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MONOMIAL IDEAL RESIDUE CLASS RINGS AND ITERATED GOLOD MAPS

JÖRGEN BACKELIN

Introduction.

Ghione and Gulliksen proved in [2, § 4, Ex. 4] that if

$$R = k[[X_1, X_2, X_3]]/(M_1, \ldots, M_r) ,$$

where k is a field, and $M_1, \ldots, M_r$ are monomials in the ring $k[[X_1, X_2, X_3]]$ of formal power series, then R is the image of a ring S under a Golod homomorphism, where S is either a Golod ring or a complete intersection (a c.i.). They were able to conclude that the Poincaré series of R, i.e. the power series

$$P_R(t) = P_R^k(t) = \sum_{i \geqq 0} \dim_k \operatorname{Tor}_i^R(k, k) t^i$$

(where k is identified with the residue field of R), is the series expansion at the origin of a rational function. The main result in this article is that such an R is always the image of a c.i. under a Golod homomorphism. Thus, by a result of Levin [3], the Poincaré series

$$P_R^M(t) = \sum_{i \geqq 0} \dim_k \operatorname{Tor}_i^R(M, k) t^i$$

of any finitely generated R-module M is rational. This result in its turn, with the help of methods from [2], proves that if

$$A = k[[X_1, X_2, X_3, X_4]]/(M_1, \ldots, M_{r'}) ,$$

where $M_1, \ldots, M_{r'}$ are monomials, then $P_A(t)$ is rational.

The article is formulated in terms of residue rings of polynomial rings $k[X_1, \ldots, X_n]$ rather than residue rings of formal power series rings. This makes it possible to employ a structure of multiple degrees, which is introduced in section 1.

In section 2, a method by Ghione and Gulliksen [2] is introduced in the form of a sequence of Golod homomorphisms, which attaches any such monomial ideal residue class ring to the corresponding polynomial ring.

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In section 3 the theorem and its corollaries are stated, and the theorem is proved in the last two sections.

0. Notations and conventions.

All rings are assumed to be unitary, associative, commutative and noetherian, and all modules are assumed to be unitary and finitely generated, hence noetherian. No distinction is made between right and left modules.

The letter n always denotes a positive integer. Boldface letters and words stand for n-tuples of integers; the corresponding ordinary letters and words, subindexed from 1 to n, stand for the entries of the n-tuples. Thus $\mathbf{deg}\, x = \boldsymbol{i}$ if and only if $\deg_\nu x = i_\nu$ for $\nu = 1,\ldots,n$.

$\mathbf{0} = (0,\ldots,0)$ (n copies).

$\boldsymbol{i}+\boldsymbol{j} = (i_1+j_1,\ldots,i_n+j_n)$.

By definition, $\boldsymbol{i} \leqq \boldsymbol{j}$ if and only if $i_\nu \leqq j_\nu$ for $\nu = 1,\ldots,n$. If $\mathbf{deg}\, x = \boldsymbol{i}$, then

$$\operatorname{Supp} x \;=\; \operatorname{Supp} \boldsymbol{i} \;=\; \{\nu \in \{1,\ldots,n\} \;:\; i_\nu \neq 0\}\;.$$

Graded objects are non-negatively graded.

If $U = U_* = \coprod_{i \geqq 0} U_i$ is a graded object, then $U_+ = \coprod_{i \geqq 1} U_i$.

The letter k denotes a fixed field.

A (k-)augmented ring is a triple (R,k,ε), where R is a ring and $\varepsilon \colon R \to k$ is an epimorphism of rings. Then $\operatorname{edim} R =$ the embedding dimension of $R = \dim_k \operatorname{Ker} \varepsilon/(\operatorname{Ker} \varepsilon)^2$.

δ_*^* denotes the Kronecker delta:

$$\delta_i^j \;=\; \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}.$$

1. SMH rings.

Local rings with a fixed residue class ring k form a special class of k-augmented rings, and many definitions in local algebra may be extended in a natural way to such rings and their corresponding modules. In the sequel I shall quote some results formulated in terms of local rings, but apply them in terms of connected graded (commutative noetherian) k-algebras. (Such an algebra $R = \coprod_{i \geqq 0} R_i$ is augmented by $\varepsilon \colon R \to R/R_+ \xrightarrow{\sim} R_0 \xrightarrow{\sim} k$.) The reader may verify that the proofs referred to hold also in this case, and that on the other hand there are local counterparts to the theorem and the corollaries in section 3.

