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# FILTRATIONS OF MEROMORPHIC $\mathbf{C}^*$ ACTIONS ON COMPLEX MANIFOLDS

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## I. Introduction.

Let $X$ be a compact complex manifold with a meromorphic $\mathbf{C}^*$ action. We prove that the condition that $X$ have no quasi-cycles is sufficient, and almost necessary, for $X$ to have a filterable plus decomposition [2]. This result can be used to show that if $X$ has no quasi-cycles, then the integral homology groups of $X$ can be calculated on the fixed point set $X^{\mathbf{C}^*}$ as in [4, Theorem 1].

## II. Statement of results.

Recall that a holomorphic $\mathbf{C}^*$ action on $X$ is called *meromorphic* if there is a meromorphic map $\mathbf{C}P^1 \times X \to X$ extending the action $\mathbf{C}^* \times X \to X$. For a meromorphic action,

$$x_0 = \lim_{\lambda \to 0} \lambda \cdot x \quad \text{and} \quad x_\infty = \lim_{\lambda \to \infty} \lambda \cdot x$$

exist for all $x$ and lie in the fixed point set $X^{\mathbf{C}^*}$. Let $X_1, \ldots, X_r$ denote the connected components of $X^{\mathbf{C}^*}$. We write $X_i \to X_j$ if there exists a non-trivial orbit $\mathbf{C}^* \cdot x$ such that $x_0 \in X_i$ and $x_\infty \in X_j$. We write $X_i < X_j$ if there exists a *quasi-chain* $X_i \to \ldots \to X_j$. The relation $<$ is a partial ordering on the set $\chi = \{X_1, \ldots, X_r\}$ of fixed point components exactly when there are no *quasi-cycles* $X_i \to \ldots \to X_i$, i.e. precisely when $X_i \not< X_i$ for any $i$. We call a one step quasi-cycle $X_i \to X_i$ a *loop*. If $X$ is connected and $<$ is a partial ordering on $\chi$, then there exists a unique minimal element $X_1$, the *source* of $X$, and a unique maximal element $X_r$, the sink of $X$, so that for any $X_i$, $i \neq 1$ or $r$, $X_1 < X_i < X_r$. In the following theorem, $X$ denotes a connected compact complex manifold with meromorphic $\mathbf{C}^*$ action and $\mathbf{R}^+$ the nonnegative reals.

THEOREM. *Suppose $<$ is a partial ordering on $\chi$. Then any nondecreasing function $g\colon \chi \to \mathbf{R}^+$ which assumes its absolute maximum uniquely on the sink*

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*gives rise to a not necessarily unique filtration of the plus decomposition of $X$. Conversely, any filtration of the plus decomposition of $X$ gives rise to a strictly increasing function on $\chi$.*

In fact, given $g$ as above with values $a_0 \leqq \ldots \leqq a_k$, any one of the filtrations defined by $g$ refines the filtration

$$X = \mathscr{G}_0 \supset \mathscr{G}_1 \supset \ldots \supset \mathscr{G}_k$$

where $\mathscr{G}_i = \{x : g(x_0) \geqq a_i\}$. This filtration is described in Lemma 1. To obtain a canonical example of a strictly increasing function $g$ on $\chi$, let $g(X_i) = \#$ steps in the longest quasi-chain from $X_1$ to $X_i$ ($g$ is well defined if $<$ is a partial ordering).

COROLLARY. *If $<$ is a partial ordering on $\chi$, then the plus decomposition of $X$ may be filtered.*

We also prove a partial converse in section V. Namely, if $X$ has no loops and has a filterable plus decomposition, then $<$ is a partial ordering on $\chi$, i.e. $X$ has no quasi-cycles. Several corollaries and examples are treated in section VI. In section VII, the principal application is briefly discussed.

Finally, we would like to remark that all our results have obvious analogues for filtrations of the minus decomposition.

## III. Remarks on the B-B decompositions.

Recall that the plus and minus cells of a meromorphic $\mathbf{C}^*$ action on $X$ are defined respectively as

$$X_j^+ = \{x \in X : x_0 \in X_j\}$$

and

$$X_j^- = \{x \in X : x_\infty \in X_j\}$$

If $X$ has no quasi-cycles, then the proof of Theorem 1 [3] shows that the basic structure for the $X_j^+$ and $X_j^-$ is the same as in the case $X$ is compact Kaehler [3] or a complete nonsingular algebraic variety [1]. That is, each $X_j^+$ (respectively $X_j^-$) is a holomorphically locally trivial affine space bundle over $X_j$ with bundle projection $p_j(x) = x_0$ (respectively $q_j(x) = x_\infty$), and, moreover, each $\overline{X_j^+}$ (resp. $\overline{X_j^-}$) is analytic and contains $X_j^+$ (respectively $X_j^-$) as a Zariski open subset.

