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MANIFOLDS WITH A SPECIAL TYPE OF CONELIKE SINGULARITIES

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0. Introduction.

Manifolds with singularities have been studied by Sullivan [12], Baas [1], and others. In particular Levitt [10] considers a certain category of such manifolds, with the underlying structure of PL-manifolds.

The manifolds of Levitt's category are constructed inductively. The first step is to consider manifolds of the form $M = M_0 \cup \bigcup_i S_i$ where M_0 is a smooth manifold, S_i is a PL-manifold with boundary, and where each S_i is equipped with a standard PL-equivalence

$$f_i : V_i \times c\Sigma_i \to S_i$$

Here is V_i a smooth manifold, and $c\Sigma_i$ the cone on a smooth sphere Σ_i.

Some of these manifolds have the underlying structure of a PL-sphere.

We can now inductively consider manifolds $M = M_0 \bigcup_i S_i$, where M_0 is a manifold with singularities constructed in a previous step, and each S_i is equipped with a PL-equivalence

$$f_i : V_i \times c\Sigma_i \to S_i ,$$

where V_i is smooth, and $c\Sigma_i$ is the cone on a previously constructed manifold, with the underlying structure of a PL-sphere.

We imitate this procedure, to construct our category of C-manifolds as follows.

Let bP_n denote the group of concordance classes of homotopy spheres that bound n-dimensional framed manifolds.

As a preliminary step we construct a category of C_1-manifolds. A C_1-manifold is a manifold of form $M = M_0 \cup \bigcup_i S_i$ where each S_i is provided with a PL-isomorphism

$$f_i : V_i \times c\Sigma_i \to S_i$$

where V_i is a smooth manifold, and Σ_i is a homotopy sphere that represents an element in bP_*.

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Let $S_{C_1}(M)$ denote the set of concordance classes of C_1-manifolds with underlying PL-manifold M. There is a forgetful map

$$i\colon S_{\mathrm{O}}(M) \to S_{C_1}(M)$$

from the set of smooth structures $S_{\mathrm{O}}(M)$ on M to $S_{C_1}(M)$.

In particular there is a forgetful map $i\colon S_{\mathrm{O}}(S^n) \to S_{C_1}(S^n)$ which factors over $S_{\mathrm{O}}(S^n)/bP_{n+1}$.

Next we extend the C_1-category inductively to a category of manifolds with singularities, the C-manifolds, such that the composite $S_{\mathrm{O}}(S^n)/bP_{n+1} \to S_{C_1}(S^n) \to S_C(S^n)$ is a close as possible to an isomorphism.

In order to do so, we construct a natural transformation $\chi_1 : S_{C_1}(M) \to N(M)$, which for a given smooth manifold M maps $S_{C_1}(M)$ into the group of cobordism classes of normal maps $M' \to M$.

We can introduce a bundle category containing "tangent bundles" of C_1-manifolds. The set of C_1-bundles of a finite complex X is classified by a space BC_1.

One can show by a formal argument that the set of C_1-structures on a smooth manifold M is classified by homotopy classes of maps of M into the fibre PL/C_1 of the natural map $BC_1 \to B\mathrm{PL}$. Again by formal arguments we can show that χ_1 induces a map $\chi_1 : \mathrm{PL}/C_1 \to G/\mathrm{O}$.

Using this map we construct the category of C_2-manifolds as a category of manifolds with singularities containing C_1. In sections 3–4 we continue this process. We obtain inductively the categories C_i. At each step the definition of the next category involves choices.

Let C be the union of the categories C_i. There is a corresponding bundle category with classifying space BC. The set of C-structures on a smooth manifold M are classified by the set of homotopy classes $[M, \mathrm{PL}/C]$.

The homotopy types of PL/C and BC depend in a curious way on the (unresolved) Kervaire invariant conjecture.

Let $A \subset \mathbf{Z}$: be the set of numbers of form $n = 2^r - 2$ such that there exists a framed manifold of dimension n with Kervaire invariant 1.

THEOREM A. *There is a category C of manifolds with singularities containing the category C_1 constructed by the inductive procedure outlined above.*

(i) *Let C' be another category, constructed by the same inductive procedure, but possibly by making different choices.*

There is a natural equivalence

$$E\colon S_C(-) \to S_{C'}(-)\,.$$

(ii) *There is a fibration*

$$\mathrm{PL}/C \to G/\mathrm{O} \to \prod_{n \varepsilon A} K(\mathbf{Z}/2, n),$$

(iii) *There is a fibration*

$$BC \to G/\mathrm{PL} \times B\mathrm{O} \to \prod_{n \varepsilon A} K(\mathbf{Z}/2, n).$$

By modifying our argument we construct other similar categories.

In particular we give an interpretation of Coker J and of BSO localized at the odd primes.

THEOREM B. *There is a category $\bar{C}$ of manifolds with singularities, unique in the same sense as C above, such that there are homotopy equivalences*

$$\lambda: \mathrm{PL}/\bar{C} \to (\mathrm{Coker}\ J)[\tfrac{1}{2}]$$

$$\varrho: \mathrm{PL}/\bar{C} \to B\mathrm{SO}[\tfrac{1}{2}].$$

We finally want to thank our advisor Ib Madsen for many helpful discussions.

1. The C_1-category.

In this paragraph the C_1-category will be defined and studied. This represents the first step in the inductive construction of the C-category.

A C_1-manifold M is a PL-manifold with extra structure. This extra structure consists of two elements: A smooth, codimension 0 submanifold M_0 with boundary, and certain conelike singularities in the complement. More precisely, the complement will be the disjoint union of manifolds, each PL-homeomorphic to $V \times D$. Here the V's are smooth manifolds, and the D's are PL-discs. The boundary $V \times S$ is assumed to be contained in the boundary of the smooth submanifold M_0, so it inherits a smooth structure. We demand that as a smooth manifold it is a product $V \times \Sigma$ where Σ is a smooth manifold, PL-equivalent to a sphere.

We can think of a C_1-manifold as a smooth manifold, with cone-singularities glued onto the boundary.

The next inductive step will be to glue singularities onto a C_1-manifold. If we do not put any restrictions on the singularities, we will essentially end up with the PL-category. This is a theorem of Levitt [10].

There are various ways to specify allowable sets of singularities. In this paper, we shall allow singularities of Milnor- and Kervaire-spheres. Specifically, recall the cyclic group bP_{2i} of $(2i-1)$-dimensional homotopy spheres which bound parallellizable manifolds. The elements of bP_* appear to be the simplest, and are the best understood exotic spheres. We allow only singularities of the form $V \times c\Sigma$, where $\Sigma \in bP_*$, and $c\Sigma$ denotes the cone on Σ. For each concordanceclass in bP_*, choose a particular representative.