In the light of Lemma 1 below, a ring which is isomorphic to some residue ring of a polynomial ring modulo an ideal generated by monomials should perhaps be called a "standard multihomogeneous" ring. In this article such a ring is called an SMH ring.

LEMMA 1. *Let* (R, k, ε) *be an augmented ring, and let* $n = \operatorname{edim} R$. *Then* R *is an* SMH *ring if and only if* (i) *and* (ii) *below hold*:

(i) R *can be given the structure of an* (n-*tuply*) *multihomogeneous ring*, i.e. *as a* k-*vector space* $R = \coprod_{\mathbf{i}} R_{\mathbf{i}}$, *where* $\mathbf{i}$ *runs through all* n-*tuples of non-negative integers, in such a way that for any* $\mathbf{i}$ *and* $\mathbf{j}$, $R_{\mathbf{i}} \cdot R_{\mathbf{j}} \subseteq R_{\mathbf{i}+\mathbf{j}}$; *and*

(ii) *the multidegree of* R *is standard in the sense that* $R_{\mathbf{0}} = k$, *and that* $\operatorname{Ker} \varepsilon$ *has a minimal set of generators* $(x_1, \ldots, x_n)$, *such that for* $i = 1, \ldots, n$, $x_i \in R_{(\delta_i^1, \delta_i^2, \ldots, \delta_i^n)}$.

If R is multihomogeneous, elements in $\bigcup_{\mathbf{i}} R_{\mathbf{i}}$ are called multihomogeneous (or mhom). If $x \in R_{\mathbf{i}}$ then $\mathbf{deg}\, x = \mathbf{i}$. (As usual the definition is ambigous for $x = 0$.)

PROOF OF LEMMA 1. If $R = k[X_1, \ldots, X_n]/(M_1, \ldots, M_r)$ is an SMH ring, let $x_1, \ldots, x_n$ be the images of $X_1, \ldots, X_n$, respectively, and let

$$\mathbf{deg}\,(x_1^{i_1} \cdot \ldots \cdot x_n^{i_n}) = (i_1, \ldots, i_n) = \mathbf{i} \quad \text{for any such } \mathbf{i} \geqq \mathbf{0}\,.$$

Conversely, if (i) and (ii) are satisfied, induction on $\sum_{\nu=1}^{n} i_\nu$ yields that

$$R_{\mathbf{i}} = k x_1^{i_1} \cdot \ldots \cdot x_n^{i_n} \quad \text{for } \mathbf{i} \geqq \mathbf{0}\,,$$

whence the k-algebra homomorphism

$$\begin{aligned} f\colon k[X_1, \ldots, X_n] &\to R \\ X_\nu &\mapsto x_\nu \end{aligned}$$

is onto, and its kernel is generated by monomials.

2. The Ghione-Gulliksen sequence.

Assume that $\mathfrak{a}$ is an ideal in $\tilde{R} = k[X_1, \ldots, X_n]$, which is minimally generated by the non-trivial monomials $M_1, \ldots, M_r$, i.e. by monomials of ordinary degree $\geqq 2$. Let $R = \tilde{R}/\mathfrak{a}$. If i is an integer such that $0 \leqq i \leqq n$, we define

$$\mathfrak{a}^i = (M_\mu : \operatorname{Supp} M_\mu \subseteq \{1, \ldots, i\})$$

and

$$R^i = \tilde{R}/\mathfrak{a}^i\,.$$

Then $R^0 = \tilde{R}$, $R^n = R$, and in a natural way there is a sequence of augmented ring epimorphisms

$$\tilde{R} = R^0 \to R^1 \to R^2 \to \ldots \to R^n = R\,. \tag{1}$$

Recall that an augmented ring epimorphism $(A, k, \varepsilon_A) \to (B, k, \varepsilon_B)$ is a Golod homomorphism if

$$P_B(t) = P_A(t)(1-t(P_A^B(t)-1))^{-1}$$

or if any other of a number of equivalent conditions are satisfied; cf. [1, Def. 3.6]. (Another of the conditions will be introduced in section 4.)

