes a connected and simply connected, real or complex nilpotent Lie group with Lie algebra $\mathfrak{g}$ of left invariant vector fields on G. The universal enveloping algebra of $\mathfrak{g}$ is denoted $\mathscr{U}(\mathfrak{g})$.

The dual space of $\mathfrak{g}$ is denoted $\mathfrak{g}^*$, and for $\mathfrak{g}$ real we denote by $(\mathfrak{g}^*)^{\mathbf{C}}$ the set of all complex-valued, real linear functionals on $\mathfrak{g}$. For $\mathfrak{g}$ real and $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$ we denote by α^c the extension of α to an element of $(\mathfrak{g}^{\mathbf{C}})^*$, where $\mathfrak{g}^{\mathbf{C}}$ denotes the complexification of $\mathfrak{g}$.

For $\alpha \in \mathfrak{g}^*$, or $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$ if $\mathfrak{g}$ is real, $S(\alpha, \mathfrak{g})$ denotes the set of subalgebras $\mathfrak{k}$ of $\mathfrak{g}$ which are subordinate to α, i.e. for which $\alpha([\mathfrak{k},\mathfrak{k}])=\{0\}$, and $M(\alpha,\mathfrak{g})$ the subset of $S(\alpha,\mathfrak{g})$ consisting of the subalgebras in $S(\alpha,\mathfrak{g})$ of maximal dimension.

Since $\mathfrak{g}$ is nilpotent we have for $\alpha \in \mathfrak{g}^*$ ($\mathfrak{g}$ real or complex) that a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ belongs to $M(\alpha,\mathfrak{g})$ if and only if it, as a subspace of $\mathfrak{g}$, is maximally totally isotropic with respect to the bilinear form $\alpha([\cdot,\cdot])$, i.e. iff

$$\forall X \in \mathfrak{g} : \; X \in \mathfrak{k} \Leftrightarrow \alpha([X,\mathfrak{k}]) = \{0\} .$$

For G complex, $\alpha \in \mathfrak{g}^*$ and $\mathfrak{k} \in S(\alpha,\mathfrak{g})$, the joint eigenspace $\mathscr{H}_{\alpha,\mathfrak{k}}(G)$ is defined in the introduction. There $\mathscr{H}(G)$ denotes the space of holomorphic functions on G, equipped with the topology of uniform convergence on compact subsets of G. If K denotes the analytic subgroup of G corresponding to $\mathfrak{k}$, there exists a character $\chi\colon K \to \mathscr{C}$ such that $\chi(\exp X)=e^{\alpha(X)}$ for all $X \in \mathfrak{k}$, and then

$$\mathscr{H}_{\alpha,\mathfrak{k}}(G) = \{f \in \mathscr{H}(G) \mid f(gk)=f(g)\chi(k) \;\forall g \in G, k \in K\} .$$

The space $\mathscr{H}_{\alpha,\mathfrak{k}}(G)$ is a closed subspace of $\mathscr{H}(G)$, invariant under the left regular representation of G on $\mathscr{H}(G)$ which then restricts to a holomorphic representation $\zeta_{\alpha,\mathfrak{k}}$ of G on $\mathscr{H}_{\alpha,\mathfrak{k}}(G)$.

Similarly, for G real, $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$ and $\mathfrak{k} \in S(\alpha,\mathfrak{g})$, the left regular representation of G on $\mathscr{E}(G)$ ($=C^\infty(G)$ equipped with its usual topology) restricts to a differentiable representation $\lambda_{\alpha,\mathfrak{k}}$ of G on the joint eigenspace $\mathscr{E}_{\alpha,\mathfrak{k}}(G)$ defined in the introduction. Also $\mathscr{E}_{\alpha,\mathfrak{k}}(G)$ is alternatively described in terms of the character on K determined by α.

An ordered basis $\Xi=\{X_1,\ldots,X_n\}$ for $\mathfrak{g}$ modulo a subalgebra $\mathfrak{k}$ is called *coexponential*, if for every $i=1,\ldots,n$, $\mathfrak{g}_i := \operatorname{span}\{X_{i+1},\ldots,X_n,\mathfrak{k}\}$ is an ideal of $\mathfrak{g}_{i-1}$. Such a basis exists for every $\mathfrak{k}$ since $\mathfrak{g}$ is nilpotent.

In the rest of this section we let G be complex, the real case is quite analogous, just exchange $\mathbf{C}$ with $\mathbf{R}$, $\mathscr{H}$ with $\mathscr{E}$ and ζ with λ; cfr. [2].

Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$, K the corresponding analytic subgroup of G and $\Xi=\{X_1,\ldots,X_n\}$ a coexponential basis for $\mathfrak{g}$ modulo $\mathfrak{k}$. Then the map

$$s:\ (x_1,\ldots,x_n)\mapsto \exp(x_1X_1)\ldots\exp(x_nX_n)K$$

is a bianalytic diffeomorphism of $\mathbf{C}^n$ onto the coset space G/K, and if Ξ' is another coexponential basis for $\mathfrak{g}$ modulo $\mathfrak{k}$ with corresponding map $s'\colon \mathbf{C}^n \to G/K$, then the composite map $s^{-1}\circ s'\colon \mathbf{C}^n\to\mathbf{C}^n$ is polynomial. Thus the identification of G/K with C^n via s gives rise to an unambiguous notion of the algebra $\operatorname{Pol}(G/K)$ of polynomial functions on G/K and the algebra $\operatorname{DP}(G/K)$ of differential operators on G/K with polynomial coefficients.

Also, if $\mathfrak{k}\in S(\alpha,\mathfrak{g})$, the map $S\colon \mathscr{H}(G)\to\mathscr{H}(\mathbf{C}^n)$ given by

$$(Sf)(x_1,\ldots,x_n) = f(\exp(x_1X_1)\ldots\exp(x_nX_n))$$

restricts to a topological isomorphism $S_{\alpha,\mathfrak{k},\Xi}$ of $\mathscr{H}_{\alpha,\mathfrak{k}}(G)$ onto $\mathscr{H}(\mathbf{C}^n)$ making $\zeta_{\alpha,\mathfrak{k}}$ equivalent to a representation $\zeta_{\alpha,\mathfrak{k},\Xi}$ of the form

$$[\zeta_{\alpha,\mathfrak{k},\Xi}(g)f](x) = e^{\alpha(p(g,x))}f(g^{-1}\cdot x)\,,$$

where $g\in G$, $f\in\mathscr{H}(\mathbf{C}^n)$, and $x\in\mathbf{C}^n$. Here p is a polynomial map $G\times\mathbf{C}^n\to\mathfrak{k}$ and $g\cdot x$ denotes the action of g on x induced by the identification of G/K with $\mathbf{C}^n$ by means of s. The formula implies that $d\zeta_{\alpha,\mathfrak{k},\Xi}(\mathscr{U}(\mathfrak{g}))$ is contained in $\operatorname{DP}(\mathbf{C}^n)$, the algebra of all differential operators on $\mathbf{C}^n$ with polynomial coefficients.