DEFINITION 1.1. A C_1-manifold M is a quadruple $(M_0, V_i, \Sigma_i, f_i)$ consisting of

(i) A smooth manifold with boundary M_0.
(ii) Smooth compact manifolds V_i, $i=1,\ldots,p$.
V_i may or may not have boundary.
(iii) Smooth spheres Σ_i, which are among the particular representatives chosen.
(iv) Disjoint inclusions $f_i\colon V_i \times \Sigma_i \to \partial M_0$ of $V_i \times \Sigma_i$ in ∂M_0 as codimension 0 submanifolds.

This definition only mentions the smooth part of the manifold. In order to get the underlying PL-manifold of M, we glue in the singularities:

DEFINITION 1.2. The underlying PL-manifold M_{PL} of a C_1-manifold M is the union

$$M_{\mathrm{PL}} = M_0 \cup \bigcup_i (V_i \times c\Sigma_i)\,.$$

The C_1-category does not possess a natural Cartesian product. There is, however, a natural product of a smooth manifold and a C_1-manifold.

Let X be a PL-manifold. A C_1-structure on X is a pair (M, h), where M is a C_1-manifold, and h is a PL-homeomorphism $h\colon M_{\mathrm{PL}} \to X$. A concordance between the two C_1-structures (M_0, h_0) and (M_1, h_1) is a C_1-structure W on $X \times I$, which restricts to (M_i, h_i) on $X \times \{i\}$.

The usual version of smoothing theory uses tangent microbundles. This approach involves the smooth structure on the product of a manifold with itself. In the C_1-category, products are not available. For this reason we have to generalize smoothing theory using thickenings (see Wall [14]), not microbundles.

A k-dimensional thickening of the CW-complex X is a compact k-dimensional manifold with boundary M, with $\pi_1(\partial M) = \pi_1(M)$ and a simple homotopy equivalence $i\colon X \to M$. Two thickenings are equivalent if there is a concordance W between M_1 and M_2, and a homotopy commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & (M_1)_{\mathrm{PL}} \\ \downarrow & & \downarrow \\ (M_2)_{\mathrm{PL}} & \longrightarrow & W_{\mathrm{PL}} \end{array}$$

where $(M_i)_{\mathrm{PL}}$ and W_{PL} denote the underlying PL-manifolds.

If M is a thickening of X, $M \times I$ will also be a thickening of X. This allows us to consider "stable thickenings".

We did not specify which structure M should have. We could choose M to be smooth, C_1 or PL. Then we obtain the concepts of smooth thickening, C_1-thickening or PL-thickening.

The set of stable thickenings is in all three cases a representable functor on finite CW-complex. See e.g. Levitt [10] for the C_1-case.

In the smooth and the PL-case there is a natural equivalence between the stable thickening functor, and the corresponding bundle theory. Let $i\colon X \to M$ be a thickening, and $\tau(M)$ be the tangent bundle of M. The equivalence maps the thickening i to the bundle $i^*(\tau(M))$. The trivial bundle on a manifold N is represented by the thickening $i\colon N \hookrightarrow D(\nu(N))$ where i is the inclusion of the zero section in the normal disc bundle.

This equivalence shows that the smooth and PL-thickening functors are classified by BO, respectively $B\mathrm{PL}$. We let BC_1 denote the classifying space for C_1-thickenings. A smooth thickening is also a C_1-thickening, and a C_1-thickening is also a PL-thickening. Thus we have maps $BO \to BC_1$ and $BC_1 \to B\mathrm{PL}$. The composition is the natural map $BO \to B\mathrm{PL}$.

These maps are only defined on finite subcomplexes. In the rest of the paper we will not worry about these questions. Maps will only be specified on finite complexes.

Let X be a PL-manifold. We consider the set $\mathscr{S}_{C_1}(X)$ of concordance classes of C_1-structures on X.

THEOREM 1.2. *There is a 1–1 correspondence between $\mathscr{S}_{C_1}(X)$ and homotopy classes of liftings of the map classifying the* PL-*tangent bundle of X*

$$\begin{array}{ccc} & & BC_1 \\ & \nearrow & \downarrow \\ X & \xrightarrow[t_{\mathrm{PL}}]{} & B\mathrm{PL} \end{array}$$

SKETCH OF PROOF. Let $f\colon M \to X$ be a C_1-structure. Then $f^{-1}\colon X \to M$ is a C_1-thickening. This thickening determines a lifting of t_{PL}. We need an inverse of this construction. Let $t_{C_1}\colon X \to BC_1$ be a lifting of t_{PL}. By stability we can represent it by a PL-embedding in a C_1-manifold $X \hookrightarrow N$. X has trivial normal bundle in N. Using the PL s-cobordism theorem, we conclude that N is PL-equivalent to $X \times I^n$ for some n. It is possible to generalize the product theorem of smoothing theory to show that there is a 1–1 correspondence between $\mathscr{S}_{C_1}(X)$ and $\mathscr{S}_{C_1}(X \times I)$. (See Levitt [10] for details.) The C_1-structure

$N \to X \times I^n$ thus determines a unique C_1-structure on X. It is easily checked that the constructions above are inverses.

The space BC_1 is not an H-space, but BO acts on it. Let μ_{BO}: $BO \times BO \to BO$ and μ_{PL}: $BPL \times BPL \to BPL$ be the familiar maps characterizing Whitney sum. From the viewpoint of thickenings μ_{BO} and μ_{BPL} classify the following constructions. Let $f_1: X \to M_1$ and $f_2: X \to M_2$ be two thickenings. Then $f_1 \times f_2: X \times X \to M_1 \times M_2$ is a thickening. Take the induced thickening over the diagonal $\Delta: X \to X \times X$. The map classifying this thickening is the sum of f_1 and f_2.

LEMMA 1.3. *There is a natural map* $\mu_{BC_1}: BO \times BC_1 \to BC_1$ *and the following diagram is homotopy commutative.*

$$\begin{array}{ccc} BO \times BO & \xrightarrow{\mu_{BO}} & BO \\ \downarrow & & \downarrow \\ BO \times BC_1 & \xrightarrow{\mu_{BC_1}} & BC_1 \\ \downarrow & & \downarrow \\ BPL \times BPL & \xrightarrow{\mu_{BPL}} & BPL \end{array}$$

PROOF. Suppose X is a PL-manifold. We construct a natural transformation

$$[X; BO \times BC_1] \to [X, BC_1] .$$

Consider $(\alpha, \beta): X \to BO \times BC_1$. Its first coordinate defines a smooth thickening $\bar{\alpha}: X \to M_0$, its second a C_1-thickening $\bar{\beta}: X \to M_{C_1}$. The restriction of the C_1-thickening $\bar{\alpha} \times \bar{\beta}: X \times X \to M_0 \times M_{C_1}$ to the diagonal determines a C_1-thickening, classified by $\gamma: X \to BC_1$. This determines the transformation. It is easy to check that it is natural, and that the diagrams of the lemma commute.