The following lemma is essentially proved in [2, section 4, Ex. 4], though the authors assumed $i=2$ in their example, since they employed only this special case. Their proof is independent of this assumption. (A proof closely related to theirs is also sketched in section 4.)

LEMMA 2. *If $0 \leqq i \leqq n-1$ in the situation above, then $R^i \twoheadrightarrow R^{i+1}$ is a Golod homomorphism.*

DEFINITION. (1) is called the Ghione–Gulliksen sequence (the G-G sequence) belonging to R.

The G-G sequence thus defined depends on the (somewhat arbitrary) order of the indexes of the variables of R. We sometimes have to reindex these variables in order to get a new G-G sequence with better properties, while R is not essentially changed.

3. The main result.

THEOREM. *Let R be an* SMH *ring with* edim $R=n \geqq 3$, *such that $R^3=R$. Then there is a $p \in \{0,1,2\}$ and a way to reindex the three first variables, such that R^p is a complete intersection, and that $R^p \to R^3=R$ is a Golod homomorphism.*

The theorem is proved in section 5.

COROLLARY 1. *If R is an* SMH *ring such that $R^3=R$ (in particular, if* edim R $=3$), *and M is a (graded, finitely generated) R-module, then $P_R^M(t)$ is rational.*

This follows by an implicit result of the proof of [3, Theorem 6.3].

COROLLARY 2. *If R is an* SMH *ring such that $R^4=R$ (in particular, if* edim R $=4$), *then $P_R(t)$ is rational.*

PROOF. By Lemma 2,

$$P_R(t) = P_{R^4}(t) = P_{R^3}(t)(1-t(P_{R^3}^{R^4}(t)-1))^{-1},$$

which is rational by Corollary 1.

4. Homology in the G-G sequence.

Through this section, let i, j, and n be integers such that $0 \leqq i < j \leqq n$, let k be a field, let $\tilde{R} = k[X_1,\ldots,X_n]$, let $M_1,\ldots,M_r$ be non-trivial monomials in $\tilde{R}$ (where $r \geqq 0$), and let $R = \tilde{R}/(M_1,\ldots,M_r)$. We shall study $\mathrm{Tor}_*^{R^i}(R^j, k)$, with respect to multidegrees. For simplicity we assume that $R^i = R/(M_1,\ldots,M_s)$ and that $R^j = R(M_1,\ldots,M_t)$, where $0 \leqq s \leqq t \leqq r$.

Following Tate ([4]), we may extend the Koszul complex $K = K_*$ of R^i to an R^i-free DGA resolution Y_* of k. Let $x_1,\ldots,x_n$ be the images of $X_1,\ldots,X_n$, respectively, in R^i. Then K may be defined as the exterior R^i-algebra on the variables $T_1,\ldots,T_n$, with the differential d given by $dT_\nu = x_\nu$ for $\nu = 1,\ldots,n$. Note that K inherits the multihomogeneous structure from R^i, if we put

$$\mathbf{deg}\, T_\nu = \mathbf{deg}\, x_\nu = (\delta_\nu^1,\ldots,\delta_\nu^n) \quad \text{for } \nu = 1,\ldots,n,$$

and extend $\mathbf{deg}$ by the condition that $\mathbf{deg}\,(a \cdot b) = \mathbf{deg}\, a + \mathbf{deg}\, b$. Then d becomes mhom, with $\mathbf{deg}\, d = \mathbf{0}$. Thus (K, d) is a mhom complex.

Let $\tilde{S} = k[X_1,\ldots,X_i]$ and $S = \tilde{S}/(M_1,\ldots,M_s)$. Then $R^i \cong S[X_{i+1},\ldots,X_n]$, whence

$$H_*(K) \cong \mathrm{Tor}^{\tilde{R}}(R^i, k) \cong \mathrm{Tor}^{\tilde{S}}(S, k),$$

whence

$$\text{(2)} \qquad \mathrm{Supp}\, a \subseteqq \{1,\ldots,i\} \quad \text{if } a \in H(K) \text{ is mhom and } a \neq 0.$$

When we successively extend K to Y by Tate's method, we may preserve these properties, and thus achieve that Y is mhom, that its differential d preserves multidegree, and that all of the variables which generate Y as an R^i-algebra, except $T_{i+1},\ldots,T_n$, have supports contained in $\{1,\ldots,i\}$.