If Ξ' is another coexponential basis for $\mathfrak{g}$ modulo $\mathfrak{k}$ then the equivalence $T=S_{\alpha,\mathfrak{k},\Xi}\circ S^{-1}_{\alpha,\mathfrak{k},\Xi'}$ between $\zeta_{\alpha,\mathfrak{k},\Xi'}$ and $\zeta_{\alpha,\mathfrak{k},\Xi}$ is of the form

$$[Tf](x) = e^{\alpha(q(x))}f(r(x)),\qquad f\in\mathscr{H}(\mathbf{C}^n)\,,$$

for certain polynomial maps $q\colon \mathbf{C}^n\to\mathfrak{k}$ and $r\colon\mathbf{C}^n\to\mathbf{C}^n$, where also r^{-1} is polynomial. In particular T defines an algebra automorphism $D\mapsto T\circ D\circ T^{-1}$ of $\operatorname{DP}(\mathbf{C}^n)$.

Notation analogous the one introduced for complex G will be used for real G.

$\operatorname{DP}(\mathbf{R}^n)$ will denote the complex algebra of all differential operators on $\mathbf{R}^n$ with polynomial coefficients.

3. Representations of $\mathscr{U}(\mathfrak{g})$. Irreducibility.

The following theorem is the analogue for complex groups of Theorem 4.1 in [2].

3.1. THEOREM. *Let G be a connected and simply connected, complex nilpotent Lie group with Lie algebra* $\mathfrak{g}$. *Let* $\alpha \in \mathfrak{g}^*$, $\mathfrak{k} \in M(\alpha, \mathfrak{g})$, *and let* Ξ *be a coexponential basis for* $\mathfrak{g}$ *modulo* $\mathfrak{k}$, *then*

$$d\zeta_{\alpha,\mathfrak{k},\Xi}(\mathscr{U}(\mathfrak{g})) = \mathrm{DP}\,(\mathbf{C}^n)\,, \tag{3.1}$$

where $n = \dim \mathfrak{g}/\mathfrak{k}$.

PROOF. If $\dim \mathfrak{g} \leqq 2$, then $\mathfrak{g}$ is abelian and so $\mathfrak{k} = \mathfrak{g}$, that is $n = 0$. In this case $\mathrm{DP}\,(\mathbf{C}^n)$ consists of the scalars only in which case (3.1) clearly holds. We proceed by induction on $\dim \mathfrak{g}$.

By the last paragraph of Section 2 it suffices to prove (3.1) for one choice of Ξ.

Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$ and observe that $\mathfrak{z} \subseteqq \mathfrak{k}$ since $\mathfrak{k} \in M(\alpha, \mathfrak{g})$.

If $\mathfrak{i} := \mathfrak{z} \cap \ker \alpha \neq \{0\}$, we consider the quotient group $\tilde{G} = G/\exp(\mathfrak{i})$ with Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{i}$. Since $\mathfrak{i} \subset \ker \alpha$, there exists $\tilde{\alpha} \in \tilde{\mathfrak{g}}^*$ such that $\tilde{\alpha}(\tilde{X}) = \alpha(X)$ for all $X \in \mathfrak{g}$, where $X \mapsto \tilde{X}$ denotes the quotient map $\mathfrak{g} \to \tilde{\mathfrak{g}}$. Clearly $\tilde{\alpha} \neq 0$, $\tilde{\mathfrak{k}} \in M(\tilde{\alpha}, \tilde{\mathfrak{g}})$, $\tilde{\Xi}$ is a coexponential basis for $\tilde{\mathfrak{g}}$ modulo $\tilde{\mathfrak{k}}$ and

$$d\zeta_{\alpha,\mathfrak{k},\Xi}(X) = d\zeta_{\tilde{\alpha},\tilde{\mathfrak{k}},\tilde{\Xi}}(\tilde{X}) \quad \text{for all } X \in \mathfrak{g}\,.$$

Hence

$$d\zeta_{\alpha,\mathfrak{k},\Xi}(\hat{\mathscr{U}}(\mathfrak{g})) = \mathrm{DP}\,(\mathbf{C}^n)$$

by the induction hypothesis applied to $\tilde{G}$.

From now on assume $\mathfrak{z} \cap \ker \alpha = \{0\}$. Then $\dim \mathfrak{z} = 1$, since $\mathfrak{z} \subseteqq \mathfrak{k}$. Let $Z \in \mathfrak{z}$ with $\alpha(Z) = 1$. Let $Y \in \mathfrak{g}$ represent a non-zero element of the center of $\mathfrak{g}/\mathfrak{z}$, then

$$\mathfrak{g}_0 := \{V \in \mathfrak{g} \mid [V, Y] = 0\}$$

is an ideal of codimension 1 in $\mathfrak{g}$. We may choose Y so that $\alpha(Y) = 0$.

The rest of the proof is divided according to whether .

$$\text{(I) } \mathfrak{k} \subseteqq \mathfrak{g}_0 \qquad \text{or} \qquad \text{(II) } \mathfrak{k} \nsubseteqq \mathfrak{g}_0\,.$$

(I) Assume $\mathfrak{k} \subseteqq \mathfrak{g}_0$ and put $\alpha_0 = \alpha \mid \mathfrak{g}_0$. Then $\alpha_0 \neq 0$ and $\mathfrak{k} \in M(\alpha_0, \mathfrak{g}_0)$. Choose $X_1 \in \mathfrak{g} \setminus \mathfrak{g}_0$ such that $[X_1, Y] = Z$ and let $\Xi_0 = \{X_2, \ldots, X_n\}$ be a coexponential basis for $\mathfrak{g}_0$ modulo $\mathfrak{k}$, thus making $\Xi = \{X_1, X_2, \ldots, X_n\}$ a coexponential basis for $\mathfrak{g}$ modulo $\mathfrak{k}$. Set $\zeta = \zeta_{\alpha,\mathfrak{k},\Xi}$ and $\zeta_0 = \zeta_{\alpha_0,\mathfrak{k},\Xi_0}$, the latter being a representation of the subgroup $G_0 = \exp(\mathfrak{g}_0)$ of G. Then $d\zeta(X_1) = -\partial/\partial x_1$, while for every $V_0 \in \mathfrak{g}_0$, $\varphi \in \mathscr{H}(\mathbf{C}^n)$ and $(x_1, \underline{x}_0) \in \mathbf{C} \times \mathbf{C}^{n-1} = \mathbf{C}^n$

$$[d\zeta(V_0)\varphi](x_1, x_0) = [d\zeta_0(e^{-x_1 \operatorname{ad} X_1} V_0)\varphi(x_1, \cdot)](\underline{x}_0)\,. \tag{3.2}$$

Since $\mathfrak{k} \in M(\alpha_0, \mathfrak{g}_0)$, $\mathfrak{k}$ contains the central elements Z and Y of $\mathfrak{g}_0$, whence $d\zeta_0(Z) = -\alpha_0(Z)I = -I$ and $d\zeta_0(Y) = -\alpha_0(Y)I = 0$. Hence by (3.2), $d\zeta(Y) = x_1$.

