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PLURISUBHARMONIC FUNCTIONS ON SMOOTH DOMAINS

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1.

In this short note we will discuss regularization of plurisubharmonic functions. More precisely, we will address the following problem:

QUESTION. Assume Ω is a bounded domain in $\mathbf{C}^n$ $(n \geqq 2)$ with smooth $(\mathscr{C}^\infty)$ boundary and that $\varrho\colon \Omega \to \mathbf{R} \cup \{-\infty\}$ is a (discontinuous) plurisubharmonic function. Does there exist a sequence

$$\{\varrho_n\}_{n=1}^{\infty}, \quad \varrho_n\colon \Omega \to \mathbf{R},$$

of $\mathscr{C}^\infty$ plurisubharmonic functions such that $\varrho_n \searrow \varrho$ pointwise?

If ϱ is continuous, the answer to the above question is yes (see Richberg [3]). On the other hand, when ϱ is allowed to be discontinuous and Ω is not required to have a smooth boundary, the answer is in general no (see [1], [2] for this and related questions).

Our result in this paper is that the answer to the above question is no. We present a counterexample in the next section. The construction leaves open what happens if we make the further requirement that Ω has real analytic boundary. Another question, suggested to the author by Grauert, is obtained by replacing Ω by a compact complex manifold with smooth boundary, and assuming continuity of ϱ.

In the next section we need of course both to construct the domain Ω *and* the function ϱ. These constructions are intertwined and therefore we need at first to define approximate solutions Ω_1 and ϱ_1 and then use both to define Ω and ϱ. The geometric properties we seek of Ω are the following. There exists an annulus $A \subset \bar{\Omega}$ such that $\partial A \subset \Omega$. Furthermore there exist concentric circles C_1, C_2, C_3 in the relative interior of A arranged by increasing radii such that $C_1, C_3 \subset \partial\Omega$ and $C_2 \subset \Omega$. Finally there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of annuli such that $A_n \to A$ and $A_n \subset \Omega$ $\forall n$. The properties we seek of ϱ are as follows. The

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function ϱ is strictly positive on C_2 and is strictly negative on ∂A. A simple application of the maximum principle now shows that smoothing is impossible.

The example we construct is in $\mathbf{C}^2$. This is with no loss of generality as one obtains then an example in $\mathbf{C}^n$ by crossing with a smooth domain in $\mathbf{C}^{n-2}$, rounding off the edges and pulling back ϱ to the new domain.

2.

All domains and functions which we will consider in $\mathbf{C}^2(z, w)$ will be invariant under rotations in the z-plane, i.e. will depend only on $|z|$. They will also be invariant under the map $(z, w) \to (1/z, w)$. Because of the latter we will describe only those points (z, w) in these domains or domains of definitions for which $|z| \leqq 1$.

If U is a domain in $\mathbf{C}^2(z, w)$, we let U_z denote the part of U over z, i.e.

$$U_z := \{(\eta, w) \in \mathbf{C}^2 : \eta = z \text{ and } (\eta, w) \in U\} .$$

Abusing notation we will also take U_z to mean the set $\{w \in \mathbf{C}; (z, w) \in U\}$. Similarly, if $\sigma: U \to \mathbf{R} \cup \{-\infty\}$ is a function, then σ_z denotes the restriction of σ to U_z.

Let A be the annulus in $\mathbf{C}^2$ given by

$$A = \{(z, w) ; w = 0 \text{ and } 1/2 \leqq |z| \leqq 2\} .$$

This is then the limit of a sequence of annuli $\{A_n\}_{n=1}$, where

$$A_n = \{(z, w) ; w = 1/n \text{ and } 1/2 \leqq |z| \leqq 2\} .$$

We will next describe a bounded domain Ω_1 in $\mathbf{C}^2$ with $\mathscr{C}^\infty$ boundary containing all A_n's (and hence A) in its closure. It will suffice to describe $\Omega_{1,z}$ for various z's. That these can be made to add up to a domain with $\mathscr{C}^\infty$ boundary will be clear throughout.

Choose a sequence of positive numbers $\{r_k\}_{k=1}^\infty$, $0 < r_1 < r_2 < \ldots < 1$, with $r_3 = 1/2$. We let $\Omega_{1,z} = \emptyset$, if $|z| \leqq r_1$ and $\Omega_{1,z}$ be a nonempty disc, concentric about the origin if $r_1 < |z| \leqq r_4$. Recall that $\Omega_{1,z} = \Omega_{1,|z|}$ for all z. If $r_2 \leqq |z| \leqq r_4$ we make the extra assumption that $\Omega_{1,z}$ has radius 2. For $|z| > r_4$ we will break the symmetry in the w-direction at first by letting $\Omega_{1,z}$ gradually approach the shape of an upper-disc. (This is a rough description to be made more precise below.) Increasing $|z|$ further we will rotate this approximate upper half disc 180° clockwise until it becomes approximately a lower half disc. Then we proceed by reversing the process, first by rotating counterclockwise back to an approximate upper half disc and then expanding this back to a disc of radius 2 near $|z| = 1$. As mentioned earlier, if $|z| > 1$, then $\Omega_{1,z} := \Omega_{1,1/z}$.