Let $\bar{Y}_* = R^j \otimes_{R^i} Y_*$ (whence $\mathrm{Tor}_*^{R^i}(R^j, k) \cong H_*(\bar{Y})$). Then $\bar{Y}$ is mhom. In fact, $\bar{Y}$ is the direct sum as a k-vector space complex of the subcomplexes $\bar{Y}_{\mathbf{i}*}$, where $\bar{Y}_{\mathbf{i}q} = (\bar{Y}_q)_{\mathbf{i}}$ for all $\mathbf{i} \geqq \mathbf{0}$ and $q \geqq 0$. If $\deg M_\mu \nleqq \mathbf{i}$ for $\mu = s+1,\ldots,t$, then $\bar{Y}_{\mathbf{i}} \cong Y_{\mathbf{i}}$, whence $H_*(\bar{Y})_{\mathbf{i}} \cong H_*(Y)_{\mathbf{i}} = 0$ if $\mathbf{i} \neq \mathbf{0}$. We have proved the second half of the following lemma:

LEMMA 3. *If $a \in \mathrm{Tor}_+^{R^i}(R^j, k)$ is non-zero and multihomogeneous, then* $\mathrm{Supp}\, a \subseteqq \{1,\ldots,j\}$, *and there is a $\mu \in \{s+1,\ldots,t\}$ such that* $\mathbf{deg}\, M_\mu \leqq \mathbf{deg}\, a$.

The first half of Lemma 3 may be proved as was (2).

As a corollary to Lemma 3 we see that if $0 \neq a \in \mathrm{Tor}_+^{R^i}(R^j, k)$ is mhom, then there are integers ν and m such that $i < \nu \leqq j$ and that $\deg_\nu a = m \geqq 1$. Because of the properties of the algebra generators of Y, any mhom cycle $z' \in \bar{Y}_+$

representing a must be on the form $z' = T_v x_v^{m-1} a' + x_v^m a''$, where $\deg_v a' = \deg_v a'' = 0$. Another representative of a is

$$z = z' - d(T_v x_v^{m-1} a'') = T_v x_v^{m-1}(a' + da'') .$$

Since $T_v^2 = 0$ and $0 = dz = -T_v x_v^{m-1} da' + x_v^m(a' + da'')$, we have $T_v z = x_v z = 0$. We thus have proved

LEMMA 4. *Assume that $a \in \mathrm{Tor}_+^{R^i}(R^j, k)$ is non-zero and multihomogeneous, and that $v \in \mathrm{Supp}\, a \cap \{i+1, \ldots, j\}$. Then, if we realize $\mathrm{Tor}^{R^i}(R^j, k)$ as $H(\bar{Y})$ as above, a may be represented by a multihomogeneous cycle $z = T_v x_v^{(\deg_v a) - 1} a' \in \bar{Y}_+$. Moreover $z\bar{Y}_{\mathbf{i}} = 0$ for any $\mathbf{i}$ such that $i_v \neq 0$.*

As a corollary we get

LEMMA 5. *If a and b are multihomogeneous elements in $\mathrm{Tor}^{R^i}(R^j, k)$, such that $ab \neq 0$, then $\mathrm{Supp}\, a \cap \mathrm{Supp}\, b \cap \{i+1, \ldots, j\} = \emptyset$.*

Recall that an augmented ring homomorphism $f\colon (A, k, \varepsilon_A) \to (B, k, \varepsilon_B)$ is called small if the induced homomorphism $f_*\colon \mathrm{Tor}^A(k, k) \to \mathrm{Tor}^B(k, k)$ of graded vector spaces is a monomorphism. Cf. e.g. ([1]).

The referee has kindly supplied the proof of the following lemma.