Furthermore, since by (3.2)

$$[d\zeta_0(V_0)\varphi(x_1,\cdot)](\underline{x}_0) = \sum_{k=0}^{\infty} \frac{x_1^k}{k!}[d\zeta((\mathrm{ad}\, X_1)^k V_0)\varphi](x_1,\underline{x}_0),$$

the series being finite because $\mathfrak{g}$ is nilpotent, we have

$$1_{\mathbf{C}} \otimes d\zeta_0(\mathscr{U}(\mathfrak{g}_0)) \subseteqq d\zeta(\mathscr{U}(\mathfrak{g}_0)).$$

So applying the induction hypothesis to G_0 we conclude $d\zeta(\mathscr{U}(\mathfrak{g})) = \mathrm{DP}\,(\mathbf{C}^n)$.

(II) Assume $\mathfrak{k} \nsubseteq \mathfrak{g}_0$. Then there exists $X \in \mathfrak{k}$ with $[X, Y] = Z$, and this implies that $Y \notin \mathfrak{k}$ since $\mathfrak{k} \in S(\alpha, \mathfrak{g})$ and $\alpha(Z) \neq 0$. We may choose X such that $\alpha(X) = 0$.

Let $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$ and set $\mathfrak{k}' = \mathbf{C}Y + \mathfrak{k}_0$. Then $\mathfrak{k}' \in S(\alpha, \mathfrak{g})$ and $\dim \mathfrak{k}' = \dim \mathfrak{k}$, so in fact $\mathfrak{k}' \in M(\alpha, \mathfrak{g})$. Also $\mathfrak{k}' \subseteqq \mathfrak{g}_0$. Note that $\mathfrak{k} = \mathbf{C}X + \mathfrak{k}_0$.

The direct sum of vector spaces $\tilde{\mathfrak{g}} = \mathbf{C}X + \mathbf{C}Y + \mathfrak{k}_0$ is a subalgebra of $\mathfrak{g}$ of codimension $n-1$, so if $\{X_1, \ldots, X_{n-1}\}$ is a coexponential basis for $\mathfrak{g}$ modulo $\tilde{\mathfrak{g}}$, then $\Xi := \{X_1, \ldots, X_{n-1}, Y\}$ is a coexponential basis for $\mathfrak{g}$ modulo $\mathfrak{k}$, while $\Xi' := \{X_1, \ldots, X_{n-1}, X\}$ is a coexponential basis for $\mathfrak{g}$ modulo $\mathfrak{k}'$. Let $\zeta = \zeta_{\alpha,\mathfrak{k},\Xi}$ and $\zeta' = \zeta_{\alpha,\mathfrak{k}',\Xi'}$, then

$$\Phi(d\zeta'(D)) = d\zeta(D) \quad \text{for all } D \in \mathscr{U}(\mathfrak{g}), \tag{3.3}$$

where Φ denotes the algebra automorphism of $\mathrm{DP}\,(\mathbf{C}^n)$ given by

$$\begin{aligned} \Phi(x_i) &= x_i, & \Phi\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_i} \qquad i = 1, \ldots, n-1 \\ \Phi(x_n) &= -\frac{\partial}{\partial x_n}, & \Phi\left(\frac{\partial}{\partial x_n}\right) &= x_n . \end{aligned} \tag{3.4}$$

The relation (3.3) is proved by induction on $\dim \mathfrak{g}/\tilde{\mathfrak{g}} = n-1$: If $\dim \mathfrak{g}/\tilde{\mathfrak{g}} = 0$ then $n = 1$ and by direct calculations $d\zeta'(X) = -\partial/\partial x_n$, $d\zeta(X) = -\alpha(Z)x_n = -x_n$, $d\zeta'(Y) = \alpha(Z)x_n = x_n$, and $d\zeta(Y) = -\partial/\partial x_n$, while $d\zeta'(V) = d\zeta(V) = -\alpha(V)I$ for all $V \in \mathfrak{k}_0$. This proves (3.3) when $\dim \mathfrak{g}/\tilde{\mathfrak{g}} = 0$.

Let $\dim \mathfrak{g}/\tilde{\mathfrak{g}} > 0$ and suppose that (3.3) holds when $\mathfrak{g}$ there is replaced by any subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ for which $\mathfrak{g}_0 \supseteqq \tilde{\mathfrak{g}}$ and $\dim \mathfrak{g}_0/\tilde{\mathfrak{g}} < \dim \mathfrak{g}/\tilde{\mathfrak{g}}$ and ζ, ζ' are replaced by corresponding representations of $G_0 = \exp\,(\mathfrak{g}_0)$. Then (3.3) follows from the recursion formulas, cfr. (3.2),

$$d\zeta(X_1) = -\frac{\partial}{\partial x_1}; \qquad d\zeta(V) = \sum_{k=0}^{\infty} \frac{(-x_1)^k}{k!} 1_{\mathbf{C}} \otimes d\zeta_2((\mathrm{ad}\, X_1)^k V)$$

$$d\zeta'(X_1) = -\frac{\partial}{\partial x_1}; \qquad d\zeta'(V) = \sum_{k=0}^{\infty} \frac{(-x_1)^k}{k!} 1_{\mathbf{C}} \otimes d\zeta_2'((\mathrm{ad}\, X_1)^k V),$$

where $V \in \mathfrak{g}_2 = \text{span}\{X_2, \ldots, X_{n-1}, \tilde{\mathfrak{g}}\}$, $\zeta_2 = \zeta_{\alpha_2, \mathfrak{k}, \Xi_2}$, and $\zeta_2' = \zeta_{\alpha_2, \mathfrak{k}', \Xi_2'}$, with $\alpha_2 = \alpha | \mathfrak{g}_2$, $\Xi_2 = \{X_2, \ldots, X_{n-1}, Y\}$, and $\Xi_2' = \{X_2, \ldots, X_{n-1}, X\}$.

Now, since $\mathfrak{k}' \subseteqq \mathfrak{g}_0$, we have by (I) $d\zeta'(\mathscr{U}(\mathfrak{g})) = \mathrm{DP}(\mathbf{C}^n)$, hence by (3.3) $d\zeta(\mathscr{U}(\mathfrak{g})) = \mathrm{DP}(\mathbf{C}^n)$.