We now return to the more precise description of $\Omega_{1,z}$ for $|z|>r_4$. Writing $w=u+iv$ in real coordinates u, v, let $v=f(u)$ be a $\mathscr{C}^\infty$ function defined for $u \in \mathbf{R}$ with $f(u)=0$ if $u \leqq 0$ or $u \geqq 2, f \geqq 0$ and $f(u)=0$ on $(0,2)$ if and only if $u=1/n$ for some positive integer n. We may assume that $|f|$, $|f'|$, $|f''|$ are very small and therefore in particular that the graph of f only intersects the boundary of any disc $\Delta(0; R)=\{|w|<R\}$ in exactly two points. If $r_4<|z|<r_5$, we let $\Omega_{1,z}$ be a subdomain of $\Delta(0; 2)$ containing those $u+iv \in \Delta(0; 3/2)$ for which $v \geqq f(u)$. When $r_5 \leqq |z| \leqq r_6$ we choose $\Omega_{1,z}$ independent of z with the properties that $\Omega_{1,z} \subset \Delta(0; 7/4) \cap \{v>f(u)\}$ and $\Delta(0; 3/2) \cap \{v>f(u)\} \subset \Omega_{1,z}$. Let $\theta(x)$ be a real $\mathscr{C}^\infty$ function on $\mathbf{R}$ with $\theta(x)=0$ if $x \leqq r_6$, $\theta(x)=\pi$ if $x \geqq r_7$, and $\theta'(x)>0$ if $r_6 < x < r_7$. Then we can rotate $\Omega_{1,z}$ 180° clockwise for $r_6 \leqq |z| \leqq r_7$ by defining $\Omega_{1,z}=e^{-i\theta(|z|)}\Omega_{1,r_6}$ for such z. Further, we let $\Omega_{1,z}=\Omega_{1,r_7}$ when $r_7 \leqq |z| \leqq r_8$. Reversing the procedure, we rotate $\Omega_{1,z}$ back 180° when $r_8 \leqq |z| \leqq r_9$ so that Ω_{1,r_9} again equals Ω_{1,r_6}. Continuing, we let $\Omega_{1,z}=\Omega_{1,r_9}$ whenever $r_9 \leqq |z| \leqq r_{10}$. Reversing the procedure between r_4 and r_5 we obtain $\Omega_{1,z}$'s, $r_{10} \leqq |z| \leqq r_{11}$ so that in particular $\Omega_{1,r_{11}}$ is the disc $\Delta(0,2)$. When $r_{11}<|z| \leqq 1$, we let $\Omega_{1,z}$ always be this same disc. This completes the construction of Ω_1.

The next step is to define an (almost) plurisubharmonic function ϱ_1. Let $\{\varepsilon_n\}_{n=1}^\infty$ be a sufficiently rapidly decreasing sequence of positive numbers, $\varepsilon_n \searrow 0$. Then

$$\sigma_1(w) := \sum_{n=1}^{\infty} \varepsilon_n \log\left|w-\frac{1}{n}\right|$$

is a subharmonic function on the complex plane and $\sigma_1(0) \in (-\infty, 0)$. Letting $\sigma(w)=\sigma_1(w)+1-\sigma_1(0)$ we obtain a subharmonic function on $\mathbf{C}(w)$ with $\sigma(0)=1$ and $\sigma(1/n)=-\infty \ \forall n \in \mathbf{Z}^+$. If the constant $K>0$ is chosen large enough, the plurisubharmonic function $\sigma(w)+K\log(|z|/r_5)$ will be strictly less than -1 at all points $(z, w) \in \Omega_1$ for which $|z| \leqq r_4$. The function ϱ_1: $\Omega_1 \to \mathbf{R}$ is defined by the equations

$$\varrho_1(z, w) = \varrho_1(1/z, w)$$

and

$$\varrho_1(z, w) = \max\{\sigma(w)+K\log(|z|/r_5), -1\}, \qquad \text{when } |z| \leqq 1 .$$

Then ϱ_1 is the restriction to Ω_1 of the similarly defined function on $\mathbf{C}^2$ and ϱ_1 is plurisubharmonic at all points (z, w) with $|z| \neq 1$. This completes the construction of ϱ_1.