The plus decomposition $X = \bigcup X_j^+$ is filterable in the sense of [2] if there exists a filtration

$$X = \mathscr{F}_1 \supset \mathscr{F}_2 \supset \ldots \supset \mathscr{F}_r \supset \mathscr{F}_{r+1} = \varnothing \tag{1}$$

such that each $\mathscr{F}_i$ is a closed invariant subvariety of $X$ and such that for each $i$, $1 \leqq i \leqq r$,

$$\mathscr{F}_i - \mathscr{F}_{i+1} = X^+_{\alpha(i)} \tag{2}$$

where $\alpha$ is a bijection of $\{1, 2, \ldots, r\}$.

## IV. Some lemmas.

The basic fact we need in this paper is the following

LEMMA 1. *If $X_k \cap \overline{X_i^+} \neq \varnothing$ and $k \neq i$, then $X_i < X_k$.*

The proof of this lemma can be given in two ways. If one assumes $X$ has no quasi-cycles, then a chain $X_i \to \ldots \to X_k$ can be constructed directly by an interesting local argument. It turns out, however, that the lemma is true without any condition on quasicycles as the following general result of Fujicki [6] shows.

THEOREM. *For each $i$, $1 \leqq i \leqq r$, there is a diagram*

$$\begin{array}{ccc} Z_i & \xrightarrow{\varphi_i} & \overline{X_i^+} \\ {\scriptstyle f_i}\downarrow & & \\ \mathscr{Q}_i & & \end{array}$$

*with the following properties*:

(a) *$f_i$ is a flat morphism of irreducible compact complex spaces $Z_i$ and $\mathscr{Q}_i$,*

(b) *$\varphi_i$ is a bimeromorphic holomorphic map of $Z_i$ onto $\overline{X_i^+}$ such that the restriction of $\varphi_i$ to each fibre $f_i^{-1}(t)$ is an imbedding,*

(c) *there is a natural meromorphic action of $\mathbf{C}^*$ on $Z_i$ making $f_i$ and $\varphi_i$ equivariant with respect to the trivial action on $\mathscr{Q}_i$, and*

(d) *there is a dense Zariski open set $U \subseteqq \mathscr{Q}_i$ such that, for every $t \in U$, $\varphi_i(f_i^{-1}(t))$ is the closure of a $\mathbf{C}^*$ orbit from $X_i$.*

This theorem is analogous to, but easier than, a result of [6] for $\mathscr{C}$ spaces. To prove it, one must use meromorphicity of the action along with Hironaka's flattening theorem [8] and the existence of the Douady space.

To prove Lemma 1, note that the fibres of $f_i$ are chains from $X_i$, so the lemma follows from the surjectivity of $\varphi_i$.

The next lemma will also be useful. Note that in this lemma, we allow the possibility $X_i < X_i$.

LEMMA 2. *Let $g: \chi \to \mathbf{R}^+$ be a nondecreasing function and set $\mathscr{G}_i = \{x : g(x_0) \geqq i\}$ where $g(x_0)$ means $g(X_i)$ if $x_0 \in X_i$. Then $\mathscr{G}_i$ is a closed invariant subvariety* *of $X$. Conversely, if, for a given $g: \chi \to \mathbf{R}^+$, all $\mathscr{G}_i$ are closed then $g$ is nondecreasing.*

PROOF. Suppose $x \in \overline{\mathscr{G}_i}$. We may assume that $x \in \overline{X_p^+}$ where $g(X_p) \geqq i$. Then if $x_0 \notin X_p$, there exists a chain from $X_p$ to $X_j$, that is, $X_p < X_j$, where $x_0 \in X_j$. Thus $g(x_0) = g(X_j) \geqq g(X_p) \geqq i$, so $x \in \mathscr{G}_i$, that is, $\mathscr{G}_i$ is closed. It is a subvariety since

$$\mathscr{G}_i = \bigcup \{\overline{X_j^+} : g(X_j) \geqq i\}$$

To prove the converse, suppose $x_0 \in X_i$ and $g(x_0) = k$. Then $x_\infty \in \mathscr{G}_k$, so $g(x_\infty) \geqq k = g(x_0)$. Therefore, if $X_i \to X_j$, then $g(X_j) \geqq g(X_i)$; hence it follows that $g$ is nondecreasing.