In accordance with usual notation we let PL/C_1 denote the homotopy theoretical fibre of the map $BC_1 \to BPL$. In analogy with smoothing theory, one would suspect that C_1-structures on a C_1-manifold M are classified by homotopy classes in $[M; PL/C_1]$. The lack of a multiplication in C_1 prevents us from proving this, but the following restricted version is true.

LEMMA 1.4. *Let M be a smooth manifold. There exists a* 1–1 *correspondence between* $\mathcal{S}_{C_1}(M)$ *and* $[M, PL/C_1]$.

PROOF. First we construct a map $[M, PL/C_1] \to \mathcal{S}_{C_1}(M)$ as follows. If t_0: $M \to BO$ is the tangent bundle, and α: $M \to PL/C_1$ a map, then

$$M \xrightarrow{(t_0,\alpha)} BO \times PL/C_1 \to BO \times BC_1 \to BC_1$$

is a lifting of the PL tangent bundle $t_{PL}: M \to BPL$, so it determines a C_1-structure on M.

Next we construct the inverse map $\mathscr{S}_{C_1}(M) \to [M, PL/C_1]$. Let $\bar{\gamma}: M \to \bar{M}$ be a C_1-structure on M, and $\gamma: M \to BC_1$ the corresponding lifting of the PL tangent bundle map t_{PL}. We have the homotopy commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{(-t_0,\gamma)} & BO \times BC_1 & \xrightarrow{\mu_{BC_1}} & BC_1 \\ & & \downarrow & & \downarrow \\ & & BPL \times BPL & \xrightarrow{\mu_{BPL}} & BPL \end{array}$$

where both t_0 and γ are liftings of t_{PL}. The composite is trivial, in fact canonically trivialized. This trivialization defines an element of $[M, PL/C_1]$.

Finally, it is easy to see that the two maps are inverses of each other.

On the homogenous space level we get an action of PL/O on PL/C_1 which lifts the action from lemma 1.3. Indeed, let M be a smooth manifold, and $\alpha: M \to PL/O$, $\beta: M \to PL/C_1$ two maps. They give $M \times M$ a C_1-manifold structure which induces a C_1-structure $\bar{\gamma}$ on the smooth tangent bundle $\tau(M)$ of M via $\tau(M) \hookrightarrow M \times M$. We get a map $\gamma: M \hookrightarrow \tau(M) \to PL/C_1$, and set $\alpha \cdot \beta = \gamma$. The action is natural and gives rise to a map

$$\mu_{PL/C_1}: PL/O \times PL/C_1 \to PL/C_1$$

such that the following diagrams homotopy commute

$$\begin{array}{ccc} PL/O \times PL/O & \longrightarrow & PL/O \\ \downarrow & & \downarrow \\ PL/O \times PL/C_1 & \longrightarrow & PL/C_1 \\ \downarrow & & \downarrow \\ BO \times BC_1 & \longrightarrow & BC_1 \end{array}$$

Given two maps $\alpha_1, \alpha_2: M \to PL/O$, then $\alpha_1 \cdot (\alpha_2 \beta)$ and $(\alpha_1 \cdot \alpha_2) \cdot \beta$ are the pullbacks of $(\alpha_1, \alpha_2, \beta)$ by

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\mathrm{id} \times \Delta} M \times M \times M \quad \text{and}$$

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\Delta \times \mathrm{id}} M \times M \times M, \quad \text{respectively}.$$

These maps are homotopic, so on classifying space level we see that

$$\begin{array}{ccc} PL/O \times PL/O \times PL/C_1 & \xrightarrow{\mu_{PL/O} \times \mathrm{id}} & PL/O \times PL/C_1 \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \mu_{PL/C_1}} & & \Big\downarrow{\scriptstyle \mu_{PL/C_1}} \\ PL/O \times PL/C_1 & \xrightarrow{\mu_{PL/C_1}} & PL/C_1 \end{array}$$

is homotopy commutative.

2. The map χ_1: PL/$C_1 \to G/\mathrm{O}$.

When we constructed the C_1-category we had to decide on a particular set of singularities. We chose the set of singularities that involves cones of spheres in bP_*. Recall that bP_{*+1} can be characterized as the kernel of the map χ_{0*}: $\pi_*(\mathrm{PL/O}) \to \pi_*(G/\mathrm{O})$, see for example Sullivan [13]. In this section we shall extend the natural map χ_0: PL/O $\to G/\mathrm{O}$ to a map χ_1: PL/$C_1 \to G/\mathrm{O}$, which will play an analogous role in the definition of the C_2-category. With later generalizations in mind, we will give a purely homotopy theoretical definition. First, however, we outline a more conceptual, geometric definition.

Let M be a smooth manifold, and $\sigma \in [M; \mathrm{PL}/C_1]$. Further, let

$$\bar{\sigma}: M_0 \cup \left(\bigcup_i V_i \times c\,\Sigma_i \right) \to M$$

be a C_1-structure represented by σ. By the assumption, Σ_i is the boundary of some parallelizable manifold $M(\Sigma_i)$. The manifold $\tilde{M} = M_0 \cup (\bigcup_i V_i \times M(\Sigma_i))$ is a smooth manifold, and there is a degree one normal map $\tilde{M} \to M$ whose normal cobordism class is classified by a homotopy class $M \to G/\mathrm{O}$. We will prove in lemma 2.2 that this defines a natural transformation, inducing a map at classifying spaces χ_1: PL/$C_1 \to G/\mathrm{O}$.

In this construction we have to specify for each given homotopy sphere Σ a particular parallelizable manifold $M(\Sigma)$, such that $\partial(M(\Sigma)) = \Sigma$. We will now do that.

Recall that bP_{4n} is a cyclic group of finite order, say θ_n. For $n > 1$ we construct the Milnor manifold M^{4n} of index 8 by plumbing together 8 copies of the tangent disc bundle of S^{2n}, see Browder [4] for details. The boundary of M^{4n} is a homotopy sphere, generating bP_{4n}.

The group bP_{4n-2} is either $\mathbf{Z}/2$ or 0. The case $bP_{4n-2} = 0$ can, according to Browder [3], only occur if n is a power of 2. We now choose for each $\Sigma \in bP_{2n}$ a framed bounding manifold $M(\Sigma)$. The only restriction we make on the choice is that if Σ is the $(4n$-$1)$-dimensional generator mentioned above, then $M(\Sigma)$ is the Milnor manifold.

We point out that $M(\Sigma)$ defines an extensions of the classifying map of Σ

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{g_\Sigma} & \mathrm{PL/O} \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{g_D} & G/\mathrm{O} \end{array} \tag{2.1}$$

Now we will give the homotopy theoretical definition. Suppose that M is a smooth manifold, $\sigma \in [M, \mathrm{PL}/C_1]$ and that $\bar{\sigma}: M_0 \cup (\bigcup_i V_i \times c\,\Sigma_i) \to M$ is a representative of the class of C_1-structures on M defined by σ. If we consider M_0, V_i, and Σ_i as pieces of PL-submanifolds of the smooth manifold M, they will all inherit a standard smooth structure from the smooth structure on M.

