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GENERALIZED CLASSES OF GROUPS, C-NILPOTENT SPACES AND "THE HUREWICZ THEOREM"

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Introduction.

The purpose of this work is to introduce a category of spaces more general than the category of nilpotent spaces, where we still have nice theorems like the relative Hurewicz theorem. This more general category is described in terms of a fixed family of groups, called class, and the action of $\pi_1(X)$ on $\pi_n(X)$, where X is a topological space. This notion of class is a generalization of the definition of a class of abelian groups, given by Serre in [9] and a Serre class of nilpotent groups given by P. Hilton, and J. Roitberg in [5].

In part I, we define the axioms which a class $\boldsymbol{C}$ must satisfy, and prove some results which are relevant to part III.

In part II, we define the category of $\boldsymbol{C}$-nilpotent spaces, where $\boldsymbol{C}$ is a generalized class of groups.

In part III, we prove the relative Hurewicz theorem module a class $\boldsymbol{C}$ of groups for $\boldsymbol{C}$-nilpotent spaces. The relative Hurewicz theorem, which we prove here, generalizes results of [1], [3], and [5]. In particular, we would like to point out Corollary 3.4 of [5].

We would like to thank the referee for many useful comments, which contributed greatly to its clarity.

I. Generalized class of groups.

Let Ab be the category of abelian groups and N the category of nilpotent groups.

DEFINITION 1.1. A family $\boldsymbol{C}$ of groups is called a class of groups if it satisfies the following property: Given $0 \to A \to B \to C \to 0$, a short exact sequence of groups, then $A, C \in \boldsymbol{C}$ if and only if $B \in \boldsymbol{C}$.

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Now let us consider the following axioms:

AXIOM I. $H_*(\pi, \mathbf{Z}) \in \boldsymbol{C}$, $* > 0$ for all $\pi \in \boldsymbol{C}$ where Z, the integers, is regarded as the trivial π-module.

AXIOM II. If $\pi, A \in \boldsymbol{C}$, where A is a π-module, then $H_*(\pi, A) \in \boldsymbol{C}$, $* > 0$.

AXIOM III. If $A \in \boldsymbol{C}$ is abelian, then $\oplus A \in \boldsymbol{C}$ where the direct sum is taken over any indexing set.

DEFINITION 1.2. A class $\boldsymbol{C}$ is called acyclic if it satisfies the Axiom I.

DEFINITION 1.3. A class $\boldsymbol{C}$ is called complete if it satisfies the Axiom III.

It is easy to see that $\boldsymbol{C} \cap \mathrm{Ab}$ is a class of abelian groups and $\boldsymbol{C} \cap \mathrm{N}$ is a Serre class of nilpotent groups. (See [9] and [5] for the definitions of class of abelian groups and Serre class of nilpotent groups, respectively.)

It is unknown whether there is a class of abelian groups which is not acyclic. (See [9].) We do not know whether there is a class which is not acyclic, and also whether there is a class which does not satisfy the Axiom II above.

PROPOSITION 1.4. *If $\boldsymbol{C}$ is complete then $\boldsymbol{C}$ satisfies the Axiom* II.

PROOF. Let us consider the bar construction $B(\pi)$ associated with the group π. (See [7], for more details about the bar construction.) Then we have for each $n \geq 0$ that $(B(\pi) \otimes_{\mathbf{Z}[\pi]} A)_n$ is a direct sum of A's indexed by the set $\pi \underbrace{\times \ldots \times}_{n} \pi$. So $(B(\pi) \otimes_{\mathbf{Z}[\pi]} A)_n \in \boldsymbol{C}$, for all n. By definition of class there follows that $H_*(\pi, A) \in \boldsymbol{C}$.

EXAMPLES OF CLASS 1). Let us take for each abelian class $\boldsymbol{C}$ of groups $S(\boldsymbol{C})$, the smallest class of groups which contains $\boldsymbol{C}$. It is easy to see that if $\boldsymbol{C}_1$ is the abelian class of all abelian groups, then $S(\boldsymbol{C}_1)$ is the class of all solvable groups. If $\boldsymbol{C}_2$ is the abelian class of all finite abelian groups, then $S(\boldsymbol{C}_2)$ is the class of all finite solvable groups. If $\boldsymbol{C}_3$ is the abelian class of all finitely generated abelian groups, then $S(\boldsymbol{C}_3)$ is not the family of all finitely generate solvable groups, because a subgroup of a finitely generated solvable group is not in general a finitely generated group. This remark was made by the referee.

2) The family of all finite groups is a class.

3) Let R be a non-empty subset of the set of all primes. We say that $n \in R$ if n is a product of primes in R. We say that a group G is a R-torsion group if for

every $x \in G$ there exists $n \in R$ such that $x^n = 1$. The family of all R-torsion groups is a class.

Now let us show that if we start with a class C of abelian groups, then $S(C)$ preserves some properties of C.

Given an abelian class C, let us call $F_1(C) = C$. Suppose we have defined F_{n-1}. Let F_n be the family of all groups which are extensions by two elements of F_{n-1}.

PROPOSITION 1.5. $\bigcup_{i=1}^{\infty} F_i = S(C)$.

PROOF. It suffices to show that $\bigcup_{i=1}^{\infty} F_i$ is a class and $\bigcup_{i=1}^{\infty} F_i \subseteqq S(C)$, because the fact that $\bigcup_{i=1}^{\infty} F_i$ is a class implies in

$$\bigcup_{i=1}^{\infty} F_i \supseteqq S(C) .$$

That $\bigcup_{i=1}^{\infty} F_i \subseteqq S(C)$ follows easily. For $F_1 \subseteqq S(C)$. Suppose by induction that $F_{n-1} \subseteqq S(C)$. By the definition of F_n, the fact that $S(C)$ is a class, and by the induction hypothesis, we have that $F_n \subseteqq S(C)$. So, $\bigcup_{i=1}^{\infty} F_i \subseteqq S(C)$.

Now let us show that $\bigcup_{i=1}^{\infty} F_i$ is a class. Let $0 \to N \to G \to Q \to 0$ be a short exact sequence. If $N, Q \in \bigcup_{i=1}^{\infty} F_i$, then there is i_0 such that $N, Q \in F_{i_0}$. So, $G \in F_{i_0+1} \subseteqq \bigcup_{i=1}^{\infty} F_i$.

Let $G \in \bigcup_{i=1}^{\infty} F_i$. So, $G \in F_{i_0}$ for some i_0. If $i_0 = 1$, then N and Q also belongs to F_1, by the hypothesis. Let us assume that the result is true for $i_0 < n$ and $G \in F_n$. By definition there exists $N_1, Q_1 \in F_{n-1}$ such that

$$0 \to N_1 \to G \to Q_1 \to 0$$

We have the following exact sequences:

$$0 \to N_1 \cap N \to N \to \frac{N}{N_1 \cap N} \to 0 \tag{1}$$

$$0 \to \frac{N_1 + N}{N_1} \to \frac{G}{N_1} \to \frac{G}{N + N_1} \to 0 \tag{2}$$

$$0 \to \frac{N_1 + N}{N} \to \frac{G}{N} \to \frac{G}{N + N_1} \to 0 \tag{3}$$

By the induction hypothesis and (2), we have $(N + N_1)/N_1 \in F_{n-1}$. Since $N_1 \cap N \subseteqq N_1$, we have $N_1 \cap N \in F_{n-1}$. Therefore by (1), $N \in F_n$. By the induction hypothesis and (2), $G/(N + N_1) \in F_{n-1}$. But $N_1 \to (N_1 + N)/N$ is onto, so $(N_1 + N)/N \in F_{n-1}$. By (3) there follows $G/N \in F_n$. So $\bigcup_{i=1}^{\infty} F_i$ is a class.

If $\pi \in S(C)$, let us call the filtration degree of π the first integer n such that $\pi \in F_n(C)$.