LEMMA 6. *$R^i \twoheadrightarrow R^j$ is small.*

PROOF. Note that $x_{i+1}, \ldots, x_j$ is an R^i-sequence, and that

$$\mathrm{Ker}\,(R^i \twoheadrightarrow R^j) \subset (x_{i+1}, \ldots, x_j) \cap (x_1, \ldots, x_n)^2 .$$

Now apply [1, Prop. 4.3].

Since $R^i \to R^j$ is small, one of the equivalent conditions for $R^i \to R^j$ to be Golod is the following (essentially (1) of [1, Def. 3.6]): For every positive integer q there is an application

$$\gamma : H_+(\bar{Y}) \times H_+(\bar{Y}) \times \ldots \times H_+(\bar{Y}) \to \bar{Y}_+ + (\mathrm{Ker}\, \varepsilon_{R^j})\bar{Y}_0$$

$$(q \text{ copies of } H_+(\bar{Y}))$$

such that

(i) For any $a \in H_+(\bar{Y})$, $\gamma(a)$ is a cycle which represents a; and
(ii) For any $q \geqq 2$ and any $a_1, \ldots, a_q \in H_+(\bar{Y})$,

$$d\gamma(a_1, \ldots, a_q) = \sum_{k=1}^{q-1} f(\gamma(a_1, \ldots, a_k))\gamma(a_{k+1}, \ldots, a_q) \tag{3}$$

where $f\colon \bar{Y} \to \bar{Y}$ is the R^j-linear mapping defined by $f(b) = (-1)^{d+1}b$ if $b \in \bar{Y}_d$, and by linear extension.

It is of course sufficient to find applications which satisfy (i) and (ii), except that they are defined only for (multiples of) elements in a k-vector space basis $\{h_\alpha\}_{\alpha \in A}$ of $H_+(\bar{Y})$. The applications then may be linearly extended to all (multiples of) elements in $H_+(\bar{Y})$.

We may now give a fast proof (essentially almost the same as that in [2]) of Lemma 2:

Assume that $j = i+1$. Let $\{h_\alpha\}_A$ be a mhom k-vector space basis of $H_+(\bar{Y})$.

By Lemma 3, $\deg_{i+1} h_\alpha > 0$ for any $\alpha \in A$. By Lemma 4, we may choose a representative $\gamma(h_\alpha) = T_{i+1}h'$ for h_α. Let $\gamma(h_{\alpha_1}, \ldots, h_{\alpha_q}) = 0$ for $q \geqq 2$. Since $T_{i+1}^2 = 0$, (3) is trivially verified.

Next, we may consider the case when $j = i+2$. Under what conditions will $R^i \to R^{i+2}$ be Golod?

LEMMA 7. *The following conditions are equivalent.*

(i) $R^i \to R^{i+2}$ *is Golod*

(ii) $(\mathrm{Tor}_+^{R^i}(R^{i+2}, k))^2 = 0$

(iii) *For any multihomogeneous elements a and b in* $\mathrm{Tor}_+^{R^i}(R^{i+2}, k)$, *such that* $\mathrm{Supp}\, a \cap \{i+1, i+2\} = \{i+1\}$ *and* $\mathrm{Supp}\, b \cap \{i+1, i+2\} = \{i+2\}$, $ab = 0$.

PROOF. It obviously follows from the definitions that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Assume that (iii) holds. Let $\{h_\alpha\}_{\alpha \in A}$ be a mhom k-vector space basis of $H_+(\bar{Y})$. For any $\alpha \in A$, choose a mhom representative $\gamma(h_\alpha)$ of h_α in the following manner:

If $\deg_{i+1} h_\alpha \neq 0$, then by Lemma 4, choose $\gamma(h_\alpha) = T_{i+1}h'_\alpha$ for some mhom $h'_\alpha \in \bar{Y}$. If $\deg_{i+1} h_\alpha = 0$, then by Lemma 3, $\deg_{i+2} h_\alpha \neq 0$, and we may choose $\gamma(h_\alpha) = T_{i+2}h'_\alpha$.