The manifold $\bar{\sigma}(M_0)$ is a codimension zero submanifold of M, so it inherits a smooth structure. Let $\tilde{M}_0$ be M_0 with this new structure. In the same way $\bar{\sigma}(V_i \times c\,\Sigma_i)$ inherits a smooth structure. The product structure theorem of smoothing theory tells us that $\bar{\sigma}(V_i \times c\,\Sigma_i)$ is concordant to $\hat{V}_i \times D_i$ where $\hat{V}_i$ is some smooth structure on V_i, and D_i is a standard disc. By restricting to the boundary we find that $\partial M_0 = \bigcup_i \hat{V}_i \times S_i$. Here S_i is a standard sphere.

The differences between the smoothings induced via $\bar{\sigma}$, and the given smoothing correspond to homotopy classes of maps

$$g_0 : M_0 \to \mathrm{PL/O}$$
$$g_{V_i} : V_i \to \mathrm{PL/O}$$
$$g_{\Sigma_i} : \Sigma_i \to \mathrm{PL/O}\,.$$

These classes are related by the equations

$$g_0 \Big|_{V_i \times \Sigma_i} = \mu_{\mathrm{PL/O}} \cdot (g_{V_i} \times g_{\Sigma_i})\,.$$

We define the homotopy class $M \to G/O$ as follows

i) On M_0 it is the map $M_0 \xrightarrow{g_0} \mathrm{PL/O} \xrightarrow{\chi_0} G/O$.

ii) On $V_i \times c\Sigma_i$ it is

$$V_i \times c\Sigma_i \xrightarrow{g_{V_i} \times g_{c\Sigma_i}} \mathrm{PL/O} \times G/O \to G/O \times G/O \to G/O$$

where $g_{c\Sigma_i}$ is the extension of g_{Σ_i} defined by 2.1.

The map χ_0 is an H-map, so the two maps agree up to canonical homotopy on $V_i \times \Sigma_i$. We can glue them together to get a welldefined element $\chi_1(\sigma) \in [M, G/O]$. It is not difficult to check that this agrees with the geometrical definition.

LEMMA 2.2. $\chi_1 : [-\,; \mathrm{PL}/C_1] \to [-\,; G/O]$ *is a natural transformation.*

PROOF. Let M, N be smooth manifolds, $\sigma : N \to \mathrm{PL}/C_1$ represent a homotopy class, and let $f : M \to N$ be a continuous map. Note that homotopic maps $N \to \mathrm{PL}/C_1$ determines the same element in $[N, G/O]$.

CASE 1. Suppose $f : M \to N = D(\xi)$ is the inclusion of M as the zerosection in a discbundle. We can represent the homotopy class of σ by the homotopic map

$$D(\xi) \xrightarrow{\pi} M \xrightarrow{f} D(\xi) \xrightarrow{\sigma} \mathrm{PL}/C_1\,.$$

The map $\sigma\circ f$ induces a C_1-structure n M

$$\overline{\sigma\circ f}\colon M_0 \cup \left(\bigcup_i (V_i \times c\Sigma_i)\right) \to M$$

$\overline{\sigma\circ f}$ induces a C_1-structure on $D(\xi)$

$$\overline{\sigma\circ f\circ\pi}: D(\xi|_{M_0}) \cup \left(\bigcup_i (D(\xi|_{V_i}) \times c\Sigma_i)\right) \to D(\xi)$$

whose classifying map is $\sigma\circ f\circ\pi$. Thus $\overline{\sigma\circ f}$ represents the restriction $f^*(\sigma)$ of σ to the zerosection. The maps $\overline{\sigma\circ f}$ and $\overline{\sigma\circ f\circ\pi}$ define $g_M \in [M, G/O]$ and $g_{D(\xi)} \in [D(\xi), G/O]$ respectively, and it is easily concluded that $g_{D(\xi)}\circ f = g_M$. This proves lemma 3.1 in this case.

CASE 2. Using case 1 we replace N by $N \times D$, where D is a large disc. Thus we can assume that $f\colon M \to N$ is an embedding. The image $f(M)$ has a normal disc bundle $D(\nu)$ in N. Then the map $f\colon M \to N$ factors as

$$M \xrightarrow{f'} D(\nu) \xrightarrow{f''} N\,.$$

Case 1 applies to the inclusion f'. The map f'' is an inclusion of a submanifold of codimension zero. Assume without loss of generality that f is of this form. Let $\sigma: N \to \mathrm{PL}/C_1$ induce the C_1-structure

$$\bar{\sigma}\colon N_0 \cup \left(\bigcup_i (V_i \times c\Sigma_i)\right) \to N\,.$$

The submanifolds $\bar{\sigma}(V_i \times c\Sigma_i)$ are also of codimension zero. As we already remarked, this structure is of the form $\hat{V}_i \times D_i$, where $\hat{V}_i$ is some smoothing of V_i. By smooth transversality we can assume that $\bar{\sigma}$ restricted to $\bar{\sigma}^{-1}(M)$ is of the form

$$\overline{\sigma\circ f}\colon (N_0 \cap \bar{\sigma}^{-1}(M)) \cup \left(\bigcup_i V_i' \times c\Sigma_i\right) \to M\,,$$

where V_i' is a smooth submanifold of V_i. This induces a C_1-structure on M. The induced element in $[M, \mathrm{PL}/C_1]$ is $\sigma\circ f$. The C_1-structures $\bar{\sigma}$ and $\overline{\sigma\circ f}$ defines $g_M \in [M; G/O]$ and $g_N \in [N, G/O]$ and $g_N\circ f = g_M$. This proves the lemma in general.

A smooth surgery problem is a degree one normal map $\tilde{M} \to M$. Normal cobordism classes of such maps are classified by elements of $[M, G/O]$. Similarly, PL-surgery classes by elements of $[M, G/\mathrm{PL}]$. The homotopy type of G/PL localized at 2 was determined by Sullivan in [13]. Essentially $G/\mathrm{PL}_{(2)}$ is a product of one copy of each of the Eilenberg-MacLane spaces $K(\mathbf{Z}/2, 4n-2)$ and $K(\mathbf{Z}_{(2)}, 4n)$ for each i. In particular there are indecomposable classes $k^{4n-2} \in H^{4n-2}(G/\mathrm{PL}; \mathbf{Z}/2)$ inducing characteristic classes for surgery problems.

The natural transformation which forgets the smooth structure of a smooth surgery problem induces a map j: $G/\mathrm{O} \to G/\mathrm{PL}$. For suitable choice of the k^{4n-2} above (related to the surgery invariant) it was proved in Brumfiel-Madsen-Milgram [8] that $j^*(k^{4n-2})=0$ if $n \neq 2^r$, but $j^*(k^{2^r-2}) \neq 0$.