PROPOSITION 1.6. *Let C be an acyclic class of abelian groups which satisfies the Axiom II. Then $S(C)$ is also an acyclic class which satisfies the Axiom II.*

PROOF. By hypothesis $H_*(\pi, \mathbf{Z}) \in C$, $*>0$ and $H_*(\pi, A) \in C$ for $\pi, A \in C$. Let us assume that $H_*(\pi, \mathbf{Z}) \in C$, $*>0$ for $\pi \in F_n(C)$ and $H_*(\pi, A) \in C$ for $\pi \in F_n(C)$ and $A \in C$. If $G \in F_{n+1}(C)$, we have a short exact sequence

$$0 \to N \to G \to Q \to 0, \quad \text{where } N, Q \in F_n .$$

By the Lyndon–Hochschild–Serre spectral sequence, we have

$$H_p(Q, H_q(N, \mathbf{Z})) \Rightarrow H_*(G, \mathbf{Z})$$

and

$$H_p(Q, H_q(N, A) \Rightarrow H_*(G, A) .$$

By the induction hypothesis we have

$$H_p(Q, H_q(N, \mathbf{Z})) \in C; \quad p+q \neq 0 \text{ and}$$
$$H_p(Q, H_q(N, A)) \in C .$$

So, $H_*(G, \mathbf{Z}) \in C$, $*>0$ and $H_*(G, A) \in C$. Since $\bigcup_{i=1}^{\infty} F_i = S(C)$ by Proposition 1.5, we have $S(C)$ acyclic and satisfying the Axiom II.

PROPOSITION 1.7. *If π is a finitely generated abelian group and $A \in C$ is abelian, then $H_*(\pi, A) \in C$.*

PROOF. Let us prove by induction on the number n of generators of π. If $n=1$, then $\pi = \mathbf{Z}$ or $\pi = \mathbf{Z}_m$, where $\mathbf{Z}_m$ is the cyclic group on m elements. For $\pi = \mathbf{Z}$, we have the well known resolution of $\mathbf{Z}$ as $\mathbf{Z}[\pi]$-module

$$0 \to \mathbf{Z}[\pi] \to \mathbf{Z}[\pi] \to \mathbf{Z} \to 0 .$$

So $H_*(\pi, A)$ is the homology of the complex $0 \to A \xrightarrow{d} A \to 0$, where d depends on the action of π on A. But certainly $H_i(\pi, A) \in C$ for all $i \geqq 0$. If $\pi = \mathbf{Z}_m$, we have the following known resolution of $\mathbf{Z}$ as $\mathbf{Z}[\pi]$-module:

$$\cdots \to \mathbf{Z}[\pi] \xrightarrow{1+t+\cdots+t^{m-1}} \mathbf{Z}[\pi] \xrightarrow{1-t} \mathbf{Z}[\pi] \to \mathbf{Z} \to 0$$

(for more details about these resolutions see [7].)

Suppose the result is true for all groups π with less than n generators. Let π have n-generators and a_1 be one of the generators. We have the following short exact sequence

$$0 \to \langle a_1 \rangle \to \pi \to \frac{\pi}{\langle a_1 \rangle} \to 0$$

where $\langle a_1 \rangle$ denotes the subgroup generated by the element a_1. The quotient $\pi/\langle a_1 \rangle$ has less than n-generators. Call $N = \langle a_1 \rangle$ and $\pi/\langle a_1 \rangle = Q$. If we use the Lyndon–Hochschild–Serre spectral sequence in homology with respect to the above exact sequence, we have:

$$E^2_{p,q} \cong H_p(Q, H_q(N, A)) \Rightarrow H_*(\pi, A).$$

But $H_q(N, A) \in C$ because N is cyclic. By the induction hypothesis, $H_p(Q, H_q(N, A)) \in C$.

By a routine argument, $E^\infty_{p,q} \in C$. But $E^\infty_{p,q}$ forms the associated graded of certain filtration of $H_n(\pi, A)$. So, $H_n(\pi, A) \in C$.

Given a class C we denote by $\bar{C}_{\mathrm{Ab}}$ the smallest abelian class which contains the abelian class $C \cap \mathrm{Ab}$ and the abelian class F' of all finitely generated abelian groups. Let us call $F'_1 = (C \cap \mathrm{Ab}) \cup F'$ and F'_n all the abelian groups which are extensions of two groups of F'_{n-1}.

PROPOSITION 1.8. $\bigcup_{i=1}^{\infty} F'_i = \bar{C}_{\mathrm{Ab}}$.

PROOF. Similar to that of Proposition 1.5.

Suppose that C is acyclic and satisfies the Axiom II.

PROPOSITION 1.9. *If $\pi \in \bar{C}_{\mathrm{Ab}}$ and $A \in C \cap \mathrm{Ab}$, then $H_*(\pi, A) \in C$. If π is a finitely generated abelian group and $A \in \bar{C}_{\mathrm{Ab}}$, then $H_*(\pi, A) \in \bar{C}_{\mathrm{Ab}}$. If the action of π on A is trivial, $A \in \bar{C}_{\mathrm{Ab}}$ and $\pi \in C$, then $H_*(\pi, A) \in C$, $* > 0$.*

PROOF. The second part is basically the Proposition 1.7. For the first part, let $\pi \in \bar{C}_{\mathrm{Ab}}$. By proposition 1.8 it suffices to show that the result is true for each $\pi \in F'_n$. If $\pi \in F'_1$ then the result is true because either C satisfies the Axiom II or by Proposition 1.7. By a routine induction argument the result follows for $\pi \in F'_n$, for each n. The third part is similar to the first one.

Let π, G be groups. Suppose we have an action of π on G, that is, a map $\theta: \pi \to \mathrm{Aut}(G)$ which is a homomorphism. Recall from [6, p. 67] the definition of $\Gamma^n_\pi(G)$ and $\Gamma^n(\pi)$.

In chapter III we will define when a homomorphism $\varphi: H \to G$ is a C-isomorphism. When H and G are abelian groups, this definition will coincide with the classical one from [9]. So, we feel free to use the notion of C-isomorphism as long the groups involved are abelian.

DEFINITION 1.10. We say that π is C-nilpotent if $\Gamma^n(\pi) \in C$, for some integer n. The C-nilpotency degree is r if $\Gamma^r(\pi) \notin C$ but $\Gamma^{r+1}(\pi) \in C$, which we denote by nil π.

DEFINITION 1.11. We say that $\theta\colon \pi \to \mathrm{Aut}\,(G)$ is C-nilpotent if $\Gamma_\pi^n(G) \in C$, for some integer n. The C-nilpotent degree is r if $\Gamma_\pi^r(G) \notin C$ but $\Gamma_\pi^{r+1}(G) \in C$, which we denote by $\mathrm{nil}_\pi G$.

PROPOSITION 1.12. *Let C be an acyclic class which satisfies the Axiom* II, *$\pi \in C$ and $\theta\colon \pi \to$ Aut (A) a C-nilpotent action, with A abelian. If C is complete or $A/(\Gamma_\pi^2(A)) \in \bar{C}_{\mathrm{Ab}}$, then $H_*(\pi, A) \in C$, $* > 0$* and *$A \to H_0(\pi, A)$* is an *C-*isomorphism.

PROOF. This is an evident generalization of Corollaries 1.2. and 1.4. of [3], as it was remarked by the referee.

PROPOSITION 1.13. *Let C be an acyclic class which satisfies the Axiom II, π a C-nilpotent group, $\Gamma^i(\pi)/\Gamma^{i+1}(\pi)$ finitely generated for $i \leqq \mathrm{nil}\,(\pi)$ and $\theta\colon \pi \to$* Aut (A) *a C-nilpotent action. If C is complete, then $H_i(\pi, A) \in \bar{C}_{\mathrm{Ab}}$, $i > 0$. If $A \in \bar{C}_{\mathrm{Ab}}$, then $H_i(\pi, A) \in \bar{C}_{\mathrm{Ab}}$, $i \geqq 0$.*

PROOF. Let us prove that if the result is true for $H_i(\Gamma^n\pi, A)$, than it is true for $H_i(\pi, A)$. The case where C is complete is left to the reader.

