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GORENSTEIN RINGS WITH TRANSCENDENTAL POINCARÉ-SERIES

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1. Introduction and definitions.

The word "ring" is used throughout the paper as shorthand for "unitary, commutative, local and noetherian ring".

For a ring (R,m), with maximal ideal m and residue class field $k=R/m$, the Poincaré-series is defined as

$$P_R(z) = \sum_{i\geqq 0} \dim_k (\mathrm{Tor}_i^R (k,k))z^i .$$

It was conjectured that this generating function would turn out to be a rational function for all rings; this hope, however, was recently crushed by David Anick [1], [2] — through Clas Löfwall–Jan-Erik Roos [10] — who constructed artinian rings (R,m) with $m^3=0$ and transcendental Poincaré-series. For easy reference let us call those of the rings that Löfwall–Roos construct in Theorem 3 of [10] (they are described in section 4), that have transcendental Poincaré-series, something, beasts say.

The aim of this paper is then to use beasts to construct artinian Gorenstein rings, also with transcendental Poincaré-series. In [7] Gulliksen noted that, if (R,m) is artinian, taking the trivial extension (defined below) $R\alpha E(k)$, where $E(k)$ is the injective hull of $k=R/m$, produces a Gorenstein ring. Since it is well-known [7] that

$$P_{R\alpha M}(z) = P_R(z)/(1-zP_R^M(z))$$

$$\Big(\text{where } P_R^M(z) = \sum_{i\geqq 0} \dim_k (\mathrm{Tor}_i^R (k,M))z^i\Big)$$

for an arbitrary trivial extension $R\alpha M$, determination of $P_{R\alpha E(k)}$ boils down to finding the value of

$$P_R^{E(k)}(z) = \sum_{i\geqq 0} \dim_k (\mathrm{Ext}_R^i (k,R))z^i ,$$

by virtue of the natural isomorphism

$$\mathrm{Tor}_*^R (k,E(k)) \cong (\mathrm{Ext}_R^* (k,R)) .$$

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Since the exponent of a beast is low, only 3, it turns out to be rather easy to determine $\operatorname{Ext}_R^*(k,R)$; the techniques of Löfwall [9] and Roos [12] are available, and indeed, *for all beasts R, $R\alpha E(k)$ has transcendental Poincaré-series.* As a byproduct of the proof I also obtain that these rings have transcendental Bass-series $\sum_{i\geq 0}\dim_k(\operatorname{Ext}_R^i(k,R))z^i$ and moreover it is possible to show the existence of a smooth, closed, simply connected manifold M such that the Poincaré–Betti-series of its loopspace $\Omega M - \sum_{i\geq 0}\dim_Q H_i(\Omega M, Q)z^i$ is transcendental.

Some more definitions:

If M is a positively graded module, over some field k, its Hilbert series is defined by

$$H_M(z) = \sum_{i\geq 0} \dim_k M_i z^i .$$

The graded dual of M is the graded module M^* defined by

$$(M^*)_i = \operatorname{Hom}_k(M_{-i}, k) .$$

The trivial extension $R\alpha M$ of R by an R-module M is the ring defined additively as $R \oplus M$ and multiplicatively by

$$(r,m)(r',m') = (rr', rm' + r'm) .$$

Note that R and $R\alpha M$ have a common residue classfield.

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2. An exact sequence.

As I mentioned in the introduction, the problem is to determine $\operatorname{Ext}_R^*(k,R)$ (or at least its Hilbert-series), or more generally to try for $\operatorname{Ext}_R^*(k,M)$ as an $\operatorname{Ext}_R^*(k,k)$-module.

The most elementary approach seems to be to use long exact sequences such as the one induced by:

$$0 \to \operatorname{soc} M \to M \to M/\operatorname{soc} M \to 0$$

and this is the content of Lemma 1 and 1′ below.