In dimensions $4n-2 \neq 2^r-2$, $bP_{4n-2}=\mathbf{Z}/2$. It is conjectured that bP_{2^r-2} is always trivial. This will be the case if k^{2^r-2} is spherical. When we study the homotopy of PL/C in section 4, the dimensions where $bP_{4n-2}=0$ have to be treated separately. We will then need the following lemma.

LEMMA 2.3. *If* $bP_{4n-2}=0$, *then* k^{4n-2} *pulls back to zero under the composite*

$$k\colon \mathrm{PL}/C_1 \xrightarrow{\chi_1} G/\mathrm{O} \xrightarrow{j} G/\mathrm{PL}\,.$$

PROOF. We use the geometrical interpretation of χ_1. Let M be a smooth manifold, and let $\bar{\sigma}\colon M_0 \cup (\bigcup_i V_i \times c\Sigma_i) \to M$ be a C_1-structure on M. Since $bP_{4n-2}=0$, none of spheres Σ_i has dimension $4n-3$.

Let $\pi\colon \tilde{M} \to M$ be the surgery problem associated to $\bar{\sigma}$. It is classified by $g_M \in [M, G/\mathrm{O}]$. On the smooth part $\bar{\sigma}(M_0)$ is the surgery problem π a PL-equivalence. In particular, it is a trivial PL-surgery problem. Over the singular part our surgery problem is the disjoint sum of surgery problems of the form

$$V_i \times (M(\Sigma_i), \partial M(\Sigma_i)) \xrightarrow{\mathrm{id} \times f_i} V_i \times (c\Sigma_i, \Sigma_i)\,,$$

where the $M(\Sigma_i)$ are parallelizable manifolds, and $\partial M(\Sigma_i)$ PL-spheres. The map $j \circ g_M$ classifying the original surgery problem considered in the PL-category, will factor over a wedge of spheres

$$M \to \bigvee_i \; (V_i \times c\Sigma_i/\Sigma_i) \to \bigvee_i \; (c\Sigma_i/\Sigma_i)\,.$$

Since no spheres of dimension $4n-2$ occur in the wedge, any cohomology class in $H^{4n-2}(G/\mathrm{PL})$ will pull back to zero over M. This proves the lemma.

Next we prove that $\chi_1\colon \mathrm{PL}/C_1 \to G/\mathrm{O}$ preserves the action of PL/O.

LEMMA 2.4. *The following diagram is homotopy commutative*

$$\begin{array}{ccc} \mathrm{PL}/\mathrm{O} \times \mathrm{PL}/C_1 & \xrightarrow{\mu_{\mathrm{PL}/C_1}} & \mathrm{PL}/C_1 \\ \Big\downarrow \chi_0 \times \chi_1 & & \Big\downarrow \chi_1 \\ G/\mathrm{O} \times G/\mathrm{O} & \xrightarrow{\mu_{G/\mathrm{O}}} & G/\mathrm{O} \end{array}$$

PROOF. The spaces PL/O and G/O are loop spaces, in fact infinite loop spaces (see Boardman-Vogt [1]). This means that the multiplication can be chosen to be strictly associative.

The map PL/O $\to G/O$ is a loop map, so we can assume that the following diagram is strictly commutative

$$(*)\qquad \begin{array}{ccc} \mathrm{PL/O}\times\ldots\times\mathrm{PL/O} & \to & \mathrm{PL/O} \\ \downarrow & & \downarrow \\ G/\mathrm{O}\times\ldots\times G/\mathrm{O} & \to & G/\mathrm{O} \end{array}$$

Now we can test the diagram in the statment of the lemma on a smooth compact manifold M. Suppose $(\alpha,\beta)\colon M \to (\mathrm{PL/O}\times\mathrm{PL/C_1})$ is a given map.

We can map M into G/O in two distinct ways. The action of PL/O on PL/C_1 allows us to compose α and β. The product $\alpha\cdot\beta$ induces a C_1-structure on M. This defines a class $\chi_1(\alpha\cdot\beta)\in[M,G/\mathrm{O}]$.

But we also know that α and β defines a smooth structure $\bar{\alpha}$ and a C_1-structure $\bar{\beta}$ on M. These structures define elements $\chi_1(\alpha)$ and $\chi_1(\beta)\in[M,G/\mathrm{O}]$. Using the multiplicative structure of G/O, we can form their product $\chi_1(\alpha)\cdot\chi_1(\beta)$. The lemma states that $\chi_1(\alpha\cdot\beta)=\chi_1(\alpha)\cdot\chi_1(\beta)$. To prove this we factor (α,β) over $M\times M$

$$(\alpha,\beta)\colon M \xrightarrow{\Delta} M\times M \xrightarrow{(\alpha\times\beta)} \mathrm{PL/O}\times\mathrm{PL}/C_1 .$$

By naturality it is enough to calculate the two classes $\chi_1\big((\alpha\times *)\cdot(*\times\beta)\big)$ and $\chi_1(\alpha\times *)\cdot\chi_1(*\times\beta)$.

Let $\bar{\beta}$ be realized by

$$\bar{\beta}\colon M_0\cup(V_i\times c\Sigma_i)\to M .$$

The C_1-structure on $M\times M$ induced by $\alpha\times\beta$ is realized by

$$\mathrm{id}\times\bar{\beta}\colon M_\alpha\times M_0\cup\Big(\bigcup_i (M_\alpha\times V_i)\times c\Sigma_i\Big)\to M\times M .$$

Here M_α is the smooth structure induced by α.

We can now compute the two maps. On the smooth part the two maps are given by the compositions

$$M_\alpha\times M_0\to \mathrm{PL/O}\times\mathrm{PL/O}\to G/\mathrm{O}\times G/\mathrm{O}\to G/\mathrm{O}$$

$$M_\alpha\times M_0\to \mathrm{PL/O}\times\mathrm{PL/O}\to \mathrm{PL/O}\to G/\mathrm{O} .$$

On the singular part they are

$$M_\alpha\times V_i\times c\Sigma_i\colon \mathrm{PL/O}\times\mathrm{PL/O}\times G/\mathrm{O}$$
$$\xrightarrow{\mu_{\mathrm{PL/O}}\times\mathrm{id}} \mathrm{PL/O}\times G/\mathrm{O}\to G/\mathrm{O}\times G/\mathrm{O}\to G/\mathrm{O}$$

$$M_\alpha \times V_i \times c\Sigma_i : \mathrm{PL/O} \times \mathrm{PL/O} \times G/\mathrm{O} \to G/\mathrm{O} \times G/\mathrm{O} \times G/\mathrm{O} \xrightarrow{\mathrm{id} \times \mu_{G/\mathrm{O}}} G/\mathrm{O} \times G/\mathrm{O} \to G/\mathrm{O}$$

Because the diagram (*) commutes, the maps agree in both cases.