Let us consider the short exact sequence:

$$0 \to \Gamma^2\pi \to \pi \to \frac{\pi}{\Gamma^2\pi} \to 0\,.$$

We have

$$H_p(\pi/\Gamma^2\pi,\ H_q(\Gamma^2\pi, A)) \ \Rightarrow\ H_i(\pi, A)\,.$$

If $H_q(\Gamma^2\pi, A) \in \bar{C}_{\mathrm{Ab}}$, by Proposition 1.9, we have that

$$H_p(\pi/\Gamma^2\pi,\ H_q(\Gamma^2\pi, A)) \in \bar{C}_{\mathrm{Ab}}\,.$$

So, it is true for $n=2$.

By a routine induction argument, we can prove for all n.

Since $\Gamma^n(\pi) \in C$ for some n, by Proposition 1.12, $H_p(\Gamma^n\pi, A) \in \bar{C}_{\mathrm{Ab}}$. So the result follows.

REMARK. If $A \in C$ in the proposition above we can in fact conclude that $H_i(\pi, A) \in C$, under assumption that $\Gamma^i\pi/\Gamma^{i+1}\pi \in \bar{C}_{\mathrm{Ab}}$.

Let $\varphi\colon G \to K$ be a group homomorphism and suppose that

$H_*(G, \mathbf{Z}) \to H_*(K, \mathbf{Z})$ is a $\boldsymbol{C}$-isomorphism with respect to some class $\boldsymbol{C}$ where homology is taken with trivial local coefficient $\mathbf{Z}$.

PROPOSITION 1.14. *Let $\boldsymbol{C}$ be an acyclic class which satisfies the Axiom* II, *K and G $\boldsymbol{C}$-nilpotent groups, θ: $K \to$ Aut (A) a $\boldsymbol{C}$-nilpotent action and $\Gamma^i K/\Gamma^{i+1}K$, $\Gamma^i G/\Gamma^{i+1}G$ belong to $\bar{\boldsymbol{C}}_{\mathrm{Ab}}$. If $\boldsymbol{C}$ is complete or*

$$\frac{\Gamma_K^i A}{\Gamma_K^{i+1} A} \in \bar{\boldsymbol{C}}_{\mathrm{Ab}} \quad \textit{for } i \leqq \operatorname{nil}_K A \,,$$

then φ_*: $H_*(G, A) \to H_*(K, A)$ is a $\boldsymbol{C}$-isomorphism.

PROOF. The proof is by induction on $\operatorname{nil}_K A$. If $\operatorname{nil}_K A = 0$, then $A \in \boldsymbol{C}$. So by the remark after the Proposition 1.13, we have that $H_*(G, A) \in \boldsymbol{C}$ and $H_*(K, A) \in \boldsymbol{C}$, thus proving the proposition. Suppose the result is true for all groups where $\operatorname{nil}_K A < n$. Let θ: $K \to$ Aut (A), where $\operatorname{nil}_K A = n$.

From the sequence:

$$0 \to \Gamma_K^2(A) \to A \to \frac{A}{\Gamma_K^2(A)} \to 0 \,,$$

we get

$$\begin{array}{ccccccccc}
H_{m+1}\left(G, \dfrac{A}{\Gamma_K^2 A}\right) & \to & H_m(G, \Gamma_K^2 A) & \to & H_m(G, A) & \to & H_m\left(G, \dfrac{A}{\Gamma_K^2 A}\right) & \to & H_{m-1}(G, \Gamma_K^2 A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{m+1}\left(K, \dfrac{A}{\Gamma_K^2 A}\right) & \to & H_m(K, \Gamma_K^2 A) & \to & H_m(K, A) & \to & H_m\left(K, \dfrac{A}{\Gamma_K^2 A}\right) & \to & H_{m-1}(K, \Gamma_K^2 A) \,.
\end{array}$$

By the induction hypothesis, the second and the fifth vertical arrows are $\boldsymbol{C}$-isomorphisms. Since the action of K on $A/\Gamma_K^2 A$ is trivial, we can use the Künneth formula. Now from the fact that $\boldsymbol{C}$ is complete or $\boldsymbol{C}$ satisfies the Axiom II and

$$\frac{\Gamma_K^i A}{\Gamma_K^{i+1} A} \in \bar{\boldsymbol{C}}_{\mathrm{Ab}} \quad \text{for } i \leqq \operatorname{nil}_K A \,,$$

the first and the fourth vertical arrows are $\boldsymbol{C}$-isomorphisms. Therefore the result follows.

II. $\boldsymbol{C}$-nilpotent spaces.

Let $\boldsymbol{C}$ be a class of groups as defined in Part I, and X a space.

DEFINITION 2.1. X is called a $\boldsymbol{C}$-nilpotent space if $\pi_1(X, x_0)$ is a $\boldsymbol{C}$-nilpotent group, and the action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ is also $\boldsymbol{C}$-nilpotent for all $x_0 \in X$ and $n \geqq 1$.

Let (Y, X) be a pair. (Y, X) is called a $\boldsymbol{C}$-nilpotent pair if $\pi_1(X, x_0)$ is $\boldsymbol{C}$-nilpotent and the action of $\pi_1(X, x_0)$ on $\pi_n(Y, X, x_0)$ is $\boldsymbol{C}$-nilpotent for all $x_0 \in X$ and $n \geqq 2$.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of π-modules, where A, B, and C are abelian groups.

PROPOSITION 2.2. *Suppose that $\boldsymbol{C}$ is complete or π is a finitely generated $\boldsymbol{C}$-nilpotent group. If the π-action on A and C are $\boldsymbol{C}$-nilpotent, then the π-action on B is also $\boldsymbol{C}$-nilpotent.*

PROOF. We certainly have that $\Gamma_\pi^n B \to \Gamma_\pi^n C$ is surjective for all n. So let us assume, without loss of generality, that $C \in \boldsymbol{C}$. If not, there is an integer n such that $\Gamma_\pi^n C \in \boldsymbol{C}$ and we have the sequence:

$$0 \to A \cap \Gamma_\pi^n B \to \Gamma_\pi^n B \to \Gamma_\pi^n C \to 0\,,$$

where $\Gamma_\pi^n C \in \boldsymbol{C}$ and the action on $A \cap \Gamma_\pi^n B$ is $\boldsymbol{C}$-nilpotent.

Now, we claim that the action of π on B has the same $\boldsymbol{C}$-nilpotency degree of the action of π on A. The proof is by induction. If the $\boldsymbol{C}$-nilpotency degree is zero, then the result follows easily. Let us assume that the result is true when the $\boldsymbol{C}$-nilpotency degree is less than n. Let us consider the following diagrams:

$$\begin{array}{ccccccccc}
 & & & & 0 & \to & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Gamma_\pi^2 B & \to & \Gamma_\pi^2 C & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H_1(\pi, C) & \to & H_0(\pi, A) & \to & H_0(\pi, B) & \to & H_0(\pi, C) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

$$0 \to \Gamma_\pi^2 A \to \Gamma_\pi^2 B \to \frac{\Gamma_\pi^2 B}{\Gamma_\pi^2 A} \to 0$$

$$H_1(\pi, C) \to \frac{\Gamma_\pi^2 B}{\Gamma_\pi^2 A} \to \Gamma_\pi^2 C \to 0\,,$$

where the map $H_1(\pi, C) \to \Gamma_\pi^2 B/\Gamma_\pi^2 A$ is given by the diagram above.

By the remark after the Proposition 1.13, $H_1(\pi, C) \in \boldsymbol{C}$. So, $\Gamma_\pi^2 B/\Gamma_\pi^2 A \in \boldsymbol{C}$. By the induction hypothesis, $\Gamma_\pi^2 B$ has $\boldsymbol{C}$-nilpotency degree $n-1$. So, B has $\boldsymbol{C}$-nilpotency degree n and the result follows.