## V. Proof of the main theorem.

Suppose $<$ is a partial ordering, and suppose a nondecreasing $g$ is given. If $g$ assumes $K$ values then we may as well suppose $g(\chi) = \{1, \ldots, K\}$. We will show, by descending induction, that all $\mathscr{G}_i$ can be filtered. First of all, set $\mathscr{F}_r = \mathscr{G}_K$, and suppose $\mathscr{G}_j$ has been filtered for $j > k$. Write

$$\mathscr{G}_{k+1} = \mathscr{F}_t \supset \mathscr{F}_{t+1} \supset \ldots \supset \mathscr{F}_r$$

and

$$\mathscr{G}_k - \mathscr{G}_{k+1} = X_{k_1}^+ \cup \ldots \cup X_{k_p}^+$$

We claim that, for some $i$,

$$A_{k_i} = \overline{X_{k_i}^+} - X_{k_i}^+ \subset \mathscr{G}_{k+1}$$

For if not, then each $A_{k_i}$ has a fixed point $y$ with $g(y) = k$. Because there are only finitely many fixed point components on $X$, this implies that there is a quasi-cycle in $\mathscr{G}_k$, which is impossible. Therefore, we may suppose that

$$A_{k_1} \cup \ldots \cup A_{k_l} \subset \mathscr{G}_{k+1}$$

and set

$$\mathscr{F}_{t-j} = \mathscr{G}_{k+1} \cup X_{k_1}^+ \cup \ldots \cup X_{k_j}^+$$

for $1 \leqq j \leqq l$. Clearly each $\mathscr{F}_{t-j}$ is closed. By the same argument, there is an $i > l$ so that $A_{k_i} \subset \mathscr{F}_{t-l}$, and hence we may continue eventually getting a filtration of $\mathscr{G}_k$. This proves the existence of a filtration.

Next, suppose given a filtration as in (1), and set $g(X_j) = \alpha^{-1}(j)$ where $\alpha$ is defined in (2). We claim that if $X_j < X_k$, then $\alpha^{-1}(j) < \alpha^{-1}(k)$. This follows immediately from

LEMMA 3. *If $X$ has no loops, and if $x \in X_j^+ - X_j$, then $x_\infty \in \mathscr{F}_{\alpha^{-1}(j)+1}$.*

The proof is obvious.

A partial converse to the theorem is provided by the next result.

PROPOSITION. *If $X$ has no loops and has a filtration of the plus decomposition, then $X$ has no quasi-cycles.*

PROOF. If $X_i < X_i$, for some $i$, then there exists a quasi-cycle $X_i \to \dots \to X_i$ of length $l > 1$. Then if $X_i^+ = \mathscr{F}_j - \mathscr{F}_{j+1}$, Lemma 3 implies $X_i$ meets $\mathscr{F}_{j+l}$ which is ridiculous.

In particular we have

COROLLARY 1. *If $X$ is a compact algebraic manifold, then $X$ has a filtration of its plus decomposition if and only if $X$ has no quasi-cycles.*

PROOF. By Sumihiro's Theorem, any fixed point of $X$ has a $\mathbf{C}^*$ invariant neighborhood that is equivariantly isomorphic to $\mathbf{C}^n$ with a linear $\mathbf{C}^*$ action. In particular, $X$ cannot have a loop.

There exist examples of meromorphic $\mathbf{C}^*$ actions on Moisezon manifolds with loops. These will be discussed elsewhere, but they are similar to Example 2 of section VI.

## VI. Corollaries and examples.

It is clear that if $\chi$ admits a function $g$ with the property that $X_i \to X_j$ implies $g(X_i) < g(X_j)$, then $<$ is a partial ordering. This motivates the following corollaries, the first of which was originally proved by Koras [9].

COROLLARY 2. *If $X$ is compact Kaehler, then the plus decomposition of $X$ is filterable.*

PROOF. This follows immediately from the existence of a Frankel Morse function, i.e. a Morse function whose critical point set is $X^{\mathbf{C}^*}$ which is strictly increasing on $\mathbf{R}^+$ orbits [4].

For $x \in X^{\mathbf{C}^*}$, set the index of $x$, Ind $(x)$, equal to $\dim_{\mathbf{C}} \{y : y_\infty = x\}$.