The lemma is proved.

3. *C*-manifolds.

In this section we give an inductive construction of the C-category. In analogy with the method used to construct the C_1-category, we obtain a general C_s-manifold by gluing conelike singularities onto a C_{s-1}-manifold. This process is welldefined once we fixed an allowable set of singularities.

In order to generalize the group

$$bP_{*+1} = \ker \{(\chi_0)_* : \pi_*(\mathrm{PL/O}) \to \pi_*(G/\mathrm{O})\}$$

we will define a map $\chi_s : \mathrm{PL}/C_s \to G/\mathrm{O}$ for each category C_s. Then we define the C_{s+1}-category using cones on the C_s-spheres corresponding to elements of

$$\ker \{(\chi_s)_* : \pi_*(\mathrm{PL}/C_s) \to \pi_*(G/\mathrm{O})\} .$$

Thus we must show that all constructions and lemmas of sections 1–2 have counterparts in the C_s-category.

Suppose inductively that we have defined categories of C_r-manifolds, $1 \leqq r \leqq s$, and corresponding classifying spaces BC_r with maps

$$BO \to BC_1 \ldots \to BC_s \to B\mathrm{PL}$$

Furthermore, suppose there are maps

$$\mu_{BC_r} : BO \times BC_r \to BC_r$$

$$\mu_{\mathrm{PL}/C_r} : \mathrm{PL/O} \times \mathrm{PL}/C_r \to \mathrm{PL}/C_r$$

satisfying the obvious analogues of lemmas 1.3 and 1.5. We will finally assume that there exist maps

$$\chi_r : \mathrm{PL}/C_r \to G/\mathrm{O}$$

such that the diagrams below commute up to homotopy

$$\begin{array}{ccc} \mathrm{PL}/C_{r-1} & \xrightarrow{\chi_{r-1}} & G/\mathrm{O} \\ \downarrow & \nearrow_{\chi_r} & \\ \mathrm{PL}/C_r & & \end{array}$$

$$\begin{array}{ccc} \mathrm{PL/O} \times \mathrm{PL}/C_r & \xrightarrow{\mu_{\mathrm{PL},C_r}} & \mathrm{PL}/C_r \\ \Big\downarrow \chi_0 \times \chi_r & & \Big\downarrow \\ G/\mathrm{O} \times G/\mathrm{O} & \longrightarrow & G/\mathrm{O} \end{array}$$

Then we define bP^s_{*+1} as follows

$$bP^s_{*+1} = \ker \{(\chi_s)_* : \pi_*(\mathrm{PL}/C_s) \to \pi_*(G/\mathrm{O})\} .$$

Choose a particular representative for each element.

DEFINITION 3.1. A C_{s+1}-manifold is a quadruple $(M_0, \{V_i\}, \{\Sigma_i\}, \{f_i\})$ consisting of

(i) A C_s-manifold with boundary M_0.

(ii) Smooth compact manifolds $V_i = 1 \ldots p$.
V_i may or may not have boundary.

(iii) C_s-spheres Σ_s, which are among the representatives chosen above.

(iv) Disjoint inclusions $f_i \colon V_i \times \Sigma_i \to \partial M_0$
of $V_i \times c\Sigma_i$ in ∂M_0 as dimension 0 submanifolds.

In the same way as we did with C_1-manifolds, we can glue together M_0 and the singular parts of M to get the underlying PL-manifold:

$$M_{\mathrm{PL}} = M_0 \cup \left(\bigcup_i V_i \times c\Sigma_i \right) .$$

It is not difficult to see that theorem 1.2 and the lemmas 1.3, 1.4, and 1.5 hold, if they are stated for C_{s+1} instead of C_1. Indeed, the proofs are quite similar.

We also have to construct a map

$$\chi_{s+1} \colon \mathrm{PL}/C_{s+1} \to G/\mathrm{O} .$$

The construction is essentially the same as our homotopy theoretical construction of χ_1. It depends on picking particular extensions to the disc for homotopy trivial maps $S \to G/\mathrm{O}$.

If $\Sigma \in bP^s_*$ is classified by a map $g_\Sigma \colon S \to \mathrm{PL}/C_s$ we can choose a map $g_{c\Sigma} \colon cS \to G/\mathrm{O}$ so that the following diagram is commutative

$$\begin{array}{ccc} S & \xrightarrow{g_\Sigma} & \mathrm{PL}/C_s \\ \downarrow & & \downarrow \chi_s \\ cS & \xrightarrow{g_{c\Sigma}} & G/\mathrm{O} \end{array}$$

In the next section we discuss the question of to what extent the choices involved will affect the category C_{s+1}. It turns out that it is possible to give conditions strong enough to guarantee that the C_{s+1}-category is essentially uniquely determined.

It is easy to generalize lemmas 2.2 and 2.4. The generalization of lemma 2.3 will be proved in section 4.

We now have the categories C_s with classifying spaces BC_s. A C_s-thickening may be considered as a C_{s+1}-thickening, so we also have maps

$$BO \to BC_1 \to \dots \to BC_s \to \dots BPL .$$

We set

$$BC = \varinjlim_s BC .$$

In the same way we define the C-category to be $\varinjlim C_s$.

For a given dimension, the direct limit system stabilizes after a finite number of steps. Recall that $bP_n = 0$, if $n \leqq 7$, so that any C_1-manifold of dimension 7 or less is a smooth manifold. Because bP_n^1 consist of C_1-manifolds that are not smooth manifolds, it has to be zero if $n \leqq 8$. This shows that any C_2-manifold of dimension 8 or less is a C_1-manifold. In general, all n dimensional C-manifolds are C_{n-7} manifolds.

At this point we need the following

LEMMA 3.3. *Let X be a finite* CW-*complex*, dim $X = m$. *Any C_s-thickening is stably equivalent to a thickening $X \to N^n$, where $n = \dim N^n \leqq 2m$.*

PROOF. In [14] Wall proves a similar result for PL-thickenings. Suppose that $X \to N'$ is any C_s-thickening. It is stably PL-equivalent to some PL-thickening $X \to N$ with $\dim N \leqq 2m$. We must show that we can give N a C_s-structure so that N and N' are stably C_s-equivalent thickenings.

The concordance W between $N \times I^r$ and N' is PL-equivalent to $N' \times I$. We can use this PL-equivalence to give W a C_s-structure which is a product of N' and I. By the C_s product structure theorem, this is concordant to the product of a C_s-structure on N and the standard structure on I^{r+1}. The C_s-structure on N allows us to consider $X \to N$ as a C_s-thickening, and W is a stable C_s-equivalence between this thickening and $X \to N'$.

Lemma 3.3 shows that for a finite complex X the sequence

$$[X, BO] \to [X, BC_1] \to \dots \to [X, BC_s] \to \dots$$

will stabilize. This shows that the space BC is the classifying space for C-thickenings.