Now we will extend this result. Suppose we have a central extension

$$0 \to A \xrightarrow{i} G \xrightarrow{p} H \to 0 .$$

Suppose we have actions, w_1, w_2, and w_3 of π on A, G, and H, respectively, such that i and p are compatible with the π actions.

If $\theta: \pi \to \mathrm{Aut}(G)$ is an action, we will assume that we have an homomorphism φ: $G \to \pi$ such that $g, g_1 g^{-1} = \theta(\varphi(g))g_1$, for all $g, g_1 \in G$. If G is abelian then φ is the trivial homomorphism. The reason for this hypothesis comes from [10, Theorem 12, 385].

From now let $\boldsymbol{C}$ be an acyclic class and π a $\boldsymbol{C}$-nilpotent group. Let us also assume that either $\boldsymbol{C}$ is complete or $\boldsymbol{C}$ satisfies the Axiom II and $\Gamma^i\pi/\Gamma^{i+1}\pi$ is a finitely generated group for $i \leqq \mathrm{nil}\,\pi$.

PROPOSITION 2.3. *If w_1 and w_3 are $\boldsymbol{C}$-nilpotent, then w_2 is also $\boldsymbol{C}$-nilpotent.*

PROOF. Following the same steps as in Proposition 2.2, we will assume without loss of generality, that $H \in \boldsymbol{C}$.

We have the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{p} & H \\ {\scriptstyle\varphi_1}\downarrow & & \downarrow{\scriptstyle\varphi_2} \\ \pi & = & \pi \end{array}$$

where $gg_1g^{-1} = w_2(\varphi_1(g))g_1$, $hh_1h^{-1} = w_3(\varphi_2(h))h_1$, by hypothesis. So $[G, G] \subseteqq \Gamma_\pi^2$ and we can see that

$$\frac{G}{\Gamma_\pi^2 G} \cong \frac{G_{\mathrm{ab}}}{\Gamma_\pi^2 G_{\mathrm{ab}}},$$

where G_{ab} denotes the abelianized group of G.

Now, we have the following commutative diagrams where every row and every column are exact:

(1) $$H_2(H, \mathbf{Z}) \xrightarrow{\theta_1} A \xrightarrow{\theta_2} G_{\mathrm{ab}} \to H_{\mathrm{ab}} \to 0$$

(2) $$0 \to \mathrm{im}\,\theta_1 \to A \to \mathrm{im}\,\theta_2 \to 0$$

$$0 \to \operatorname{im} \theta_2 \to G_{ab} \to H_{ab} \to 0 \tag{3}$$

$$\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & & \Gamma_\pi^2(G) \arrow[r] \arrow[d] & \Gamma_\pi^2(H) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\
 & H_0(\pi, A) \arrow[r] \arrow[d] & H_0(\pi, G_{ab}) \arrow[r] \arrow[d] & H_0(\pi, H_{ab}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{4}$$

The sequence (1) is the low-dimensional homology exact sequence. See [11]. The sequences (2) and (3), of course, are obtained from (1).

Now call N the kernel of the map

$$H_0(\pi, A) \to H_0(\pi, G_{ab}) .$$

We will show that $N \in \boldsymbol{C}$. We have:

$$H_0(\pi, \operatorname{im} \theta_1) \to H_0(\pi, A) \to H_0(\pi, \operatorname{im} \theta_2)$$

$$H_1(\pi, H_{ab}) \to H_0(\pi, \operatorname{im} \theta_2) \to H_0(\pi, G_{ab})$$

from the sequences (2) and (3), respectively.

The map $H_0(\pi, A) \to H_0(\pi, G_{ab})$ is the composite of the following two maps:

$$H_0(\pi, A) \to H_0(\pi, \operatorname{im} \theta_2) \to H_0(\pi, G_{ab}) .$$

So we have:

$$H_0(\pi, \operatorname{im} \theta_1) \to N \to \operatorname{im} (H_1(\pi, H_{ab}) \to H_0(\pi, \operatorname{im} \theta_2)) ,$$

But $H_2(H, \mathbf{Z}) \in \boldsymbol{C}$ implies $\operatorname{im} \theta_2 \in \boldsymbol{C}$. Furthermore, $H_1(\pi, H_{ab}) \in \boldsymbol{C}$, by Proposition 1.13. So, $N \in \boldsymbol{C}$.

Now by a routine argument we have:

$$0 \to \Gamma_\pi^2 A \to \Gamma_\pi^2 G \to \frac{\Gamma_\pi^2 G}{\Gamma_\pi^2 A} \to 0$$

$$N \to \frac{\Gamma_\pi^2 G}{\Gamma_\pi^2 A} \to \Gamma_\pi^2 H \to 0 .$$

Therefore, $\Gamma_\pi^2 G / \Gamma_\pi^2 A \in \boldsymbol{C}$. Call $\varphi_1'\colon \Gamma_\pi^2 G \to \pi$ the restriction of the map φ_1 which factors through $\Gamma_\pi^2 G / \Gamma_\pi^2 A$. So by the induction hypothesis $\Gamma_\pi^2 G$ also has $\boldsymbol{C}$-nilpotency degree $n-1$ and the result follows.

COROLLARY 2.4. *If $X \subseteqq Y$ are both **C**-nilpotent, then (Y, X) is a **C**-nilpotent pair.*

PROOF. $\pi_1(X)$ is **C**-nilpotent by hypothesis. Now let us consider the long exact sequence in homotopy of the pair (Y, X):

$$\pi_n(X) \to \pi_n(Y) \to \pi_n(Y, X) \to \pi_{n-1}(X) \to \pi_{n-1}(Y).$$

It is well known that this sequence is a sequence of π_1-modules. So, let us consider the short exact sequence of $\pi_1(X)$-modules

$$0 \to \operatorname{im}(\pi_n(X) \to \pi_n(Y)) \to \pi_n(Y, X) \to \operatorname{Ker}(\pi_{n-1}(X) \to \pi_{n-1}(Y)) \to 0.$$

In order to show that $\pi_1(X)$ acts **C**-nilpotently in $\pi_n(Y, X)$, we use the above sequence and Proposition 2.2 for the case $n > 2$ and Proposition 2.3 for $n = 2$.

PROPOSITION 2.5. *Let $A \in \mathbf{C}$ be abelian. Then $H_*(K(A, m)) \in \mathbf{C}$, $* > 0$ and $m \geqq 1$.*

PROOF. See Lemma 2.17 of [6, p. 69].

PROPOSITION 2.6. *If π acts **C**-nilpotently on A and B, then π acts **C**-nilpotently on $A \otimes B$ and on* Tor (A, B), *where A and B are abelian groups.*

PROOF. It is a routine argument using induction on the **C**-nilpotency degree and Proposition 2.2.

PROPOSITION 2.7. *If $\theta \colon \pi \to$* Aut (A) *is **C**-nilpotent then π acts **C**-nilpotently on $H_*(K(A, m))$ for all $m \geqq 1$, where A is an abelian group.*

PROOF. If the **C**-nilpotency degree is zero, then the result is true by Proposition 2.5. Let us assume that the result is true when the **C**-nilpotency degree is less than n. We have the fibration

$$\begin{array}{ccc} K(\Gamma_\pi^2 A, m) \to & K(A, m) \\ & \downarrow \\ & K(A/\Gamma_\pi^2 A, m). \end{array}$$

By the Serre's spectral sequence, we have:

$$H_p(K(A/\Gamma_\pi^2 A, m), H_q(K(\Gamma_\pi^2 A, m))) \Rightarrow H_{p+q}(K(A, m)).$$

Since A is abelian we have homology with trivial local coefficients. By the Künneth formula, the induction hypothesis and Proposition 2.6., the result follows.

REMARK. If $m=1$ and A is not abelian the situation is more complicated.
Now we will describe a C-nilpotent space in a different way.
Let $\tilde{X}$ be the univeral cover of X.