This finishes the proof of the theorem.

As a corollary we obtain the following extension of Theorem 4.1 of [2], cf. Remark 1 below, the extension being that the functiomal α may be general complex-valued, not just of the special form $\alpha = c\beta$ with $c \in \mathbf{C}$ and $\beta \in \mathfrak{g}^*$.

3.2. THEOREM. *Let G be a connected and simply connected, real nilpotent Lie group with Lie algebra $\mathfrak{g}$.*

Let $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$, $\mathfrak{k} \in S(\alpha, \mathfrak{g})$ and let Ξ be a coexponential basis for $\mathfrak{g}$ modulo $\mathfrak{k}$. If $\mathfrak{k}^{\mathbf{C}} \in M(\alpha^c, \mathfrak{g}^{\mathbf{C}})$ then

$$d\lambda_{\alpha, \mathfrak{k}, \Xi}(\mathscr{U}(\mathfrak{g})^{\mathbf{C}}) = \mathrm{DP}(\mathbf{R}^n),$$

where $n = \dim \mathfrak{g}/\mathfrak{k}$.

PROOF. Let G^c denote the complexification of G, i.e. the connected and simply connected complex Lie group with Lie algebra $\mathfrak{g}^{\mathbf{C}}$, and consider the representation $\zeta := \zeta_{\alpha^c, \mathfrak{k}^{\mathbf{C}}, \Xi}$ of G^c on $\mathscr{H}(\mathbf{C}^n)$. It is easily seen that the restriction map $f \mapsto f|_{\mathbf{R}^n}$ of $\mathscr{H}(\mathbf{C}^n)$ into $\mathscr{E}(\mathbf{R}^n)$ intertwines the representations $\zeta | G$ and $\lambda_{\alpha, \mathfrak{k}, \Xi}$ of G. The theorem is now consequence of Theorem 3.1.

REMARK 1. If $\mathfrak{g}$ is real and $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$ is of the form $\alpha = c\beta$ for some $c \in \mathbf{C}$ and $\beta \in \mathfrak{g}^*$, then for $\mathfrak{k} \in S(\alpha, \mathfrak{g})$ we have $\mathfrak{k} \in M(\alpha, \mathfrak{g})$ if and only if $\mathfrak{k}^{\mathbf{C}} \in M(\alpha^c, \mathfrak{g}^{\mathbf{C}})$. Hence Theorem 3.1 above contains Theorem 4.1 of [2].

REMARK 2. For $\mathfrak{g}$ real and $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$ there may in general not exist a $\mathfrak{k} \in S(\alpha, \mathfrak{g})$ for which $\mathfrak{k}^{\mathbf{C}} \in M(\alpha^c, \mathfrak{g}^{\mathbf{C}})$, as shown by the following Example 1 (see also Remark 5.3 of [2]).

EXAMPLE 1. Let $\mathfrak{g}$ be a real nilpotent Lie algebra and let $\alpha \in \mathfrak{g}^*$. Let $\mathfrak{g}_0$ denote the underlying real Lie algebra of $\mathfrak{g}^{\mathbf{C}}$ and let α_0 denote $\alpha^{\mathbf{C}}$ considered as an element of $(\mathfrak{g}_0^*)^{\mathbf{C}}$. Note that the elements $1 \otimes_{\mathbf{R}} 1 \otimes_{\mathbf{R}} X + i \otimes_{\mathbf{R}} i \otimes_{\mathbf{R}} X$, $X \in \mathfrak{g}$, span an ideal $\mathfrak{i}$ of $\mathfrak{g}_0^{\mathbf{C}}$ and that $\mathfrak{i} \subseteqq \ker \alpha_0^c$ and $\mathfrak{i} + \bar{\mathfrak{i}} = \mathfrak{g}_0^{\mathbf{C}}$.

Now, let $\mathfrak{k} \in S(\alpha_0, \mathfrak{g}_0)$ and suppose $\mathfrak{k}^{\mathbf{C}} \in M(\alpha_0^c, \mathfrak{g}_0^{\mathbf{C}})$. Then $\alpha_0^c([\mathfrak{k}^{\mathbf{C}}, \mathfrak{i}]) = \{0\}$, so $\mathfrak{i} \subseteqq \mathfrak{k}^{\mathbf{C}}$ and thus $\bar{\mathfrak{i}} \subseteqq \mathfrak{k}^{\mathbf{C}}$, whence $\mathfrak{k}^{\mathbf{C}} = \mathfrak{g}_0^{\mathbf{C}}$, that is $\mathfrak{k} = \mathfrak{g}_0$.

Hence if $\alpha([\mathfrak{g}, \mathfrak{g}]) \neq \{0\}$, there does not exist $\mathfrak{k} \in S(\alpha_0, \mathfrak{g}_0)$ for which $\mathfrak{k}^{\mathbf{C}} \in M(\alpha_0^c, \mathfrak{g}_0^{\mathbf{C}})$.

EXAMPLE 2. If $\mathfrak{g}$ is the $2n+1$ dimensional Heisenberg algebra, or more generally if $\mathfrak{g}$ is nilpotent and $\dim[\mathfrak{g},\mathfrak{g}] \leqq 1$, then for every $\alpha \in (\mathfrak{g}^*)^{\mathbf{C}}$ there exists $\mathfrak{k} \in S(\alpha,\mathfrak{g})$ such that $\mathfrak{k}^{\mathbf{C}} \in M(\alpha^c,\mathfrak{g}^{\mathbf{C}})$, in fact then $\mathfrak{k} \in M(\alpha,\mathfrak{g})$ implies that $\mathfrak{k}^{\mathbf{C}} \in M(\alpha^c,\mathfrak{g}^{\mathbf{C}})$.

3.3. COROLLARY. *Let G be a connected and simply connected, complex (respectively real) nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $\alpha \in \mathfrak{g}^*$ (respectively $(\mathfrak{g}^*)^{\mathbf{C}}$) and $\mathfrak{k} \in S(\alpha,\mathfrak{g})$. Then, if $\mathfrak{k} \in M(\alpha,\mathfrak{g})$ (respectively $\mathfrak{k}^{\mathbf{C}} \in M(\alpha^c,\mathfrak{g}^{\mathbf{C}})$), the representation $\zeta_{\alpha,\mathfrak{k}}$ (respectively $\lambda_{\alpha,\mathfrak{k}}$) of G is both topologically and operator irreducible.*

PROOF. For real G this follows from Theorem 3.2 as in [2] and for complex G it follows from Theorem 3.1 via the Lemma below.