In the first case $\gamma(h_\alpha)\bar{Y}_{\boldsymbol{i}} = 0$ for any $\boldsymbol{i}$ such that $i_{i+1} \neq 0$, and correspondingly in the second case. Thus, if $\alpha, \beta \in A$ are such that

$$\mathrm{Supp}\, h_\alpha \cap \mathrm{Supp}\, h_\beta \cap \{i+1, i+2\} \neq \varnothing\,,$$

then $\gamma(h_\alpha)\gamma(h_\beta) = 0$. If this is the case, choose $\gamma(h_\alpha, h_\beta) = 0$. If $\alpha, \beta \in A$ and

$$\mathrm{Supp}\, h_\alpha \cap \mathrm{Supp}\, h_\beta \cap \{i+1,\, i+2\} = \varnothing\,,$$

then $h_\alpha h_\beta = 0$ by (iii), whence $\gamma(h_\alpha)\gamma(h_\beta) = dw_{\alpha,\beta}$, say, for some mhom $w_{\alpha,\beta} \in \bar{Y}_+$. $w_{\alpha,\beta}$ may be corrected by a boundary in a manner similar to that in the proof of the first half of Lemma 4, in such a way that we may assume $w_{\alpha,\beta} = T_{i+1}w'_{\alpha,\beta}$, say. Then, put $\gamma(h_\alpha, h_\beta) = (-1)^{1+d}w_{\alpha,\beta}$, where we assume $h_\alpha \in H_d(\bar{Y})$. For any $q \geqq 3$ and $\alpha_1, \ldots, \alpha_q \in A$, put $\gamma(h_{\alpha_1}, \ldots, h_{\alpha_q}) = 0$.

We now have set up a system of applications, which fulfills the prescribed conditions necessary to prove (i) of Lemma 7. (To see that (3) holds, note that $i+1,\ i+2 \in \operatorname{Supp}\gamma(h_\alpha, h_\beta)$ if $\gamma(h_\alpha, h_\beta) = \pm w_{\alpha,\beta} \neq 0$. Hence by Lemma 4, $\gamma(h_\alpha, h_\beta)\gamma(h_\delta) = \gamma(h_\alpha)\gamma(h_\beta, h_\delta) = 0$ for any $\alpha, \beta, \delta \in A$. Furthermore, since $T_{i+1}^2 = 0$, $\gamma(h_\alpha, h_\beta)\gamma(h_\delta, h_\varepsilon) = 0$ for any $\alpha, \beta, \delta, \varepsilon \in A$.)

Thus the lemma is proved.

5. Proof of the theorem.

We may assume that

$$R = \tilde{R}/\mathfrak{a} \simeq (k[X_1, X_2, X_3]/(M_1, \ldots, M_r))[X_4, \ldots, X_n],$$

where $\tilde{R} = k[X_1, \ldots, X_n]$ and where $\{M_1, \ldots, M_r\}$ is a minimal set of non-trivial monomials which generate $\mathfrak{a}$.

If R is a Golod ring, i.e. if $R^0 = \tilde{R} \to R$ is Golod, then the theorem is satisfied with $p=0$.

Assume in the sequel that R is not a Golod ring. Since the projective dimension of the $\tilde{R}$-module R is $\leqq 3$, this is equivalent to the condition $\operatorname{Tor}_+^{\tilde{R}}(R,k)^2 \neq 0$, as is well known. We may identify mhom components in a non-zero product, and thus find two mhom elements a and b in $\operatorname{Tor}_+^{\tilde{R}}(R,k)$, such that $a \cdot b \neq 0$. By Lemma 5, $\operatorname{Supp} a \cap \operatorname{Supp} b = \varnothing$, and by Lemma 3, $\operatorname{Supp} a$ and $\operatorname{Supp} b$ are non-empty but contained in $\{1,2,3\}$. Thus, after reindexing the three first variables we may assume that either

$$\text{(4)} \qquad \operatorname{Supp} a = \{1\} \quad \text{and} \quad \operatorname{Supp} b = \{2\},$$

or

$$\text{(5)} \qquad \operatorname{Supp} a = \{1\} \quad \text{and} \quad \operatorname{Supp} b = \{2,3\}.$$

If (4) holds, then R^2 is a c.i.. By Lemma 2, $R^2 \to R^3 = R$ is Golod, whence the theorem is satisfied with $p=2$.