PROPOSITION 2.8. *X is C-nilpotent if and only if $\pi_1(X)$ is C-nilpotent and the action of $\pi_1(X)$ on $H_*(\tilde{X})$ is C-nilpotent.*

PROOF. We follow the same steps as Lemma 2.18 of [6, p. 70]. Suppose X is C-nilpotent. So, $\pi_1(X)$ is certainly C-nilpotent. To show that $\pi_1(X)$ acts C-nilpotently on $H_*(\tilde{X})$, it suffices to show that $\pi_1(X)$ acts C-nilpotently on $H_*(\tilde{X}_k)$, where $\tilde{X}_k'$ is the kth stage of the postnikov system of $\tilde{X}$. The result is certainly true for $k=1$. Suppose the result is true for $k=n$. Then we have

$$\begin{array}{ccc} K(\pi_{n+1}(X), n+1) & \to & \tilde{X}_{n+1} \\ & & \downarrow \\ & & \tilde{X}_n \end{array}$$

The E^2-term of the Serre's spectral sequence is $H_p(\tilde{X}_n, H_q(K(\pi_{n+1}(X), n+1)))$. Since we have trivial local coefficients, by the induction hypothesis, Propositions 2.6 and 2.7 we have that $\pi_1(X)$ acts C-nilpotently on $H_*(\tilde{X}_{n+1})$. So, the result follows.

Now suppose $\pi_1(X)$ is C-nilpotent and the action of $\pi_1(X)$ on $H_*(\tilde{X})$ is C-nilpotent. Since $H_2(\tilde{X}) \cong \pi_2(\tilde{X}) \cong \pi_2(X)$, we have that the action of $\pi_1(X)$ on $\pi_2(X)$ is also C-nilpotent. Let us suppose by induction that the action of $\pi_1(X)$ is C-nilpotent on $H_*(X^k)$, where X^k is the k-connective cover of X. So, $\pi_1(X)$ acts C-nilpotently on $\pi_k(X)$. Let us consider the fibration.

$$\begin{array}{ccc} K(\pi_k X, k-1) & \to & X^{k+1} \\ & & \downarrow \\ & & X^k \end{array}$$

By using the Serre's spectral sequence we find that $\pi_1(X)$ acts C-nilpotently on X^{k+1}. So, the result follows.

Now let us consider $\bar{X}$ a regular cover X, where $\pi_1(\bar{X}) = \pi_0$, $\pi_1(X) = \pi$.

PROPOSITION 2.9. *π acts C-nilpotently on $H_*(\pi_0)$.*

PROOF. Let r be the C-nilpotency degree of π. Then it follows that $\Gamma_\pi^{r+1}(\pi_0) \in C$. Let us consider the Lyndon–Hochschild–Serre spectral sequence associated with the short exact sequence:

$$0 \to \Gamma_\pi^{r+1}(\pi_0) \to \pi_0 \to \frac{\pi_0}{\Gamma_\pi^{r+1}(\pi_0)} \to 0$$

$$E_{p,q}^2 \cong H_p(\pi_0/\Gamma_\pi^{r+1}(\pi_0), H_q(\Gamma_\pi^{r+1}(\pi_0))) .$$

Since $H_q(\Gamma_\pi^{r+1}(\pi_0)) \in C$ for $q \geqq 0$, we have $E_{p,q}^2 \in C$ for $q > 0$ by Proposition 1.9. But π acts nilpotently on $\pi_0/\Gamma_\pi^{r+1}(\pi_0)$, so it acts nilpotently on

$$H_p(\pi_0/\Gamma_\pi^{r+1}(\pi_0)) \cong E_{p,0}^2 .$$

So the result follows.

PROPOSITION 2.10. *If X is C-nilpotent then π_0 is C-nilpotent and the action of π/π_0 on $H_*(\tilde{X})$ is also C-nilpotent.*

PROOF. It is clear that π_0 is C-nilpotent because π is C-nilpotent.

From the hypothesis that $\Gamma^i\pi/\Gamma^{i+1}\pi$ is finitely generated, it is not hard to see that $\Gamma^i\pi_0/\Gamma^{i+1}\pi_0 \in \bar{C}_{\mathrm{Ab}}$. Now let us consider the Serre's spectral sequence:

$$E_{p,q}^2 \cong H_p(\pi_0, H_q(\tilde{X})) \Rightarrow H_*(\tilde{X}) .$$

To show the result it suffices to prove that π/π_0 acts C-nilpotent on $H_p(\pi_0, H_q(\tilde{X}))$. The proof is by induction on the C-nilpotent degree of π_0 on $H_q(\tilde{X})$.

Suppose it is zero. Then the result is trivial because, by Proposition 1.9, $H_p(\pi_0, H_q(\tilde{X})) \in C$.

Now suppose the result is true if the C-nilpotency degree is less than n. Let the C-nilpotency degree of π_0 on $H_q(\tilde{X})$ be n. We have:

$$0 \to \Gamma_{\pi_0}^2(H_q(\tilde{X})) \to H_q(\tilde{X}) \to \frac{H_q(\tilde{X})}{\Gamma_{\pi_0}^2(H_q(\tilde{X}))} \to 0$$

a sequence of π_0-modules. Then we have

$$H_n(\pi_0, \Gamma_{\pi_0}^2 H_q(\tilde{X})) \to H_n(\pi_0, H_q(\tilde{X})) \to H_n(\pi_0, H_q(\tilde{X})/\Gamma_{\pi_0}^2 H_q(\tilde{X})) .$$

The action of π/π_0 on $H_n(\pi_0, \Gamma_{\pi_0}^2 H_q(\tilde{X}))$ is C-nilpotent by induction hypothesis. Since π_0 acts trivially on $H_q(\tilde{X})/\Gamma_{\pi_0}^2(H_q(\tilde{X}))$, by Künneth formula and Proposition 2.6, it suffices to show that π acts C-nilpotently on $H_*(\pi_0)$. But this is true by Proposition 2.9. Therefore the result follows.

III. The relative Hurewicz theorem.

Now we will prove the relative Hurewicz theorem.

Let C be a class as defined in part I.

DEFINITION 3.1. Let $\varphi: K \to G$ be a homomorphism of groups. We say that φ is C-surjective if there is a normal subgroup G_1 of G where $G_1 \in C$ such that the image of the composite map

$$K \xrightarrow{\varphi} G \xrightarrow{p} G/G_1$$

has the following property: there exists a series of normal subgroups

$$\operatorname{im}(p \circ \varphi) = \bar{K}_0 \subseteqq \bar{K}_1 \subseteqq \dots \subseteqq \bar{K}_m = G/G_1$$

and

$$\frac{K_{i+1}}{K_i} \in C, \quad i=0,1,\dots,m-1 \text{ where } K_i = p^{-1}(\bar{K}_i)\,.$$

We say that φ is C-injective if $\operatorname{Ker}(\varphi) \in C$.

We say that φ is C-isomorphism if φ is C-injective and C-surjective.

REMARK. If G is nilpotent and K is a subgroup of G, a series of normal subgroups always exists, see [8] for more details.

PROPOSITION 3.2. *Let $\varphi: K \to G$ be a homomorphism of C-nilpotent groups. If $H_1(K) \to H_1(G)$ is C-surjective, then φ is also C-surjective.*

PROOF. Since K and G are C-nilpotent, there exists an integer n such that $\Gamma^n(K)$, $\Gamma^n(G) \in C$. So, let us consider the groups $K/\Gamma^n(K)$, $G/\Gamma^n(G)$ and

$$\bar{\varphi} : \frac{K}{\Gamma^n(K)} \to \frac{G}{\Gamma^n(G)}\,.$$

We have that $K/\Gamma^n(K)$, $G/\Gamma^n(G)$ are nilpotent groups.

Now let us consider the following commutative diagram:

$$\begin{array}{ccc} H_1(K) & \xrightarrow{P_{1*}} & H_1(K/\Gamma^n(K)) \\ {\scriptstyle\varphi_4}\downarrow & & \downarrow{\scriptstyle\varphi_5} \\ H_1(G) & \xrightarrow{P_{2*}} & H_1(G/\Gamma^n(G)) \end{array}$$

Since P_{2*} is an epimorphism and φ_4 is C-surjective, then φ_5 is also C-surjective. Now by Theorem 3.1. of [4] the map

$$\bar{\varphi}: \frac{K}{\Gamma^n(K)} \to \frac{G}{\Gamma^n(G)}$$

is C-surjective in the sense of nilpotent classes. But $\Gamma^n(G) \in C$, so the result follows by definition of a C-surjective map.