COROLLARY 3. *Suppose $X$ satisfies the index condition* Ind $(x_0) <$ Ind $(x_\infty)$ *for all $x \in X - X^{\mathbf{C}^*}$. Then the plus decomposition is filterable.*

COROLLARY 4. *Suppose $X^{\mathbf{C}^*}$ is finite and for all $j, k$, $X_j^+ \pitchfork X_k^-$. Then again the plus decomposition is filterable.*

PROOF. By a theorem of Smale [10], there exists a Frankel Morse function. In fact, Smale's Morse function has the property that $f(x) = \operatorname{Ind}(x)$ for $x \in X^{\mathbf{C}^*}$. This shows that the index condition is generically satisfied.

EXAMPLE 1. Not every Kaehler manifold satisfies the weak index condition $\operatorname{Ind}(x_0) \leqq \operatorname{Ind}(x_\infty)$. Hence Corollaries 1 and 2 are independent. To see an example of this, let $Y = \mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^2$ with the action

$$\lambda \cdot ([Z_0, Z_1], [W_0, W_1, W_2]) = ([Z_0, \lambda Z_1], [W_0, W_1, W_2]),$$

and let $X$ be the space obtained from $Y$ by blowing up $([1,0],[1,0,0])$. There are three fixed point components $X_1 < X_2 < X_3$, where $\operatorname{Ind}(X_2) = 2$ and $\operatorname{Ind}(X_3) = 1$, so the index condition does not hold.

EXAMPLE 2. An example of a nonfilterable plus decomposition is obtained from the following $X$ that has quasi-cycles but no loops. Let $Y = \mathbf{C}\mathrm{P}^1 \times Z$ where $Z = \mathbf{C}\mathrm{P}^1 \times \mathbf{C}\mathrm{P}^1$. Choose two smooth curves $C_1$ and $C_2$ in $Z$ which are homologous to the diagonal and intersect transversely at two points $a, b$. Let $\mathbf{C}^*$ act on $Y$ by acting on the first factor only. Let $C_1 \cup C_2 \subseteqq \{0\} \times Z$, and blow $C_1$ and $C_2$ up with the Hironaka twist [8, Appendix B]. We get an action of $\mathbf{C}^*$ in this blown up space $X$ with four fixed point components:

(a) the source $X_1$ biholomorphic to $Z$,
(b) the sink $X_4$ biholomorphic to $Z$,
(c) $X_2$ biholomorphic to $C_1$, and
(d) $X_3$ biholomorphic to $C_2$.

Note $X_2 < X_3 < X_2$, but there are no loops. Incidentally, $\operatorname{Ind}(X_2) = \operatorname{Ind}(X_3) = \operatorname{Ind}(X_4)$ so the weak index condition is satisfied.

It is worthwhile noting that since $X$ is an algebraic manifold, Sumihiro's theorem guarantees that $X$ cannot have loops. Hence $X$ has a quasi-cycle of minimal length (two).

## VII. Applications.

If $X$ is a compact Kaehler manifold with meromorphic $\mathbf{C}^*$ action, then it was proved in [4] that there exist isomorphisms of homology

$$\mu_k: \bigoplus_j H_{k-2m_j}(X_j, \mathbf{Z}) \to H_k(X, \mathbf{Z}), \tag{3}$$

where $m_j$ is the complex fibre dimension of $X_j^+$. The upshot of the main theorem of this paper is

COROLLARY 4. (3) *remains true if* $X$ *has no quasi-cycles.*

The proof is basically the same as in [4]. All one needs to do extra is to verify the Frankel–Morse inequalities

$$b_k(X) \leqq \sum_j b_{k-2m_j}(X_j) \tag{4}$$

We reproduce a proof of (4) due to Koras [9]. By using the exact sequence

$$\to H_c^k(\mathscr{F}_i - \mathscr{F}_{i+1}) \to H^k(\mathscr{F}_i) \to H^k(\mathscr{F}_{i+1}) \to H_c^{k+1}(\mathscr{F}_i - \mathscr{F}_{i+1}) \to \dots$$

and the fact that

$$\begin{aligned} H_c^k(X_j^+) &\cong H_{2m_j+2f_j-k}(X_j^+) \cong H_{2m_j+2f_j-k}(X_j) \\ &\cong H^{k-2m_j}(X_j), \end{aligned}$$

where $f_j = \dim_{\mathbf{C}} X_j$, one gets $b_k(X) \leqq \sum b_{k-2m_j}(X_j)$.

A different proof of the corollary is given in [5].

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