3.4. LEMMA. (a) *If $V \neq \{0\}$ is a closed subspace of $\mathscr{H}(\mathbf{C}^n)$ invariant under the action of every $D \in \mathrm{DP}(\mathbf{C}^n)$, then $V=\mathscr{H}(\mathbf{C}^n)$.*

(b) *If A is a densely defined, closed linear operator in $\mathscr{H}(\mathbf{C}^n)$ commuting with every $D \in \mathrm{DP}(\mathbf{C}^n)$, then $A \in \mathbf{C}I$.*

PROOF. (a) Let $f \in V$, $f \neq 0$. Since not all the derivatives of f vanish at 0 we may assume $f(0)=1$. Let $\mu \in V^{\perp} \subseteqq \mathscr{H}'(\mathbf{C}^n)$. Then the function

$$F(a) := \langle \mu_z, f(e^a z)\rangle$$

is holomorphic in $a \in \mathbf{C}$ and

$$\left[\left(\frac{d}{da}\right)^k F\right]_{a=0} = \langle \mu, D^k f\rangle = 0 \quad \text{for all } k=0,1,2,\ldots$$

where $D=z_1(\partial/\partial z_1)+\ldots+z_n(\partial/\partial z_n) \in \mathrm{DP}(\mathbf{C}^n)$. So $F\equiv 0$, and since $f(e^a z) \to f(0)=1$ uniformly in z on compact sets as $a \to -\infty$ along $\mathbf{R}$, it follows that $1 \in V$. Thus V contains all the polynomials on $\mathbf{C}^n$ and is therefore dense in $\mathscr{H}(\mathbf{C}^n)$.

(b) Let $f \neq 0$ be in the domain $D(A)$ of A. Then $f \neq 0$ on some open, connected set Ω in $\mathbf{C}^n$ and $(1/f)Af \in \mathscr{H}(\Omega)$. Since A commutes with multiplication by the polynomials on $\mathbf{C}^n$ and these are dense in $\mathscr{H}(\mathbf{C}^n)$, A commutes with multiplication by every $g \in \mathscr{H}(\mathbf{C}^n)$. So in Ω

$$\frac{\partial}{\partial z_i}\left(\frac{1}{f}Af\right) = \frac{1}{f^2}\left(f\cdot\frac{\partial}{\partial z_i}Af - \frac{\partial f}{\partial z_i}\cdot Af\right) = 0, \quad i=1,\ldots,n,$$

because A commutes with $\partial/\partial z_i$. Thus for some constant $c(f) \in \mathbf{C}$, $Af=c(f)f$ in Ω and therefore in $\mathbf{C}^n$ by uniqueness of analytic continuation. If $f, g \in D(A) \setminus \{0\}$, then $c(f)=c(g)$ since $c(g)fg=fAg=A(fg)=c(f)fg$. This proves (b).

4. Invariant differential operators. Reducibility.

In this section we shall work with the following non-standard definition.

4.1. Definition. A subalgebra $\mathfrak{h}$ of a (nilpotent) Lie algebra $\mathfrak{g}$ is called *maximal* if there exist $\alpha \in \mathfrak{g}^* \setminus \{0\}$ and $\mathfrak{k} \in M(\alpha, \mathfrak{g})$ such that $\mathfrak{h} = \mathfrak{k} \cap \ker \alpha$.

4.2. Lemma. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $\alpha \in \mathfrak{g}^*$, $\mathfrak{k} \in S(\alpha, \mathfrak{g})$. Assume $\alpha|_{\mathfrak{k}} \neq 0$ and set $\mathfrak{h} := \mathfrak{k} \cap \ker \alpha$. Then $\mathfrak{h}$ is maximal (if and) only if $\mathfrak{k} \in M(\alpha, \mathfrak{g})$.*

Proof. Suppose that $\mathfrak{h}$ is maximal: $\mathfrak{h} = \mathfrak{k}' \cap \ker \alpha'$ where $\alpha' \in \mathfrak{g}^* \setminus \{0\}$ and $\mathfrak{k}' \in M(\alpha', \mathfrak{g})$. Then, since $\mathfrak{g}$ is nilpotent, $\mathfrak{k}'$ equals the normalizer $\mathfrak{n}(\mathfrak{h})$ of $\mathfrak{h}$ in $\mathfrak{g}$. (Namely, let $\mathfrak{i}$ be a maximal element in the set of ideals of $\mathfrak{g}$ contained in $\mathfrak{h}$ and let Z represent a non-zero element in the center of $\mathfrak{g}/\mathfrak{i}$. Then $\mathfrak{k}' = \operatorname{span}\{Z\} + \mathfrak{h}$ and thus $[\mathfrak{n}(\mathfrak{h}), \mathfrak{k}'] \subseteq \mathfrak{h} \subseteq \ker \alpha$, from which it follows that $\mathfrak{n}(\mathfrak{h}) \subseteq \mathfrak{k}'$, whence $\mathfrak{n}(\mathfrak{h}) = \mathfrak{k}'$.) Also $\mathfrak{h} \neq \mathfrak{k} \subseteq \mathfrak{n}(\mathfrak{h})$ so in fact $\mathfrak{k} = \mathfrak{n}(\mathfrak{h}) = \mathfrak{k}'$. In particular $\mathfrak{k} \in M(\alpha', \mathfrak{g})$. Now, $\alpha(Z) \neq 0$ and $\alpha'(Z) \neq 0$, so we may replace α' by $\alpha(Z)\alpha'(Z)^{-1}\alpha'$ and thus assume that $\alpha' = \alpha$ on $\mathfrak{k}$. It follows that $\mathfrak{k} \in M(\alpha, \mathfrak{g})$, because, since $\mathfrak{g}$ is nilpotent and $\mathfrak{k} \in M(\alpha', \mathfrak{g})$, $\mathfrak{k}$ satisfies the Pukanszky condition relative to α': $\mathfrak{k} \in M(\alpha' + \varphi, \mathfrak{g})$ for all $\varphi \in \mathfrak{k}^\perp$, cf. [1, Chap. IV, sec. 3, pp. 69–70]. (The argument in [1] is for real $\mathfrak{g}$, but it works as well for complex $\mathfrak{g}$. Or use that if $\mathfrak{g}$ is complex, $\alpha \in \mathfrak{g}^*$ and $\mathfrak{k}$ a subalgebra of $\mathfrak{g}$, then $\mathfrak{k} \in M(\alpha, \mathfrak{g})$ if and only if $\mathfrak{k}_r \in M(\operatorname{Re} \alpha, \mathfrak{g}_r)$, where $\mathfrak{k}_r$ and $\mathfrak{g}_r$ denote $\mathfrak{k}$ and $\mathfrak{g}$ considered as real Lie algebras).