The following lemma is suggested by Theorem 1 in Roos [12]; it gives the Yoneda-action of $\operatorname{Ext}_R^*(k,k)$ on $\operatorname{Hom}_R(k,M)$ as a subset of $\operatorname{Ext}_R^*(k,M)$ — *M an arbitrary R-module.*

LEMMA 1. *There exists natural mappings α and β such that the following sequence is exact:*

$$0 \to \mathrm{Hom}_R(k,M) \otimes \mathrm{Hom}_R(k,k) \xrightarrow{Y} \mathrm{Hom}_R(k,M) \xrightarrow{\beta} \mathrm{Hom}_R(k, M/\mathrm{soc}\, M) \to$$
$$\to \dots \xrightarrow{\beta} \mathrm{Ext}_R^{n-1}(k, M/\mathrm{soc}\, M) \xrightarrow{\alpha} \mathrm{Hom}_R(k,M) \otimes \mathrm{Ext}_R^n(k,k) \xrightarrow{Y}$$
$$\xrightarrow{Y} \mathrm{Ext}_R^n(k,M) \xrightarrow{\beta} \mathrm{Ext}_R^n(k, M/\mathrm{soc}\, M) \to \dots$$

where $\otimes$ is $\otimes_k$ and Y the Yoneda product.

The proof is entirely similar to that of Theorem 1 in Roos [12]; it uses the isomorphisms

$$\mathrm{Hom}_R(k,M) \otimes \mathrm{Ext}_R^n(k,k) \cong \mathrm{Hom}_R(k, \mathrm{soc}\, M) \otimes \mathrm{Ext}_R^n(k,k) \cong \mathrm{Ext}_R^n(k, \mathrm{soc}\, M)$$

and the long exact sequence induced by $0 \to \mathrm{soc}\, M \to M \to M/\mathrm{soc}\, M \to 0$.

If $B = \mathrm{Ext}_R^*(k,k)$ then by the naturality of the Yoneda product the lemma can be reformulated as:

LEMMA 1'. *To every R-module M there exists an exact sequence of B-modules:*

$$(1) \quad 0 \to s^{-1}(S_M) \to s^{-1}(\mathrm{Ext}_R^*(k, M/\mathrm{soc}\, M)) \to \mathrm{Hom}_R(k,M) \otimes B \xrightarrow{Y}$$
$$\xrightarrow{Y} \mathrm{Ext}_R^*(k,M) \to S_M \to 0\,,$$

where

$$S_M = \coprod_{n \geq 0} \mathrm{Ext}_R^n(k,M)/(\mathrm{Hom}_R(k,M)\, \mathrm{Ext}_R^n(k,k))$$

and $s^{-1}(M)$ of a graded module M is the same module pushed up one step, that is $(s^{-1}(M))_q = M_{q-1}$.

3. An application of Lemma 1.

Assume that M is an artinian module.

From Lemma 1' one then obtains a finite complex, denoted by $\boldsymbol{E}(M)$:

$$\dots \to s^{-2}(\mathrm{Hom}_R(k, M/\mathrm{soc}^2 M) \otimes B) \to s^{-1}(\mathrm{Hom}_R(k, M/\mathrm{soc}\, M) \otimes B) \to$$
$$\to \mathrm{Hom}_R(k,M) \otimes B \to \mathrm{Ext}_R^*(k,M) \to 0\,,$$

if $\mathrm{soc}^2 M$ denotes $\{x \in M : m^2 x = 0\}$ etc.

Put $A = \langle \mathrm{Ext}_R^1(k,k) \rangle$ — the subalgebra of B generated by the one-dimensional elements. A is a Hopf algebra — it is primitively generated — and so B is a faithfully flat extension of A ([11]). The maps in $\boldsymbol{E}(M)$ are defined as

$$(2) \quad \mathrm{soc}^n M/\mathrm{soc}^{n-1} M \otimes B \ni \bar{x} \otimes b \mapsto \sum_i x_i x \otimes T_i b \in \mathrm{soc}^{n-1} M/\mathrm{soc}^{n-2} M \otimes B\,,$$

where x_i denotes representatives for a basis of m/m^2 and T_i is the dual basis of

$$\mathrm{Hom}_k\,(m/m^2, k) \;=\; \mathrm{Ext}^1_R\,(k,k)\,.$$

The nice thing here is that these B-linear maps obviously come from A-linear maps:

$$\mathrm{soc}^n\, M/\mathrm{soc}^{n-1}\, M \otimes A \to \mathrm{soc}^{n-1}\, M/\mathrm{soc}^{n-2}\, M \otimes A\,,$$

with the same definition, by extending them by $-\otimes_A B$. Altogether this means that there exists a complex $\boldsymbol{E}_A(M)$ of free A-modules, such that exactness of $\boldsymbol{E}_A(M)$ is equivalent with exactness of $\boldsymbol{E}(M)$.