Since a C-thickening is a PL-thickening we have a map of classifying spaces $BC \to B\mathrm{PL}$ whose fibre is denoted PL/C. Then $\mathrm{PL}/C = \varinjlim \mathrm{PL}/C_s$ and the maps χ_s induce a mapping $\chi: \mathrm{PL}/C \to G/\mathrm{O}$. By the construction,

$$\chi_*: \pi_*(\mathrm{PL}/C) \to \pi_*(G/\mathrm{O})$$

is injective.

4. How to choose trivializations.

In the last section the category C_{s+1} was defined using the map $\chi_s: \mathrm{PL}/C_s \to G/\mathrm{O}$. The construction of the map χ_s depended on choices. In this section, we first limit the element of choice, and then prove that the remaining choices are essentially equivalent.

The map $(\chi_0)_*: \pi_*(\mathrm{PL}/\mathrm{O}) \to \pi_*(G/\mathrm{O})$ has been studied by Kervaire-Milnor [9], Sullivan [13], Brumfiel [5], [6], [7] and others. The image of $(\chi_0)_*$ is a complicated (unknown) subgroup of $\pi_*(G/\mathrm{O})$, but the cokernel is much simpler.

There is a fibration

$$\mathrm{PL}/\mathrm{O} \xrightarrow{\chi_0} G/\mathrm{O} \xrightarrow{j} G/\mathrm{PL}.$$

The homotopy groups of G/PL are wellknown. They are periodical with period 4 as follows

$$\begin{cases} \pi_{4n}(G/\mathrm{PL}) = \mathbf{Z} & \pi_{4n+1}(G/\mathrm{PL}) = 0 \\ \pi_{4n+2}(G/\mathrm{PL}) = \mathbf{Z}/2 & \pi_{4n+3}(G/\mathrm{PL}) = 0\,. \end{cases}$$

Since $\operatorname{coker}(\chi_{0*}) = \operatorname{im}(j_*)$, this group has to be cyclic. In dimension $4n$ we can describe a generator of $\operatorname{coker}(\chi_{0*}) \subset \pi_*(G/\mathrm{O})$ as follows.

Recall that G/O is the space classifying normal cobordism classes of degree one normal maps. Also recall the Milnor manifold M^{4n} with index 8 mentioned in section 2. The boundary ∂M is a homotopy sphere that generates bP_{4n}. Consider the manifold N which is a connected sum along the boundary of $|bP_{4n}|$ copies of M. Then N is framed, and its boundary ∂N is a standard smooth sphere S. The union with a standard disc $\tilde{N} = N \cup_{\partial N} D$ has a natural smooth structure. There is a natural degree one normal map $\bar{f}: \tilde{N} \to S$ which is classified by $f: S \to G/\mathrm{O}$. The group $\operatorname{coker}(\chi_0)$ is generated by f.

In dimensions $4n-2$ the long exact sequence of the fibration $\mathrm{PL}/\mathrm{O} \to G/\mathrm{O} \to G/\mathrm{PL}$ specializes to

$$0 \to (\operatorname{coker} \chi_0)_{4n-2} \to \mathbf{Z}/2 \to bP_{4n-2} \to 0\,.$$

If $n \neq 2^r$, $bP_{4n-2} = \mathbf{Z}/2$, so $(\operatorname{coker}\chi_0)_{4n-2} = 0$. If $n = 2^r$ it is conjectured that $bP_{4n-2} = 0$. Let $A \subset \mathbf{Z}$ be the set of dimensions $4n-2$ for which $(\operatorname{coker}\chi_0)_{4n-2} = \mathbf{Z}/2$.

Now we are able to compute $(\operatorname{coker}\chi_{1*})_n$.

The image of

$$\chi_{1*}: \pi_{4n}(\mathrm{PL}/C_1) \to \pi_{4n}(G/\mathrm{O})$$

contains the image of χ_{0*} in this dimension, but also the element $f: S \to G/\mathrm{O}$ generating $(\operatorname{coker}\chi_{0*})_{4n}$ for the following reason:

Take $|bP_{4n}|$ copies of the boundary ∂N^{4n} of the Milnor manifold. The connected sum along the boundary of the cones of these homotopy spheres is a C_1*-manifold* $\tilde{D}$. The boundary of $\tilde{D}$ is a standard sphere, so we can take the union with a standard disc $\tilde{D} \cup_S D$. This defines a C_1*-structure* S_α on a PL $4n$-sphere. From the geometrical definition of χ_1 it is clear that

$$S_\alpha \to \mathrm{PL}/C_1 \xrightarrow{\chi_1} G/\mathrm{O}$$

is the element f. This is where we use that the trivialization of the generator of bP_* is chosen in a particular way.

From lemma 2.3 follows now that $(\operatorname{coker}\chi_{1*})_n = \mathbf{Z}/2$ if $n \in A$, $(\operatorname{coker}\chi_{1*})_n = 0$ else.

In 2 we discussed classes $k \in H^{2^r-2}(G/\mathrm{PL}, \mathbf{Z}/2)$ which are mapped to non-zero elements in $H^{2^r-2}(G/\mathrm{O})$. Consider the map

$$p: G/\mathrm{O} \xrightarrow{\prod k} \prod_{n \in A} K(\mathbf{Z}/2, n)\,.$$

It is proved in Madsen-Milgram [11] that each map $k: G/\mathrm{PL} \to K(\mathbf{Z}/2, 2^r-2)$ is twice deloopable, so p is also twice deloopable.

DEFINITION 4.1. The homotopy fibre of p is Φ_0.

Φ_0 is a loop space. Because $\mathrm{PL}/\mathrm{O} \to G/\mathrm{O} \to G/\mathrm{PL}$ is trivial we can lift $\mathrm{PL}/\mathrm{O} \to G/\mathrm{O}$ to a loop map $\mathrm{PL}/\mathrm{O} \to \Phi_0$. Let $\varphi: \Phi_0 \to G/\mathrm{O}$ be the inclusion.

LEMMA 4.2. *We can choose* χ_s: $\mathrm{PL}/C_s \to G/\mathrm{O}$ *so that it factors* $\mathrm{PL}/C_s \xrightarrow{\tilde{\chi}_s} \Phi_0 \xrightarrow{\varphi} G/\mathrm{O}$.

PROOF. We know from lemma 2.3 that $p\chi_1 \sim 0$, so we can assume that $\chi_1 = \varphi\tilde{\chi}_1$. In the construction of χ_s we only used that $\mathrm{PL}/\mathrm{O} \to G/\mathrm{O}$ is a loop map. Suppose inductively that $\chi_s = \varphi\tilde{\chi}_s$. If

$$\alpha \in \ker[\chi_{s*}: \pi_*\mathrm{PL}/C_s \to \pi_* G/\mathrm{O}]\,,$$

then

$$\alpha \in \ker[\tilde{\chi}_{s*}: \pi_*(\mathrm{PL}/C_s) \to \pi_* \varphi_0],$$

since φ is injective on homotopy groups. We can construct a map

$$\tilde{\chi}_{s+1}: \mathrm{PL}/C_{s+1} \to \Phi_0$$

in the same way as we constructed χ_{s+1} in section 3. Then put $\chi_{s+1} = \varphi\tilde{\chi}_{s+1}$.