For a closed subgroup H of G we denote by $\boldsymbol{D}(G/H)$ the algebra of all G-invariant differential operators on the homogeneous space G/H. Assume that H is connected and proper and let $\mathfrak{h} \subseteq \mathfrak{g}$ be its Lie algebra. Then $\mathfrak{h}$ is properly contained in its normalizer $\mathfrak{n}(\mathfrak{h})$ in $\mathfrak{g}$, and any $Z \in \mathfrak{n}(\mathfrak{h}) \setminus \mathfrak{h}$ defines a nonzero G-invariant vector field on $\gamma(Z)$ on G/H by

$$[\gamma(Z)f](gH) = \left.\frac{d}{dt}\right|_{t=0} f(g \exp(tZ)H) .$$

4.3. Theorem. *Let G be a connected and simply connected, real or complex nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let H be a proper, connected subgroup of G with Lie algebra $\mathfrak{h}$ and let $Z \in \mathfrak{n}(\mathfrak{h}) \setminus \mathfrak{h}$.*

Then, if $\mathfrak{h}$ is not maximal (in the sense of Definition 4.1), there exists $D \in \boldsymbol{D}(G/H)$ such that D commutes with $\gamma(Z)$ and such that for every $a \in \mathbf{C}$ and $b \in \mathbf{C} \setminus \{0\}$ there exists a polynomial $p_{a,b}$ on G/H for which

(1) $De^{p_{a,b}} = ae^{p_{a,b}}$

(2) $\gamma(Z)e^{p_{a,b}} = be^{p_{a,b}}$.

PROOF. We prove the theorem for complex G, the real case is analogous.

Set $\mathfrak{k}:=\mathbf{C}Z+\mathfrak{h}$ and choose $\alpha \in \mathfrak{g}^*$ such that $\alpha(\mathfrak{h})=\{0\}$ and $\alpha(Z)=1$. Then $\mathfrak{k} \in S(\alpha,\mathfrak{g})$ and $\mathfrak{h}=\mathfrak{k}\cap\ker\alpha$. Since $\mathfrak{h}$ is not maximal we have by Lemma 4.2 that $\mathfrak{k} \notin M(\alpha,\mathfrak{g})$. Set $n=\dim\mathfrak{g}/\mathfrak{k}$.

First consider the case in which $\mathfrak{k}$ does not contain the center $\mathfrak{z}$ of $\mathfrak{g}$, and let $\{X_1,\ldots,X_n,Z\}$ be a coexponential basis for $\mathfrak{g}$ modulo $\mathfrak{h}$ with $X_n \in \mathfrak{z}\setminus\mathfrak{k}$. Then, denoting the corresponding coordinates on G/H by $x_1,\ldots,x_n,z$, we have $\gamma(X_n)=\partial/\partial x_n$ and $\gamma(Z)=\partial/\partial z$. Hence the conclusions of the theorem are satisfied with $D=\gamma(X_n)$ and $p_{a,b}=ax_n+bz$.

This takes care of the cases in which $\dim\mathfrak{g}\leqq 2$. We continue by induction on $\dim\mathfrak{g}$ and may in the proof of the induction step assume that $\mathfrak{z}\subseteqq\mathfrak{k}$.

If $\mathfrak{i}:=\mathfrak{z}\cap\mathfrak{h}\neq\{0\}$ we consider the quotient group $\tilde{G}=G/\exp(\mathfrak{i})$ with Lie algebra $\tilde{\mathfrak{g}}=\mathfrak{g}/\mathfrak{i}$. Denote the quotient maps $G\to\tilde{G}$ and $\mathfrak{g}\to\tilde{\mathfrak{g}}$ by $\sim$. Let Ξ be a coexponential basis for $\mathfrak{g}$ modulo $\mathfrak{h}$ and identify G/H and $\tilde{G}/\tilde{H}$ with $\mathbf{C}^{n+1}$ by means of Ξ and $\tilde{\Xi}$ respectively. Then $g\cdot x=\tilde{g}\cdot x$ for all $g \in G$ and $x \in \mathbf{C}^{n+1}$, so $\boldsymbol{D}(G/H)=\boldsymbol{D}(\tilde{G}/\tilde{H})$. Also $\tilde{\gamma}(\tilde{Z})=\gamma(Z)$. Let $\tilde{\alpha} \in \tilde{\mathfrak{g}}^*$ be given by $\tilde{\alpha}(\tilde{X})=\alpha(X)$ for all $X \in \mathfrak{g}$. Then $\tilde{\mathfrak{k}} \in S(\tilde{\alpha},\tilde{\mathfrak{g}})$, $\tilde{\alpha}|_{\tilde{\mathfrak{k}}}\neq 0$, and $\tilde{\mathfrak{h}}=\tilde{\mathfrak{k}}\cap\ker\tilde{\alpha}$. It is easily seen that $\mathfrak{k} \notin M(\alpha,\mathfrak{g})$ implies $\tilde{\mathfrak{k}} \notin M(\tilde{\alpha},\tilde{\mathfrak{g}})$, so by Lemma 4.2 $\tilde{\mathfrak{h}}$ is not maximal in $\tilde{\mathfrak{g}}$. This case is thus concluded by an application of the induction hypothesis to $\tilde{G}$.

From now on we assume that $\mathfrak{z}\cap\mathfrak{h}=\{0\}$. Then $\mathfrak{k}=\mathfrak{z}+\mathfrak{h}$ (direct sum), so $\dim\mathfrak{z}=1$. If $Z_1 \in \mathfrak{z}$ and $\alpha(Z_1)=1$ then $\gamma(Z)=\gamma(Z_1)$, so we may and will assume that $Z \in \mathfrak{z}$. We now choose Y and define $\mathfrak{g}_0$ as in the proof of Theorem 3.1 and divide the proof into the two cases

(I) $\mathfrak{k}\subseteqq\mathfrak{g}$ and (II) $\mathfrak{k}\nsubseteqq\mathfrak{g}_0$.

(I) Assume $\mathfrak{k}\subseteqq\mathfrak{g}_0$ and set $\alpha_0=\alpha|\mathfrak{g}_0$. As easily seen $\mathfrak{k} \notin M(\alpha_0,\mathfrak{g}_0)$, so by Lemma 4.2, $\mathfrak{h}$ is not maximal in $\mathfrak{g}_0$. Let Ξ_0 be a coexponential basis for $\mathfrak{g}_0$ modulo $\mathfrak{h}$ and extend it with some $X \in \mathfrak{g}\setminus\mathfrak{g}_0$ to a coexponential basis Ξ for $\mathfrak{g}$ modulo $\mathfrak{h}$. Identifying G_0/H with $\mathbf{C}^n$ by means of Ξ_0, G/H with $\mathbf{C}^{n+1}$ by means of Ξ and then G/H with $\mathbf{C}\times G_0/H$ in the natural way: $\mathbf{C}^{n+1}\cong\mathbf{C}\times\mathbf{C}^n$, we may consider the elements of $\boldsymbol{D}(G_0/H)$ as differential operators on G/H. Moreover, since $G=\exp(\mathbf{C}X)G_0$ and

$$\exp(tX)g_0\cdot(x,\underline{y}) = (x+t,g_0(x)\cdot\underline{y}) \text{ for all } t \in \mathbf{C},\ g_0 \in G_0 \text{ and } (x,\underline{y}) \in \mathbf{C}\times\mathbf{C}^n,$$

where $g_0(x)=\exp(-xX)g_0\exp(xX) \in G_0$, we have $\boldsymbol{D}(G_0/H)\subseteqq\boldsymbol{D}(G/H)$. Also $\gamma_0(Z)=\gamma(Z)$. Hence we conclude this case by an application of the induction hypothesis to G_0.