Let us consider the following situation. Let $\theta: \pi \to \mathrm{Aut}(G)$ an action, where π and G are groups, not necessarily abelian, and $\varphi: G \to \pi$ a homomorphism which satisfies

$$gg_1g^{-1} = \theta(\varphi(g))g_1, \quad \text{for all } g, g_1 \in G \text{ as in part II}.$$

Now let $\boldsymbol{C}$ be a class which is complete or π is finitely generated.

PROPOSITION 3.3. *Suppose that π and the action $\theta: \pi \to \mathrm{Aut}(G)$ are $\boldsymbol{C}$-nilpotent. If $G/\Gamma_\pi^2(G) \in \boldsymbol{C}$ then $G \in \boldsymbol{C}$.*

PROOF. Let us assume for the moment that the result is true when G is abelian.

From the short exact sequence

$$0 \to \Gamma_\pi^2(G) \to G \to \frac{G}{\Gamma_\pi^2(G)} \to 0$$

we get

$$(1) \qquad H_2\left(\frac{G}{\Gamma_\pi^2(G)}\right) \to \frac{\Gamma_\pi^2(G)}{[G, \Gamma_\pi^2(G)]} \to H_1(G) \to H_1\left(\frac{G}{\Gamma^{2\pi}(G)}\right) \to 0$$

which is the low dimensional homology sequence. See [11].

By the equation $gg_1g^{-1} = \theta(\varphi(g))g_1$, it follows $\Gamma_\pi^2(G) \supseteq [G, G]$. So from the sequence (1) above we obtain

$$H_2\left(\frac{G}{\Gamma_\pi^2(G)}\right) \to \frac{\Gamma_\pi^2(G)}{[G, \Gamma_\pi^2(G)]} \to \frac{\Gamma_\pi^2(G)}{[G, G]}.$$

But G_{ab} is abelian and satisfies the hypothesis of the propositon. So $G_{ab} \in \boldsymbol{C}$. Since

$$H_2(G/\Gamma_\pi^2(G)) \in \boldsymbol{C},$$

it follows that

$$\frac{\Gamma_\pi^2(G)}{[G, \Gamma_\pi^2(G)]} \in \boldsymbol{C}.$$

From the fact that $\Gamma_\pi^3(G) \supseteq [G, \Gamma_\pi^2(G)]$ it follows that

$$\frac{\Gamma_\pi^2(G)}{\Gamma_\pi^3(G)} \in \boldsymbol{C}.$$

Since $\Gamma_\pi^2(G)$ has $\boldsymbol{C}$-nilpotency degree one less than the $\boldsymbol{C}$-nilpotency of G, by a routine induction argument, we conclude that $\Gamma_\pi^2(G) \in \boldsymbol{C}$ and therefore $G \in \boldsymbol{C}$.

So it remains to show the result for G abelian. The proof of this case is easier and we omit it.

PROPOSITION 3.4. *Let $\varphi: K \to G$ be C-surjective. Then $H_1(K) \to H_1(G)$ is also C-surjective.*

PROOF. Let $G_1 \in C$ be the normal subgroup of G given by the definition of a C-surjective map. Then we have the short exact sequence

$$0 \to G_1 \to G \to \frac{G}{G_1} \to 0$$

and from it we obtain

$$\frac{G_1}{[G_1, G]} \to H_1(G) \to H_1\left(\frac{G}{G_1}\right) \to 0 .$$

Since $G_1 \in C$ we have $G_1/[G_1, G] \in C$. It follows that $H_1(G) \to H_1(G/G_1)$ is a C-isomorphism.

Now we claim that $H_1(K_i) \to H_1(G/G_1)$ is C-surjective, where the K_i's, $i=0,\ldots,n$ are given by the definition of a C-surjective map.

If $i=n$ the result is certainly true. So, let us assume that it is true for $i>j$. We have

$$0 \to K_j \to K_{j+1} \to \frac{K_{j+1}}{K_j} \to 0 .$$

But $H_1(K_{j+1}/K_j) \in C$. So $H_1(K_j) \to H_1(K_{j+1})$ is a C-surjective map. By induction, $H_1(K_{j+1}) \to H_1(G/G_1)$ is also a C-surjective map. So, $H_1(K_0) \to H_1(G/G_1)$ is a C-surjective map. Now we have the diagram:

$$\begin{array}{ccc} H_1(K) & \xrightarrow{\varphi_3} & H_1(G) \\ \varphi_1\downarrow & & \downarrow\varphi_4 \\ H_1(K_0) & \xrightarrow{\varphi_2} & H_1(G/G_1) \end{array}$$

We have that φ_1 is an epimorphism and φ_2 is a C-surjective map. So, $\varphi_4 \circ \varphi_3$ is also a C-surjective map. But φ_4 is a C-isomorphism. So, φ_3 is a C-surjective map.

Remember that C is a class of groups which satisfies the Axiom II. Furthermore we will consider that if C is not complete, then X and Y are of finite type.

Let $\bar{\pi}_n(Y, X)$ be the quotient of $\pi_n(Y, X)$ by the action of $\pi_1(X)$, h_n the Hurewicz map $\pi_n(Y, X) \to H_n(Y, X)$ and $\bar{h}_n$ the factorization of h_n through $\bar{\pi}_n(Y, X)$.

PROPOSITION 3.5. *Let Y and X be $\boldsymbol{C}$-nilpotent spaces. If $\pi_i(Y, X) \in \boldsymbol{C}$, $1 < i < n$, $n \geqq 2$, and $\pi_1(X) \to \pi_1(Y)$ is a $\boldsymbol{C}$-surjective map, then*

$$\bar{h}_n : \bar{\pi}_n(Y, X) \to H_n(Y, X)$$

is a $\boldsymbol{C}$-isomorphism.

Before proving this proposition let us state the main theorem and prove it from Proposition 3.5.

THE RELATIVE HUREWICZ THEOREM. *Let $\boldsymbol{C}$ be an acyclic class, (Y, X) a pair where X and Y are $\boldsymbol{C}$-nilpotent spaces. If* a) *X, Y are of finite type and $\boldsymbol{C}$ satisfies the Axiom* II *or* b) *$\boldsymbol{C}$ satisfies the Axiom* III, *then the three conditions below are equivalent for $n > 2$ and the first two are equivalent for $n > 1$.*

a$_n$) *$\pi_1(X) \to \pi_1(Y)$ is a $\boldsymbol{C}$-surjective map and $\pi_i(Y, X) \in \boldsymbol{C}$ for $1 < i < n$,*

b$_n$) *$H_i(Y, X) \in \boldsymbol{C}$, $1 \leqq i < n$,*

c$_n$) *$\pi_1(X) \to \pi_1(Y)$ is a $\boldsymbol{C}$-isomorphism and $H_i(\tilde{Y}, \tilde{X}) \in \boldsymbol{C}$, $i < n$, where $\tilde{Y}$ and $\tilde{X}$ are the universal cover of Y and X, respectively.*

If one of the three conditions above holds, then $\bar{h}_n$: $\bar{\pi}_n(Y, X) \to H_n(Y, X)$ is a C-isomorphism.

PROOF. Let us prove that a$_n$ is equivalent to b$_n$ by induction. If $n=2$, a_2 is equivalent to b_2 by Propositions 3.2 and 3.4. Suppose a$_{n-1}$ is equivalent to b$_{n-1}$. Now, let us prove that a$_n$ and b$_n$ are equivalent. Suppose that a$_n$ holds. By Proposition 3.5, we have that $\bar{\pi}_n(Y, X) \to H_n(Y, X)$ is a $\boldsymbol{C}$-isomorphism. The fact that $\pi_n(Y, X) \in \boldsymbol{C}$ implies that $\bar{\pi}_n(Y, X) \in \boldsymbol{C}$. So, $H_n(Y, X) \in \boldsymbol{C}$ and b$_n$ holds. Conversely, let us assume that b$_n$ holds. By Proposition 3.3 we have $\pi_n(Y, X) \in \boldsymbol{C}$, and thus a$_n$ holds.