Of course, the complex is not exact in general. In case $M=R$, R a beast this is true however and also in the following

PROPOSITION. *The complex*

$$0 \to s^{-1}(B) \to m/m^2 \otimes B \to \mathrm{Ext}^*_R\,(k, R/m^2) \to 0 \tag{3}$$

is exact for all rings R, with the exception of those having $A=\langle \mathrm{Ext}^1_R\,(k,k)\rangle$ strictly commutative (which is equivalent to A being an exterior algebra).

PROOF. Note that by the explicit formula (2) the kernel of the first nontrivial map in $\boldsymbol{E}_A(R/m^2)$ is precisely $\mathrm{soc}\, A=\mathrm{Hom}_A\,(k,A)$. By Lemma 1′ it is then enough to prove $\mathrm{soc}\, A=0$ if and only if A is not strictly commutative. (Observe that A is graded as a subalgebra of $\mathrm{Ext}^*_R\,(k,k)$. A graded algebra is strictly commutative if $ab=(-1)^{\deg a \deg b}ba$ and $a^2=0$ if $\deg a$ is odd for all elements a and b in the algebra.) Suppose a belongs to $\mathrm{soc}\, A$.

The algebra A, as a cocommutative (see Gulliksen–Levin [14]), connected, locally finite and graded Hopf algebra, is the enveloping algebra of a graded Liealgebra: $A=U(g)$. The graded version of Poincaré–Birkhoff–Witt's theorem (PBW) says:

If a k-basis of g is well-ordered, a k-basis for $U(g)$ is given by all elements: $\bar{u}=\prod u_i^{e_i}$ where $u_1<u_2\ldots$ and $e_i=0,1$ if $\deg u_i$ is odd and an arbitrary natural number otherwise. (The difference in the exponents compared to the usual PBW orginates in the existence in the case of a graded Lie algebra of a function $g \ni x \mapsto x^2 \in g$ for $\deg x$ odd; which upon taking the enveloping algebra of g coincides with the algebra square. For all this see Milnor–Moores paper [11].)

Assume now that there exists an a such that a annihilates the augmentation ideal of A (which is generated by the T_i) and let $b=\sum k_i T_i$, $k_i \in k$. Then $b^2=0$. For suppose $b^2 \neq 0$ and choose a wellordering of a k-basis of g such that $b<b^2$ (possible since b and b^2 linear dependent implies here $b^2=0$) and such that these two are smaller than the rest. Then $b\bar{u}$ is a new basiselement, for every

basiselement $\bar{u}$; and if $b\bar{u}=b\bar{v}$ then $\bar{u}=\bar{v}$. So $bc=0$ implies $c=0$, and hence the existence of a gives $b^2=0$. Using $(T_i+T_j)^2=T_i^2+[T_i,T_j]+T_j^2$ we then get that all graded Lie commutators in A are zero. The converse, finding an a if A is strictly commutative is left for the reader as an exercise. The proposition is proved.

The condition of strict commutativity of A corresponds, in terms of the underlying ring, to the following condition:

If $\hat{R}$ is the completion of R, and $S/a \to \hat{R}$ a presentation of R as a quotient of a regular ring (S,n) with the same embedding dimension as (R,m) (that is $S/n = R/m$ and $\dim(m/m^2)=\dim(n/n^2)$), then $a\subset n^3$. (This follows easily from the explicit study of A in Löfwall [9] or from Sjödin [13].)

Suppose now that R is a ring with $m^3=0$ and $\operatorname{soc} R=m^2$. Then $\boldsymbol{E}_A(R)$ has the form:

$$0 \to s^{-2}(A) \to s^{-1}(m/m^2\otimes A) \to m^2\otimes A \to \operatorname{Ext}_R^*(k,R) \to 0\,. \tag{4}$$

Suppose further that R is a Gorenstein ring. Then $\operatorname{Ext}_R^*(k,R)=k$, and it is easy to see that this implies that (4) is exact at $\operatorname{Ext}_R^*(k,R)$. If the embedding dimension R is greater than 1 (so that R, as a Gorenstein ring cannot be of the form S/n^3, for (S,n) regular) then the sequence is also exact at the beginning; by its construction (see Lemma 1′) this suffices to make it exact as a whole. Thus we have a resolution of k over B, and B has rational Hilbert series and global dimension 2. *In particular the Poincaré series of R is rational.* (If the embedding dimension of $R=1$, the global dimension of R is infinite and its Poincaré series $1/(1-z)$.