(II) Assume $\mathfrak{k}\nsubseteqq\mathfrak{g}_0$. Then there exists $X \in \mathfrak{h}$ with $[X,Y]=Z$. It follows that $Y\notin\mathfrak{k}$, because $\mathfrak{k}\subseteqq\mathfrak{n}(\mathfrak{h})$ and $Z \notin \mathfrak{h}$.

Define $\mathfrak{h}'=\mathbf{C}Y+\mathfrak{h}\cap\mathfrak{g}_0$ and $\mathfrak{k}'=\mathbf{C}Y+\mathfrak{k}\cap\mathfrak{g}_0$. Then $\mathfrak{k}'=\mathbf{C}Z+\mathfrak{h}'$, $\mathfrak{k}' \in S(\alpha,\mathfrak{g})$, $\alpha|_{\mathfrak{k}'}$

$\neq 0$, and $\mathfrak{h}' = \mathfrak{k}' \cap \ker \alpha$. Since $\dim \mathfrak{k}' = \dim \mathfrak{k}$ and $\mathfrak{k} \notin M(\alpha, \mathfrak{g})$ we have $\mathfrak{k}' \notin M(\alpha, \mathfrak{g})$, so by Lemma 4.2, $\mathfrak{h}'$ is not maximal in $\mathfrak{g}$.

Choose coexponential bases, Ξ for $\mathfrak{g}$ modulo $\mathfrak{k}$ and Ξ' for $\mathfrak{g}$ modulo $\mathfrak{k}'$, as in the proof of Theorem 3.1 and set $\zeta_b = \zeta_{b\alpha, \mathfrak{k}, \Xi}$ and $\zeta'_b = \zeta_{b\alpha, \mathfrak{k}', \Xi'}$ for $b \in \mathbf{C} \setminus \{0\}$. Then as in the proof of Theorem 3.1 we have

$$\Phi_b(d\zeta'_b(D)) = d\zeta_b(D) \quad \forall D \in \mathscr{U}(\mathfrak{g}), \tag{4.1}$$

where Φ_b denotes the automorphism of DP $(\mathbf{C}^n)$ given by

$$\begin{aligned} \Phi_b(x_i) &= x_i, & \Phi_b\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n-1 \\ \Phi_b(x_n) &= -\frac{1}{b}\frac{\partial}{\partial x_n}, & \Phi_b\left(\frac{\partial}{\partial x_n}\right) &= b x_n . \end{aligned} \tag{4.2}$$

Identify G/H and G/H' with $\mathbf{C}^{n+1}$ via the coexponential bases $\{\Xi, Z\}$ and $\{\Xi', Z\}$ respectively and denote the corresponding actions of G on $\mathscr{H}(\mathbf{C}^{n+1})$ by ζ and ζ'. Denote the coordinates on $\mathbf{C}^{n+1}$ by $x_1, \ldots, x_n, z$. Then $\gamma(Z) = \gamma'(Z) = \partial/\partial z$, where $\partial/\partial z$ is central in both $\boldsymbol{D}(G/H)$ and $\boldsymbol{D}(G/H')$, since $Z \in \mathfrak{z}$.

Now $\mathfrak{k}' \subseteq \mathfrak{g}_0$, so by (I) there exist $D' \in \boldsymbol{D}(G/H')$, commuting with $\gamma'(Z)$, and polynomials $p'_{a,b}$ on $G/H' = \mathbf{C}^{n+1}$ such that (1) and (2) are satisfied with D' and $p'_{a,b}$ in place of D and $p_{a,b}$. Note that $\boldsymbol{D}(G/H')$ and $\boldsymbol{D}(G/H)$ are contained in DP $(\mathbf{C}^{n+1})$.

Construction of D: Since the operator D' commutes with $\gamma'(Z) = \partial/\partial z$, it is of the form

$$D' = \sum_{k=0}^{K'} D'_k \otimes \left(\frac{\partial}{\partial z}\right)^k \quad \text{with } D'_k \in \mathrm{DP}\,(\mathbf{C}^n)$$

and leaves each of the eigenspaces

$$\mathscr{H}_b(G/H') := \{ f \in \mathscr{H}(G/H') \mid \gamma'(Z) f = bf \} = \mathscr{H}(\mathbf{C}^n) \otimes e^{bz},$$

$b \in \mathbf{C}$, invariant. The map $\varphi \in \mathscr{H}(\mathbf{C}^n) \mapsto \varphi \otimes e^{bz} \in \mathscr{H}_b(G/H')$ is an equivalence between ζ'_b and the restriction of ζ' to $\mathscr{H}_b(G/H')$, and it carries the operator

$$D'(b) := \sum_{k=0}^{K'} b^k D'_k \in \mathrm{DP}\,(\mathbf{C}^n)$$

into the restriction of D' to $\mathscr{H}_b(G/H')$. Hence $D'(b)$ commutes with ζ'_b and thus with $d\zeta'_b$. By (4.1) the operator

$$D(b) := \Phi_b(D'(b)) = \sum_{k=0}^{K'} b^k \Phi_b(D'_k) \in \mathrm{DP}\,(\mathbf{C}^n)$$

therefore commutes with $d\zeta_b$ and thus with ζ_b.

It follows from (4.2) that there exist $m \in \mathbf{N}$ and $D_k \in \mathrm{DP}(\mathbf{C}^n)$, $k=1,\ldots,K$, such that

$$b^m D(b) = \sum_{k=0}^{K} b^k D_k \quad \text{for all } b \in \mathbf{C} \setminus \{0\} .$$

Now define

$$D := \sum_{k=0}^{K} D_k \otimes \left(\frac{\partial}{\partial z}\right)^k \in \mathrm{DP}(\mathbf{C}^{n+1}) . \tag{4.3}$$

Then D commutes with ζ on each of the joint invariant subspaces $\mathscr{H}(\mathbf{C}^n) \otimes e^{bz}$, $b \in \mathbf{C} \setminus \{0\}$. Since these span a dense subspace of $\mathscr{H}(\mathbf{C}^{n+1})$, it follows that D commutes with ζ, that is $D \in \mathbf{D}(G/H)$. Clearly $[D, \gamma(Z)] = 0$.