We have two main problems left. The annuli A_n all lie partly in the boundary of Ω_1, so Ω_1 has to be bumped slightly so that they all lie in the interior. However, this bumping should not change the extent to which A lies in the boundary. The other main problem is the failure of plurisubharmonicity of ϱ_1

at $|z|=1$. We will change ϱ_1 near $|z|=1$ so that it will equal $\max\{\sigma(w), -1\}$ in a neighbourhood of this set. In order to deal with both these problems, we will at first construct a subharmonic function $\tau(w)$ which can be used for patching purposes.

Our first approximation to τ will be τ_1. The domain of τ_1 will be

$$D := \{w\ ;\ |w|<2,\ w \notin (-2,0],\ w \notin \{1/n\}\}\ .$$

The properties we will require of τ_1 are that $\tau_1(u+iv)=0$ when $v \geqq f(u)$, $\tau_1(u+iv) \geqq 1$ when $v \leqq 0$, τ_1 is $\mathscr{C}^\infty$ and τ_1 is strongly subharmonic at all points $u+iv$ with $v<f(u)$.

Let K_0 denote the compact set $\{w=u+iv;\ |w| \leqq 2 \text{ and } v \geqq f(u)\}$. Since K_0 is polynomially convex, there exists a $\mathscr{C}^\infty$ subharmonic function $\lambda_0\colon \mathbf{C} \to [0,\infty\rangle$ which vanishes precisely on K_0 and which is strictly subharmonic on $\mathbf{C}-K_0$. Choose an increasing sequence of compact sets

$$F_1 \subset \operatorname{int} F_2 \subset F_2 \subset \operatorname{int} F_3 \subset \ldots \subset D, \qquad D = \bigcup F_l\ .$$

Letting $K_l = K_0 \cup F_l$ we may even assume that each bounded component of $\mathbf{C} - K_l$ clusters at some $1/n$ and in particular therefore that there are only finitely may of these components. With these choices it is possible for each $l \geqq 1$ to find a non-negative $\mathscr{C}^\infty$ function λ_l such that $\lambda_l | K_l \equiv 0$, $\lambda_l \geqq 1$ and strongly subharmonic on $\{u+iv \in K_{l+2} - \operatorname{int} K_{l+1};\ v \leqq 0\}$ and λ_l fails to be subharmonic only on a relatively compact subset of $(\operatorname{int} K_{l+3} - K_{l+2}) \cap \{v<0\}$. But then, if $\{C_l\}_{l=0}^\infty$ is a sufficiently rapidly increasing sequence,

$$\tau_1 := \sum_{l=0}^{\infty} C_l \lambda_l$$

has all the desired properties.

We next want to push the singularities of τ_1 at the points $1/n$ over to the origin. First, let us choose discs $\Delta_n = \Delta(1/n, \varrho_n)$ small enough so that $\sigma(w) + K \log 1/r_5 < -1$ on each Δ_n.

We will first perturbe τ_1 inside each Δ_n. We can make a small pertubation of the situation by making a small translation parallell to the v-axis in the negative direction in a smaller disc about $1/n$ patched with the identity outside a slightly larger disc in Δ_n to obtain a new $\mathscr{C}^2$ function $\tau_2 \geqq 0$ and a new $\mathscr{C}^\infty$ function $v = f_1(u)$ with the properties that $f_1 \leqq f$, $f_1 < f$ near $1/n$, $f_1 = f$ away from $1/n$ and $\tau_2 = 0$ when $v \geqq f_1(u)$, $\tau_2 \geqq 1$ when $v \leqq 0$ except in very small discs about $1/n$ and

$$\tau_3 = \begin{cases} 0 & \text{when } v \geqq f_1(u) \\ \tau_2 + (v - f_1(u))^2 & \text{otherwise} \end{cases}$$

is strongly subharmonic when $v < f_1(u)$.

The singularities of τ_1 at the points $1/n$ have thus been moved down to the points $\varrho_n = 1/n + if_1(i/n)$. Let $\Delta'_n = \Delta(1/n, \varrho'_n)$, $0 < \varrho'_n \ll \varrho_n$ be discs on which $\tau_3 \equiv 0$. We may assume that $p_n \notin \bar{\Delta}'_n$. Let γ be a curve from p_1 to 0 passing in the lower half plane through all the p_n's and avoiding all the $\bar{\Delta}'_n$'s. We can assume say that γ is linear between p_n and p_{n+1}. Let V be a narrow tubular neighbourhood of $\gamma - \{0\}$ also lying in the lower half-plane and avoding all the $\bar{\Delta}'_n$'s. The restriction $\tau_3 \mid V$ is $\mathscr{C}^\infty$, subharmonic and $\geqq 1$ except for singularities at each p_n. Let $\tau_4 \geqq 1$ be a $\mathscr{C}^\infty$ function on V which agrees with $\tau_3 \mid V$ on $V \cap V'$, V' some open set containing $\partial V - \{0\}$. A construction similar to the one for τ_1 yields a $\mathscr{C}^\infty$ subharmonic function $\tau_5 \geqq 0$ on $\mathbf{C} - (0)$ which vanishes outside V and is such that $\tau_4 + \tau_5$ is subharmonic on V. Finally, let τ: $\{(w) < 2,\ w \notin [-2, 0]\} \to \mathbf{R}^+$ be the $\mathscr{C}^\infty$ subharmonic function given by $\tau = \tau_3$ outside V and $\tau = \tau_4 + \tau_5$ on V. Then $\tau = 0$ on each Δ'_n and $\tau(w) = 0$ when $v \geqq f_1(u)$ except possibly on a concentric disc Δ''_n, $\Delta'_n \subset\subset \Delta''_n \subset\subset \Delta_n$. Also, $\tau(w) \geqq 1$ when $v \leqq 0$, $w \notin \bigcup \Delta''_n$. This completes the construction of the patching function τ.