In the sequel, assume (5) to hold (and that (4) does not hold for *any* choises of a and b). R^1 is a c.i., and I am going to prove that $R^1 \to R$ is Golod, whence the theorem is satisfied with $p=1$.

Let $\mathbf{deg}\, a = (\lambda, 0, 0, \ldots, 0)$ and $\mathbf{deg}\, b = (0, \mu, \nu, 0, \ldots, 0)$. Any mhom element in the Koszul complex of R that represents ab has $x_1^{\lambda-1} x_2^{\mu-1} x_3^{\nu-1}$ for a factor. Since $ab \neq 0$, we thus must have

$$\text{(6)} \qquad \mathbf{deg}\, M_q \nleqq (\lambda-1, \mu-1, \nu-1, 0, \ldots, 0) \quad \text{for } 1 \leqq q \leqq r.$$

By Lemma 3 (with $i=0$ and $j=3$) there must be some of the ideal generators, say M_1 and M_2, such that $\mathbf{deg}\, M_1 \leqq \mathbf{deg}\, a$ and $\mathbf{deg}\, M_2 \leqq \mathbf{deg}\, b$. Since the set $\{M_1, \ldots M_r\}$ of generators is minimal, we have $\deg_1 M_q < \deg_1 M_1$ for any q s.t. $2 \leqq q \leqq r$. This together with (6) implies

(7) $$\text{If } \operatorname{Supp} M_q \cap \{2,3\} = \{2\}, \quad \text{then } \deg_2 M_q \geqq \mu ,$$

and

(8) $$\text{If } \operatorname{Supp} M_q \cap \{2,3\} = \{3\}, \quad \text{then } \deg_3 M_q \geqq \nu .$$

Since the image of M_2 in R is 0,

(9) $$x_2^\mu x_3^\nu = 0 .$$

In order to prove that $R^1 \twoheadrightarrow R^3$ is Golod it is by Lemma 7 sufficient to prove that if $g, h \in \operatorname{Tor}_+^{R^1}(R^3, k)$ are mhom and $\deg_3 g = \deg_2 h = 0$, then $gh = 0$. Adopt the definition of $\bar{Y}, T_1, T_2$, etc. from section 4, with $i=1$ and $j=3$. Then, as we shall see,

(10) there is a mhom $g' \in \bar{Y}_+$ such that $x_2^\mu g'$ represents g ,

and

(11) there is a mhom $h' \in \bar{Y}_+$ such that $x_3^\nu h'$ represents h .

Let $\deg_2 g = \varrho$. By (7) and Lemma 3, $\varrho \geqq \mu$. By Lemma 4, g may be represented by some $x_2^{\varrho-1} T_2 g'' \in \bar{Y}_+$. If If $\varrho > \mu$, (10) follows. Assume that $\varrho = \mu$. Let $\mathbf{i} = \mathbf{deg}\, x_2^{\mu-1} g''$. Then $i_2 = \mu - 1$ and $i_3 = 0$, whence by Lemma 3 and (7), $\bar{Y}_{\mathbf{i}*} \xrightarrow{\sim} Y_{\mathbf{i}*}$. Furthermore $\mathbf{deg}\, g'' \neq \mathbf{0}$ since the choise $b = g$ would yield (4). Thus, since $x_2^{\mu-1}$ is a non-zero divisor in R^1 and $x_2^{\mu-1} g''$ is a cycle, g'' is a cycle and hence a boundary; say $g'' = dg'$. Thus g is represented by the element $x_2^{\mu-1} T_2 g'' + d(x_2^{\mu-1} T_2 g') = x_2^\mu g'$.

(11) is proved similarly.

By (10), (11) and (9) $gh = 0$ indeed.

Note added in proof. The assumption $R^4 = R$ of Corollary 2 is unnecessary: all SMH rings have rational Poincaré series (J. Backelin, *Les anneaux locaux à relations monomiales ont des séries de Poincaré–Betti rationnelles*, C. R. Acad. Sci. Paris Sér. I Math. 295 (1982), 607–610).

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MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
BOX 6701
S-113 85 STOCKHOLM
SWEDEN