Construction of $p_{a,b}$: By (2) for $p'_{a,b}$ we have

$$p'_{a,b}(x_1,\ldots,x_n,z) = q'_{a,b}(x_1,\ldots,x_n) + bz$$

for some polynomial $q'_{a,b}$.

Since the operator $D'(b) \in \mathrm{DP}(\mathbf{C}^n)$ commutes with $d\zeta'_b(Y) = bx_n$, it is of the form

$$D'(b) = \sum_{i=0}^{N} D'_i(b) x_n^i, \quad \text{where } D'_i(b) \in \mathrm{DP}(\mathbf{C}^{n-1}) \otimes 1_{\mathbf{C}} .$$

It follows that

$$(D'(b)f)(x_1,\ldots,x_{n-1},0) = (D'_0(b)f)(x_1,\ldots,x_{n-1},0)$$

for all $f \in \mathscr{H}(\mathbf{C}^n)$, so if we set

$$q_{a,b}(x_1,\ldots,x_{n-1},x_n) = q'_{a,b}(x_1,\ldots,x_{n-1},0) ,$$

then

$$D'_0(b) e^{q_{a,b}} = a e^{q_{a,b}}$$

by (1) for D' and $p'_{a,b}$. Since $q_{a,b}$ is independent of x_n and since

$$D(b) = \Phi_b(D'(b)) = \sum_{i=0}^{N} D'_i(b) \left(-\frac{1}{b}\frac{\partial}{\partial x_n}\right)^i ,$$

this implies that

$$D(b) e^{q_{a,b}} = a e^{q_{a,b}} .$$

Hence (1) and (2) are satisfied with D given by (4.3) and $p_{a,b}$ given by $p_{a,b} = q_{c,b} + bz$, where $c = b^{-m}a$. This finishes the proof of the theorem.

As a corollary we obtain that the maximality conditions on $\mathfrak{k}$ relative to α, imposed as sufficient conditions in Theorems 3.1 and 3.2 and Corollary 3.3, are also necessary ones:

4.4. COROLLARY. *Let G be a connected and simply connected, complex (respectively real) nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $\alpha \in \mathfrak{g}^*$ (respectively $(\mathfrak{g}^*)^{\mathbf{C}}$), $\mathfrak{k} \in S(\alpha, \mathfrak{g})$ and Ξ be a coexponential basis for $\mathfrak{g}$ modulo $\mathfrak{k}$.*

Then, if $\mathfrak{k} \notin M(\alpha, \mathfrak{g})$ (respectively $\mathfrak{k}^{\mathbf{C}} \notin M(\alpha^c, \mathfrak{g}^{\mathbf{C}})$), there exist $D \in \mathrm{DP}(\mathbf{C}^n)$ (respectively $\mathrm{DP}(\mathbf{R}^n)$), where $n = \dim \mathfrak{g}/\mathfrak{k}$, and polynomials p_a, $a \in \mathbf{C}$, on $\mathbf{C}^n$ (respectively $\mathbf{R}^n$) such that

(1) *D commutes with $\zeta_{\alpha, \mathfrak{k}, \Xi}$ (respectively $\lambda_{\alpha, \mathfrak{k}, \Xi}$)*

(2) *$De^{p_a} = ae^{p_a}$ for all $a \in \mathbf{C}$.*

In particular $\zeta_{\alpha, \mathfrak{k}}$ (respectively $\lambda_{\alpha, \mathfrak{k}}$) is neither topologically nor scalarly irreducible.

PROOF. The corollary for real G follows from that for complex G as in the proof of Theorem 3.2. So let G be complex. Set $\mathfrak{h} = \mathfrak{k} \cap \ker \alpha$.

If $\mathfrak{h} = \mathfrak{k}$, then $D = \partial/\partial x_n$ and $p_a = ax_n$ will do.

If $\mathfrak{h} \neq \mathfrak{k}$, let $Z \in \mathfrak{k} \setminus \mathfrak{h}$ with $\alpha(Z) = 1$. Then $Z \in \mathfrak{n}(\mathfrak{h}) \setminus \mathfrak{h}$ and $\mathfrak{k} = \mathbf{C}Z + \mathfrak{h}$. By Lemma 4.2, $\mathfrak{h}$ is not maximal, since $\mathfrak{k} \notin M(\alpha, \mathfrak{g})$. Identifying G/H with $\mathbf{C}^{n+1}$ by means of $\{\Xi, Z\}$ we have that $\gamma(Z) = \partial/\partial x_{n+1}$ and that the map $\varphi \mapsto \varphi \otimes e^{x_{n+1}}$, $\varphi \in \mathscr{H}(\mathbf{C}^n)$, is an equivalence between $\zeta_{\alpha, \mathfrak{k}, \Xi}$ and the action of G on $\{f \in \mathscr{H}(G) \mid \gamma(Z)f = f\}$. The existence of D and p_a now follows from Theorem 4.3 since $\mathbf{D}(G/H) \subseteq \mathrm{DP}(G/H)$.

Finally we obtain the following extension and converse of Theorem 6.1 [2], where it was proved for real G that if H is a connected subgroup of G whose Lie algebra $\mathfrak{h}$ is maximal in $\mathfrak{g}$ in the sense of Definition 4.1 then $\mathbf{D}(G/H)$ is generated by a single vector field.

4.5. THEOREM. *Let G be a connected and simply connected, real or complex nilpotent Lie group with Lie algebra $\mathfrak{g}$ and let H be a connected subgroup of G with Lie algebra $\mathfrak{h}$.*

Then the algebra $\mathbf{D}(G/H)$ is generated by a single vector field if and only if $\mathfrak{h}$ is of the form $\mathfrak{h} = \mathfrak{k} \cap \ker \alpha$ for some $\alpha \in \mathfrak{g}^* \setminus \{0\}$ and $\mathfrak{k} \in M(\alpha, \mathfrak{g})$.

PROOF. Sufficiency of the condition on $\mathfrak{h}$ was proved for real G as Theorem 6.1 of [2]. The proof for complex G is quite analogous to that in [2], now being based on the irreducibility result Corollary 3.3 (the part for complex G) above.

Necessity of the condition on $\mathfrak{h}$ follows immediately from Theorem 4.3.

REFERENCES

1. P. Bernat et al., *Représentations des groupes de Lie résolubles*, Dunod, Paris, 1972.
2. J. Jacobsen and H. Stetkær, *Eigenspace representations of nilpotent Lie groups*, Math. Scand. 48 (1981), 41–55.
3. A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys 17 (1962), 53–104.

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