The proof that a$_n$ and c$_n$ are equivalent is classical. There remains to be proved that a$_n$ implies $\bar{h}_n$ is a $\boldsymbol{C}$*-isomorphism. But this is exactly the Proposition* 3.5.

In order to prove the Proposition 3.5, we will show:

PROPOSITION 3.6. *Let Y be a $\boldsymbol{C}$-nilpotent space of finite type. Then $H_n(\bar{Y}) \in \bar{\boldsymbol{C}}_{\mathrm{Ab}}$ and $\pi_m(\bar{Y}) \in \bar{\boldsymbol{C}}_{\mathrm{Ab}}$ for $n > 0$ and $m > 1$, where $\bar{Y}$ is a regular cover of Y.*

PROOF. Let us assume that $H_i(\bar{Y}) \in \bar{\boldsymbol{C}}_{\mathrm{Ab}}$ for $i < n_0$. We are going to show that $H_{n_0}(\bar{Y}) \in \bar{\boldsymbol{C}}_{\mathrm{Ab}}$. We have the spectral sequence

$$H_p(\pi/\pi_0, H_q(\bar{Y})) \Rightarrow H_{n_0}(Y), \quad p+q=n_0 ,$$

where $\pi_0 = \pi_1(\bar{Y})$.

Since $H_q(\bar{Y}) \in \bar{C}_{\mathrm{Ab}}$ for $q<n_0$, by Proposition 1.13, $E^n_{p,q} \in \bar{C}_{\mathrm{Ab}}$. So, $H_0(\pi/\pi_0, H_{n_0}(\bar{Y})) \in \bar{C}_{\mathrm{Ab}}$. Now by Propositions 2.10 and 3.3 it follows that $H_{n_0}(\bar{Y}) \in \bar{C}_{\mathrm{Ab}}$. If $n_0=0$ the result is certainly true.

The fact that $\pi_m(\bar{Y}) \in \bar{C}_{\mathrm{Ab}}$ for $m>1$ it is classical.

Now let us prove the Proposition 3.5. Let us assume that the hypothesis a) holds. The case where b) holds is simpler and we omit it.

Since $n>1$ we have that

$$\pi_1(X) \xrightarrow{\varphi} \pi_1(Y)$$

is a C-surjective map. By the definition of a C-surjective map we have

$$\operatorname{im}\varphi + G_1 = K_0 \subseteqq K_1 \subseteqq \dots \subseteqq K_m = G ,$$

where

$$\frac{K_{i+1}}{K_i} \cong \frac{\bar{K}_{i+1}}{\bar{K}_i} \in C .$$

Now let us take the following tower of spaces

$$\begin{tikzcd}
 & Y_m \arrow[d] \\
 & \vdots \arrow[d] \\
X \arrow[ruu, "f_m"] \arrow[r] \arrow[rd, "f"'] & Y_1 \arrow[d] \\
 & Y_0 = Y
\end{tikzcd}$$

where $\pi_1(Y_i) \cong K_{m-i}$ and Y_{i+1} is the cover space of Y_i, which corresponds to the subgroup K_{m-i-1} of K_{m-i}. Let f_i: $X \to Y_i$ be a map which makes the diagram above commutative. Now we claim that $H_j(Y_{i+1}) \to H_j(Y_i)$ is a C-isomorphism for all j. Taking the spectral sequence of the regular cover we have:

$$E^2_{p,q} \cong H_p(K_{m-i}/K_{m-i-1}, H_q(Y_{i+1})) \Rightarrow H_{p+q}(Y_i) .$$

The Proposition 3.6 shows that $H_q(Y_{i+1}) \in \bar{C}_{\mathrm{Ab}}$, and consequently by Proposition 1.12 it follows that $E^2_{p,q} \in C$, $p>0$, and

$$H_q(Y_{i+1}) \to H_0(K_{m-i}/K_{m-i-1}, H_q(Y_{i+1}))$$

is a C-isomorphism. So $H_j(Y_{i+1}) \to H_j(Y_i)$ is also a C-isomorphism.

Now let us consider the diagram

$$\begin{tikzcd}
\bar{\pi}(Y_m, X) \arrow[r] \arrow[d] & H_n(Y_m, X) \arrow[d] \\
\bar{\pi}_n(Y, X) \arrow[r] & H_n(Y, X)
\end{tikzcd}$$

In order to show that $\bar{h}_n$: $\bar{\pi}(Y, X) \to H_n(Y, X)$ is a C-isomorphism, it suffices to show that $\bar{h}_n$: $\bar{\pi}_n(Y_m, X) \to H_n(Y_m, X)$ is a C-isomorphism, since the two vertical arrows are C-isomorphisms.

Now let us consider the diagram

$$\begin{tikzcd}
 & \bar{Y} \arrow[d, "p"] \\
X \arrow[ur, "\bar{f}"] \arrow[r, "f_m"'] & Y_m
\end{tikzcd}$$

where $\bar{Y}$ is the cover of Y_m which corresponds to the subgroup im φ. We claim that p_*: $H_*(\bar{Y}) \to H_*(Y_m)$ is a C-isomorphism. For let us consider the diagram

$$\begin{tikzcd}
 & \tilde{Y} \arrow[dl] \arrow[dr] & \\
\bar{Y} \arrow[rr] \arrow[d] & & Y_m \arrow[d] \\
K(\mathrm{im}\,\varphi, 1) \arrow[rr] & & K(\mathrm{im}\,\varphi + G_1, 1)
\end{tikzcd}$$

where $\tilde{Y}$ is the universal cover of both $\bar{Y}$ and Y_m. Since G_1 is normal in G, then G_1 is normal in im $\varphi + G_1$ and

$$\frac{\mathrm{im}\,\varphi + G_1}{G_1} \cong \frac{\mathrm{im}\,\varphi}{G_1 \cap \mathrm{im}\,\varphi}.$$

By the Lyndon–Hochschild–Serre spectral sequence associated with the sequence

$$0 \to G_1 \cap \mathrm{im}\,\varphi \to \mathrm{im}\,\varphi \to \frac{\mathrm{im}\,\varphi}{G_1 \cap \mathrm{im}\,\varphi} \to 0$$

we have that

$$H_*(\mathrm{im}\,\varphi) \to H_*(\mathrm{im}\,\varphi/(G_1 \cap \mathrm{im}\,\varphi))$$

is a C-isomorphism. For,

$$H_p(\operatorname{im}\varphi/(G_1 \cap \operatorname{im}\varphi), H_q(G_1 \cap \operatorname{im}\varphi)) \in C \qquad \text{for } q>0$$

by Proposition 1.9. It is also true that

$$H_*(\operatorname{im}\varphi + G_1) \to H_*((\operatorname{im}\varphi + G_1)/G_1)$$

is a C-isomorphism by the same argument as above. So, it follows that

$$H_*(K(\operatorname{im}\varphi, 1)) \to H_*(K(\operatorname{im}\varphi + G_1, 1))$$

is a C-isomorphism. Now let us consider the induced map between the two spectral sequences from the above fibrations:

$$H_p(\operatorname{im}\varphi, H_q(\tilde{Y})) \to H_p(\operatorname{im}\varphi + G_1, H_q(\tilde{Y})) .$$