All this was proved in another way by Löfwall from results in [9]. Note that the proposed Gorenstein rings $R\alpha E(k)$ of the introduction all have nilpotence degree 4.

Let us now return to the complex (4). From the proof of the proposition it is clear that the first *homology group* is $\operatorname{Hom}_A(k,A)$ and actually it is also true that the second is $\operatorname{Ext}_A^1(k,A)$: Consider the start of a resolution of k over A given by

$$L:\quad k \leftarrow A \leftarrow (m/m^2)^*\otimes A \xleftarrow{\alpha} (m^2)^*\otimes A\,,$$

where α is

$$(m^2)^*\otimes A \xrightarrow{\mu^*\otimes 1_A} (m/m^2)^*\otimes(m/m^2)^*\otimes A \xrightarrow{1\otimes m} (m/m^2)^*\otimes A\,.$$

Here μ is the multiplication map $m/m^2\otimes m/m^2 \to m^2$ and m is just multiplication in A with $(m/m^2)^*$ identified with $\operatorname{Ext}_R^1(k,k)$. That this is a beginning of a resolution is clear, since $\operatorname{im}(\mu^*)$ just is the relations in degree 2 of A, and these generate all relations, [9]. Now, after some calculations it is easy to see that $\tilde{E}_A(R) = \operatorname{Hom}_A(\tilde{L}, A)$ and since $\tilde{L}$ was the beginning of a projective resolution of k over A this implies that

$$H_i(\tilde{E}(R)) = \operatorname{Ext}_A^i(k, A) \qquad i = 0, 1 .$$

(Here $\sim$ denotes the operation on a complex done by taking away the bottom term.)

4. The number of the beast.

In [10], the beasts are created by first constructing their $\langle \operatorname{Ext}_R^1(k,k) \rangle$ as an explicit extension:

$$k \to U(h) \to A \to U(f) \to k ,$$

where h is an abelian Lie algebra and f is the product of two free finitely generated graded Lie algebras (and some other conditions on the extension, which guarantee that A is finitely presented etc.). Then standard techniques of [9] makes it easy to find a ring R with $\operatorname{Ext}_R^1(k,k) = A$. If such a ring has a transcendental Poincaré series it is precisely what was called a beast in the introduction. From the construction in Theorem 3 of [9], it then follows that h *must have infinite k-dimension.* This will be used in the sequel.

First note that *for a beast* $\operatorname{soc} R = m^2$: Translate the existence of an element $x \in m/m^2$ annihilating m into a condition on A: the existence of an element $T \in A_1$ (the elements of degree 1 of A) not involved in any nontrivial relation. Then, by using that g, the underlying liealgebra of A, as a vectorspace is the direct sum of three Lie subalgebras and the freeness respectively abelianness of these, it is easy to see that such an element cannot exist.

Obviously A is not an exterior algebra. Therefore the beginning of the complex $E(R)$ is exact and if we prove that $\operatorname{Ext}_A^1(k, A) = 0$ it follows that the whole complex is exact (by the remark at the end of the preceding section and since $\operatorname{soc} R = m^2$ and so $E_A(R)$ has the form of (4)). This will follow from the fact that g contains an infinite dimensional abelian Lie subalgebra and will be done in 2 steps.

STEP 1. Suppose $k \to D \to A \to C \to k$ is an exact sequence of graded Hopf-algebras. Then there is a suitable spectral sequence:

$$E_2^{pq} = \operatorname{Ext}_C^p(k, \operatorname{Ext}_D^q(k, A)) \Rightarrow \operatorname{Ext}_A^*(k, A)$$

(see Cartan–Eilenberg [6, p. 349]). Suppose now that $\operatorname{Ext}_D^*(k, D) = 0$. A is free as a D-module, but we can not conclude directly that $\operatorname{Ext}_D^*(k, A) = 0$ since the D-rank of A is infinite. However it is possible to use the grading of A; since A is a graded locally finite D-module there is an isomorphism

$$(\mathrm{Ext}_D^n(k, A))^* \cong \mathrm{Tor}_D^n(k, A^*)$$

by Cartan [5] (note that $\mathrm{Ext}_D^n(k, A)$ has a natural grading stemming from the grading of D).