The construction of Ω can now be completed. A point $(z, 1/n) \in A_n$ lies in the boundary of Ω_1 only when $|z|$ or $1/|z|$ is in $[r_5, r_6] \cup [r_7, r_8] \cup [r_9, r_{10}]$. This set is contained in the open set

$$\{(z, w)\ ;\ |z| \text{ or } 1/|z| \in (r_4, r_{11}) \text{ and } w \in \Delta'_n\} =: U_n .$$

We let Ω be a domain with $\mathscr{C}^\infty$ boundary which agrees with Ω_1 outside $\bigcup U_n$ and which contains all A_n's in its interior.

Next we define the plurisubharmonic function ϱ: $\Omega \to \mathbf{R}$. Let $\sigma' = \max\{\sigma, -1\}$ and choose a constant $L \gg 1$ such that $\varrho_1 \leqq L - 1$ on $\bar{\Omega}$. If $|z| \leqq r_6$, let $\varrho_z := \varrho_{1,z}$. For $r_5 \leqq |z| \leqq r_6$, this definition agrees with $\varrho_z = \max\{\varrho_{1,z}, \sigma' + L\tau\}$, since τ is then 0 and $\varrho_1 = \sigma' + K \log(|z|/r_5)$. If $r_6 < |z| \leqq r_8$, let

$$\varrho_z := \max\{\varrho_{1,z}, \sigma' + L\tau\} .$$

For $r_7 \leqq |z| \leqq r_8$, this definition agrees with $\varrho_z = \sigma' + L\tau$. To see this, observe that if $w \in \Delta''_n$, then $\varrho_{1,z} = -1$ and $\sigma' = -1$ while $\tau \geqq 0$. If on the other hand $w \notin \bigcup \Delta''_n$, then $v < 0$ and $\sigma' + L\tau \geqq -1 + L \geqq \varrho_1$. If $r_8 < |z| \leqq r_{10}$, let $\varrho_z := \sigma' + L\tau$. For $r_9 \leqq |z| \leqq r_{10}$ this definition agrees with $\varrho_z = \sigma'$ since $\tau = 0$. Also, if $r_{10} \leqq |z| \leqq 1$, let $\varrho_z := \sigma'$, and if $|z| > 1$, let $\varrho_z := \varrho_{1/z}$. Then ϱ is plurisubharmonic on Ω,

$$\varrho(e^{i\theta}, 0) = 1 \qquad \forall \theta \in \mathbf{R}$$

and

$$\varrho(e^{i\theta}/2, 0) = \varrho(2e^{i\theta}, 0) = -1 \qquad \forall \theta \in \mathbf{R} .$$

If there exists a sequence of $\mathscr{C}^\infty$ plurisubharmonic functions ϱ_m: $\Omega \to \mathbf{R}$, $\varrho_m \searrow \varrho$, then there exists an m for which

$$\varrho_m(e^{i\theta}/2, 0),\ \varrho_m(2e^{i\theta}, 0) < 0 \qquad \forall\,\theta \in \mathbf{R}\,.$$

Hence, for all large enough n,

$$\varrho_m(e^{i\theta}/2, 1/n),\ \varrho_m(2e^{i\theta}, 1/n) < 0 \qquad \forall\,\theta \in \mathbf{R}\,.$$

By the maximum principle applied to the annuli $A_n \subset \Omega$, it follows that $\varrho_m(e^{i\theta}, 1/n) < 0\ \forall\,\theta \in \mathbf{R}$ and all large enough n. Hence, by continuity of ϱ_m, $\varrho_m(e^{i\theta}, 0) \leqq 0\ \forall\,\theta \in \mathbf{R}$. This contradicts the assumption that $\varrho_m \geqq \varrho$ and therefore completes the counterexample.

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