This map is a C-isomorphism by Proposition 1.14. So $H_*(\bar{Y}) \to H_*(Y_m)$ is also a C-isomorphism. As before, it suffices to show that $\bar{\pi}_n(\bar{Y}, X) \to H_n(\bar{Y}, X)$ is a C-isomorphism. If $n=2$ we have $\pi_1(\bar{Y}, X)=0$. So by the classical Hurewicz theorem (see [10, p. 395]) we have

$$\bar{\pi}_2(\bar{Y}, X) \cong H_2(\bar{Y}, X)$$

and we proved it. From now on, let n be greater than 2. Let us consider the next stage of the Moore–Postnikov decomposition of the map f, i.e.,

$$\begin{tikzcd} & \hat{Y} \arrow[d, "\hat{p}"] \\ X \arrow[ur, "\hat{f}"] \arrow[r, "\bar{f}"'] & \bar{Y} \end{tikzcd}$$

such that

$$\pi_2(X) \to \pi_2(\hat{Y}) \text{ is onto },$$

$$\pi_2(\hat{Y}) \to \pi_2(\bar{Y}) \text{ is } 1-1 ,$$

$$\pi_1(X) \to \pi_1(\hat{Y}) \text{ is an isomorphism },$$

and $\pi_i(\hat{Y}) \to \pi_i(\bar{Y})$ is an isomorphism for $i>2$. Then the fibre of $\hat{p}$: $\hat{Y} \to \bar{Y}$ is just $K(\pi_2(Y, X), 1)$. Now let us consider the Serre's spectral sequence associated with the fibration

$$\begin{array}{ccc} K(\pi_2(Y,X),1) & \to & \hat{Y} \\ & & \downarrow \\ & & \bar{Y} \end{array}$$

$E^2_{p,q}=H_p(\bar{Y}, H_q(K(\pi_2(Y,X),1))) \Rightarrow H_*(\hat{Y},Z)$. If $q>0$, then $H_q(K(\pi_2(Y,X),1)) \in \boldsymbol{C}$ because the class is acyclic. Call $A=H_q(K(\pi_2(Y,X),1))$. Now we claim that $H_p(\bar{Y},A)$ also belongs to $\boldsymbol{C}$. To prove this, let $\bar{Y}_r$ be the rth stage of the Postnikov decomposition of Y. For $\bar{Y}_1 \cong K(\pi_1(\bar{Y}),1)$, we have by Proposition 1.13, that $H_*(\pi_1(\bar{Y}),A) \in \boldsymbol{C}$. Suppose, by the induction hypothesis, that we have proved that $H_*(\bar{Y}_{r-1},A) \in \boldsymbol{C}$. Let us consider the fibration

$$\begin{array}{ccc} K(\pi_r(\bar{Y}),r) & \rightarrow & \bar{Y}_r \\ & & \downarrow \\ & & \bar{Y}_{r-1} \end{array}$$

Then we have the Serre's spectral sequence

$$H_p\big(\bar{Y}_{r-1}, H_q(K(\pi_r(\bar{Y}),r),A)\big) \Rightarrow H_*(\bar{Y}_r,A)\,.$$

But $H_q(K(\pi_r(\bar{Y}),r),A)$ is the homology with trivial local coefficients. By Proposition 3.6, $\pi_r(\bar{Y}) \in \bar{\boldsymbol{C}}_{\mathrm{Ab}}$. From this fact and by Künneth formula there follows that $H_q(K(\pi_r(\bar{Y}),r),A) \in \boldsymbol{C}$. So, by a routine argument, $H_*(\bar{Y}_r,A) \in \boldsymbol{C}$.

As r goes to infinite, it follows that $H_*(\bar{Y}_r,A)$ converges to $H_*(\bar{Y},A)$, so we have proved.

We conclude that $\hat{p}$: $H_*(\hat{Y}) \rightarrow H_*(\bar{Y})$ is a $\boldsymbol{C}$-isomorphism. Now let $n=3$. We have $\pi_1(\hat{Y},X)=0$, $\pi_2(\hat{Y},X)=0$. So by the classical Hurewicz theorem,

$$\bar{\pi}_3(\hat{Y},X) \xrightarrow{\cong} H_3(\hat{Y},X)$$

is an isomorphism.

We have the diagram below

$$\begin{array}{ccccccccc} \pi_3(X) & \rightarrow & \pi_3(\hat{Y}) & \rightarrow & \pi_3(\hat{Y},X) & \rightarrow & \pi_2(X) & \rightarrow & \pi_2(\hat{Y}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_3(X) & \rightarrow & \pi_3(\bar{Y}) & \rightarrow & \pi_3(\bar{Y},X) & \rightarrow & \pi_2(X) & \rightarrow & \pi_2(\bar{Y}) \end{array}$$

where the first, second and fourth vertical maps are isomorphisms and the fifth is $1-1$. So $\pi_3(\hat{Y},X) \xrightarrow{\simeq} \pi_3(\bar{Y},X)$. Since $H_3(\hat{Y},X)$ is $\boldsymbol{C}$-isomorphic to $H_3(\bar{Y},X)$, we have the result for $n=3$.

From now on let $n>3$. Let us consider the fibration

$$(\tilde{\hat{Y}},\tilde{X}) \rightarrow (\hat{Y},X) \rightarrow K(\pi,1)$$

where, $\tilde{\hat{Y}},\tilde{X}$ are the universal cover of $\hat{Y},X$, respectively, and $\pi \cong \pi_1(X) \cong \pi_1(\hat{Y})$. We have $\pi_i(\tilde{\hat{Y}},\tilde{X})=0$, for $i=1,2$ and $\pi_j(\tilde{\hat{Y}},\tilde{X}) \cong \pi_j(\hat{Y},X)$, for $j>2$. So $\pi_j(\tilde{\hat{Y}},\tilde{X}) \in \boldsymbol{C}$, $j<n$. Let us consider the Serre's spectral sequence

$$E^2_{p,q} \cong H_p(\pi, H_q(\tilde{\hat{Y}},\tilde{X})) \Rightarrow H_*(\hat{Y},X)\,.$$

If $q<n$ then $H_q(\tilde{\hat{Y}},X) \in \boldsymbol{C}$ by the classical relative Hurewicz theorem module a Serre class. By Proposition 1.13, we have $E^{\infty}_{p,q} \in \boldsymbol{C}$, for $p>0$ and $q<n$. So, it follows that

$$H_0(\pi, H_n(\tilde{\hat{Y}}, \tilde{X})) \to H_n(\hat{Y}, X)$$

is a C-isomorphism. But

$$\pi_n(\tilde{\hat{Y}}, \tilde{X}) \xrightarrow{h_n} H_n(\tilde{\hat{Y}}, \tilde{X}),$$

the Hurewicz map, is a C-isomorphism. We have the projection map which induces an isomorphism

$$p_{\#} : \pi_n(\tilde{\hat{Y}}, \tilde{X}) \to \pi_n(\hat{Y}, X).$$

If we compose $p_{\#}^{-1}$ with h_n we get a map from $\pi_n(\tilde{\hat{Y}}, \tilde{X})$ to $H_n(\tilde{\hat{Y}}, \tilde{X})$ which satisfies

$$h_n \circ p_{\#}^{-1}(\alpha^{-1}\beta) = \alpha \circ h_n(p_{\#}^{-1}(\beta)),$$

for all $\alpha \in \pi_1(X)$ and $\beta \in \pi_n(\hat{Y}, X)$. (See [2, Propositon 1.3]). Now according to Proposition 1.4 of [2] we have that

$$H_0(\pi, \pi(\hat{Y}, X)) \to H_0(\pi, H_n(\tilde{\hat{Y}}, \tilde{X}))$$

is a C-isomorphism. So, $\bar{\pi}_n(\hat{Y}, X) \to H_n(\hat{Y}, X)$ is a C-isomorphism.

REMARK. Examples of C-nilpotent spaces which are not nilpotent, can be obtained by the following result: Given any group π, any sequence $\{A_n\}_{n \geqq 2}$ of abelian groups and a sequence $\theta_n \colon \pi \to \mathrm{Aut}\,(A_n)$, $n \geqq 2$, of homomorphisms then there exists a space X such that $\pi_1(X) \cong \pi$, $\pi_n(X) \cong A_n$, for $n > 1$, and the action of $\pi_1(X)$ on $\pi_n(X)$ is θ_n. See [10, Ex. A-3, p. 460].

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