Suppose $\{a_i\} i \in I$ is a D-basis of A. Then

$$A = \bigoplus_{i \in I, q} Da_i \cap A_q \quad \text{and so} \quad A^* = \bigoplus_q \Big(\bigoplus_{i \in I} Da_i \cap A_q\Big)^*$$

$$= \text{(because of } A\text{'s local finiteness)}$$

$$= \bigoplus_{i \in I, q} (Da_i \cap A_q)^* = \bigoplus_{i \in I} (Da_i)^* .$$

But Tor has no compunction about commuting with direct limits, and since

$$\mathrm{Tor}_D^n(k, D^*) \cong (\mathrm{Ext}_D^n(k, D))^* = 0$$

it follows that $\mathrm{Ext}_D^n(k, A) = 0$.

It is easy to see that this argument, and the spectral sequence, respect the gradings induced on the homology modules by the gradings of the Hopf-algebra. Thus we have proved

LEMMA. *If $k \to D \to A \to U \to k$ is an exact sequence of graded Hopf-algebras, then*

$$\mathrm{Ext}_D^m(k, D) = 0 \qquad m < n$$

$$(\mathrm{Ext}_D^n(k, D))_q = 0 \qquad q \leqq q_0$$

implies

$$\mathrm{Ext}_A^m(k, A) = 0 \qquad m < n \text{ and}$$

$$(\mathrm{Ext}_A^n(k, A))_q = 0 \qquad q \leqq q_0 .$$

STEP 2. So the proof is reduced to showing $\mathrm{Ext}_{U(h)}^*(k, U(h)) = 0$. Since h is abelian, $U(h)$ is the tensor product of an exterior algebra generated by the odd degree elements in h, and a polynomial algebra on the elements of even degree. For a finitely generated exterior algebra E

$$(*) \qquad (\mathrm{Ext}_E^n(k, E))_m = \begin{cases} k & \text{if } n = 0,\ m = \max q : E_q \neq 0 . \\ 0 & \text{otherwise} \end{cases}$$

(here n denotes the homological grading, while m is the natural grading mentioned above). For a finitely generated polynomial algebra on the other hand

$$(**)\qquad \mathrm{Ext}_P^n(k,P) = \begin{cases} k & \text{if } n = \#\text{ generators of } P\,. \\ 0 & \text{otherwise}\,. \end{cases}$$

This is easy to prove by induction and the spectral sequence above. Since h is infinite dimensional abelian, it contains either an infinite exterior algebra or an infinitely generated polynomial algebra P. Suppose the last case is true; the argument is nearly identical in the other case. Suppose Q is a finitely generated polynomial subalgebra of P. Then the Lemma and (**) gives

$$\mathrm{Ext}_P^n(k,P) = 0 \quad \text{if } n < \#\text{ generators of } Q\,.$$

Since there by assumption are Q's with an arbitrary number of generators it is clear that $\mathrm{Ext}_P^*(k,P)=0$ and so, by the Lemma again $\mathrm{Ext}_{U(h)}^*(k,U(h))=0$.

In conclusion, we have thus proved the following theorem:

THEOREM. *If R is a beast, $\mathrm{Ext}_k^*(k,A)=0$ and so $E(R)$ is a resolution of* $\mathrm{Ext}_R^*(k,R)$. *This implies the relation*

$$(P_{R\alpha E(k)}(Z))^{-1} = (P_R(Z))^{-1} - Z^3 H_R(-1/Z)\,,$$

(where $H_R(Z)$ is the usual Hilbert-series of R), between the Poincaré-series of R and $R\alpha E(k)$. In particular $P_R(Z)$ is transcendental iff $P_{R\alpha E(k)}(Z)$ is, and there exist Gorenstein rings with transcendental Poincaré-series.

COROLLARY 1. *All beasts have transcendental Bass series. (The Bass series of R is just the Hilbert series of* $\mathrm{Ext}_R^*(k,R)$.)

5. A topological corollary.

The problem in algebraic topology analogous to the question of the existence of Gorenstein rings with transcendental Poincaré-series, is whether there exists a simple connected, smooth closed manifold M such that the "Poincaré–Betti" series of ΩM (the loop space of M)

$$H_M(z) = \sum_{i\geq 0} \dim_{\mathbf{Q}}(H_i(\Omega M,\mathbf{Q}))z^i$$

is transcendental. With the help of some references and the preceding paragraphs we can now easily show that such manifolds exists. The proof uses some facts from rational homotopy theory and an idea from Roos [12].

Given an artinian, graded Gorenstein ring $R=\sum_{i\geq 0} R_i$, we can view it as a locally finite, connected, commutative, graded differential algebra R^e (CGDA) by doubling the grading (i.e. putting $R_i^e = R_{i/2}$ if i is even and $R^{ei}=0$ otherwise) and setting the differential equal to the zero map. To CGDA's S Bousfield–

Gugenheim [4] construct corresponding topological spaces $F(S)$, such that for the de-Rham–Sullivan functor A from topological spaces to CDGA's we have

$$S = A(F(S)) .$$

The sought manifolds will be constructed in this way, starting with the Gorenstein rings constructed in the last section. Only 2 facts about the functor F will be needed. First that $H^*(F(S))=S$ (as algebras) and so since R is a Gorenstein ring, $F(R^e)$ is a Poincaré $\mathbf{Q}$-duality space in the sense of Barge [3]. The other important fact is that $F(S)$ is a formal space (see Halperin–Stasheff [8]) and so (ibid. 8.13) the Eilenberg–Moore spectral sequence

$$E_2^{pq} = \operatorname{Ext}^{p}_{H^*(F(S),\mathbf{Q})}(\mathbf{Q},\mathbf{Q}) \Rightarrow H_*(\Omega F(S),\mathbf{Q})$$

degenerates.

I want to use this fact to show that the Poincaré-series of R and the Poincaré–Betti series of $F(S)$ are rationally related and in particular, that the second series us transcendent of the first is; at least in the case where R is a trivial Gorenstein extension of a beast. This can be done in much the same way as Roos [12] uses the degeneracy of the Eilenberg–Moore spectral sequence in a similar case. I take the notation from and refer to [12] for a more detailed exposition.

From the degeneration of the spectral sequence it follows that

$$\text{(1)} \qquad H_{F(S)}(z) = \sum_{n\geqq 0}\sum_{p=0}^{\infty} \dim_{\mathbf{Q}}(\operatorname{Ext}_{H^*(F(S),\mathbf{Q})}(\mathbf{Q},\mathbf{Q})_{p+n})z^n$$

(the grading on the Ext-groups is inherited from the homological grading on $H^*(F(S),Q)=S$).

Suppose now that R is a beast over $\mathbf{Q}$. It then has a natural grading, in degrees 0 to 2, and we give

$$T = R\alpha I(k) = \mathbf{Q}+m/m^2+m^2+(m^2)^*+(m/m^2)^*+\mathbf{Q} .$$

The obvious grading on degrees 0 to 5. Then T^e is also graded and we define the double Poincaré-series as

$$P_{R^e}(x,y) = \sum_{p\geqq 0}\sum_{q\geqq 0} \dim_{\mathbf{Q}}(\operatorname{Ext}^p_{R^e}(\mathbf{Q},\mathbf{Q})_q)x^py^q$$

and correspondingly for graded R^e-modules.

Then (1) above gives that $H_{F(T^e)}(z)=P_{T^e}(z^{-1},z)$, while obviously $P_{T^e}(z) = P_{T^e}(z,1)$.

In [12] it is shown that $P_R(z^{-1},z)$ and $P_R(z,1)$ are transcendent for exactly the same rings R, under the condition that these are artinian with $m^3=0$. Furthermore it is easy to see that if $T=R\alpha I(k)$ then

$$P_T(x,y) = P_R(x,y)/1 - xP_R^{I(k)}(x,y)$$

and that from the exactness of the complex $E(R)$ in the last paragraph we obtain the formula

$$P_{R^e}^{I(k)}(x,y) = x^2 H_{R^e}(-y/x) P_{R^e}(x,y) .$$

This means that

$$(P_{R^e\alpha I(k)}(x,y))^{-1} = (P_R(x,y))^{-1} - x^3 H(-y/x)$$

and by the result of Roos [12] mentioned at the beginning of this paragraph we get that $P_{R^e\alpha I(k)}(z^{-1},z)$ is transcendent whenever $P_R(z,1)$ is, in particular in the present case when R is a beast.

Thus we have a Poincaré Q-complex $F(R^e I(k))$ of dimension 10 for every beast. According to Barge [3, Thèoréme 1], there exists smooth structures on it and so we have proved the following corollary:

COROLLARY. *There exists simply connected smooth closed manifolds whose loop spaces have transcendental Poincaré–Betti series.*

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