If we let s increase, in the limit we obtain a map $\tilde{\chi}: \mathrm{PL}/C \to \Phi_0$. This map induces an isomorphism on all homotopy groups, so it is a homotopy equivalence. We can formulate this as follows:

THEOREM 4.3. *Let $A \subset \mathbf{Z}$ be the set of numbers of form $n = 2^r - 2$ such that there exists a framed manifold of dimension n with Kervaire invariant* 1. *Let C be the manifold category constructed above. Then there is a canonical fibration.*

$$\mathrm{PL}/C \to G/O \to \prod_{n \in A} K(\mathbf{Z}/2, n)$$

In particular it follows that PL/C is independent of the choices we made in the construction of the C-category. But we can prove more. Assume that by making different choices, we have arrived at two different categories C and C'. We will construct a natural equivalence $E: S_C(-) \to S_{C'}(-)$. The map E is constructed inductively. We can assume that $C_s = C'_s$, but that $\tilde{\chi}_s, \tilde{\chi}'_s$: PL/$C_s$ $\to \Phi$ might be different maps.

Let Σ be any C_s-structure on S^{n-1} which is in bP^s_n. The two different trivializations of the classifying map $\Sigma \to \mathrm{PL}/C_s$ give us a map $f: S^n \to \Phi$. Since $\pi_*(\mathrm{PL}/C_s) \to \pi_*(G/O)$ is an epimorphism can we lift f to $f': S^n \to \mathrm{PL}/C_s$. This corresponds to a C_s-structure $\bar{f}$ on S^n.

Let $\Sigma \times I$ be a trivial concordance. Take the connected sum of this with $\bar{f}$. We obtain a C_s-concordance $W(\Sigma)$ from Σ to itself.

Now we can define the map E. If M has a C_s structure M_α:

$$\bar{\alpha}: M_0 \cup \left(\bigcup_i V_i \times c\Sigma_i\right) \to M$$

we define $M_{E(\alpha)}$ to be

$$\overline{E(\alpha)}: M_0 \cup \left(\bigcup_i V_i \times W(\Sigma_i)\right) \cup \left(\bigcup_i V_i \times c\Sigma_i\right) \to M_0 \cup \left(\bigcup_i V_i \times c\Sigma_i\right) \xrightarrow{\bar{\alpha}} M,$$

where

$$(V_i \times W(\Sigma_i)) \cup (V_i \times c\,\Sigma_i) \to (V_i \times c\Sigma_i)$$

is the obvious PL-homeomorphism.

The following diagram is commutative

$$\begin{array}{ccccc} M_{E(\alpha)} & \longrightarrow & \mathrm{PL}/C_s & \xrightarrow{\bar{\chi}'} & \\ \downarrow & & & & \Phi \\ M_\alpha & \longrightarrow & \mathrm{PL}/C_s & \xrightarrow{\bar{\chi}} & \end{array}$$

Interchanging the roles of χ and χ' we get a map $E':\mathscr{S}_{C_s}(-) \to \mathscr{S}_{C_s}(-)$ which is the inverse of E. We can extend E inductively to get an isomorphism $E\colon \mathscr{S}_C(M) \to \mathscr{S}_{C'}(M)$.

THEOREM 4.4. *Let C and C' be two categories constructed using the inductive procedure above. Then there exist S a natural equivalence*

$$E\colon S_C(-) \to S_{C'}(-)\,.$$

5. The homotopy type of BC.

In this section we determine the homotopy type of BC. We also remark that the machine of sections 1–3 can be generalized. We consider the special case when G/O is replaced with the localization of coker J at odd primes.

For any C-manifold M we will define a map $\eta_M\colon M \to G/\mathrm{PL} \times B\mathrm{O}$. This will give us a map on classifying space level

$$\eta = (\eta_1 \times \eta_2)\colon BC \to G/\mathrm{PL} \times B\mathrm{O}\,,$$

which splits BC as a product.

If M is a smooth manifold, let $(\eta_1)_M$ be a trivial map, and $(\eta_2)_M$ the smooth tangent bundle.

Let N be any C_s-manifold with trivial PL-tangent bundle. Then N has some framed structure, and the C_s-structure of N is given by a map $f\colon N \to \mathrm{PL}/C_s$.

Suppose we have defined η on C_s-manifolds so that for all N of the type above, η_N is equal to the composite

$$N \xrightarrow{f} \mathrm{PL}/C_s \xrightarrow{\chi} G/\mathrm{O} \xrightarrow{(j,p)} G/\mathrm{PL} \times B\mathrm{O}\,,$$

where j and p are the natural maps. It is easy to see that this is valid for $s=0$, that is $\mathrm{PL}/C_s = \mathrm{PL}/\mathrm{O}$, if η is defined as above.

Let Σ be a sphere with a C_s-structure. It has a trivial PL-tangent bundle. If $\chi(\Sigma)\sim 0$ can we extend η_Σ from Σ to $c\Sigma$ as the composite

$$c\Sigma \xrightarrow{\chi} G/\mathrm{O} \xrightarrow{(j,p)} G/\mathrm{PL} \times B\mathrm{O}\,.$$

Then we can define η in the obvious way on all C_{s+1}-manifold. In particular, on $V \times c\Sigma$ it is the composite

$$V \times c\Sigma \xrightarrow{\chi_0 \times \chi_{s+1}} \mathrm{PL}/\mathrm{O} \times G/\mathrm{O} \to G/\mathrm{O} \to G/\mathrm{PL} \times B\mathrm{O}\,.$$

To complete the induction, let N be a framed manifold. It has a C_s-structure induced by $f: N \to \mathrm{PL}/C_s$. As usual we can decompose N

$$N = N_0 \cup \left(\bigcup_i V_i \times c\Sigma_i \right).$$

Recall from section 2 that the smooth structure on N induces a smooth structure on N_0 and V_i. This smooth structure is framed, since the smooth structure on N is. We must prove that η_N is equal to the composite

$$N \xrightarrow{f} \mathrm{PL}/C_s \xrightarrow{\chi} G/\mathrm{O} \to G/\mathrm{PL} \times B\mathrm{O} .$$

By the induction hypothesis this is true on N_0. On $V_i \times c\Sigma_i$ it is also true, since the composite $V_i \to \mathrm{PL}/\mathrm{O} \to G/\mathrm{O} \to B\mathrm{O}$ is just the tangent bundle of V_i, and since we have a commutative diagram

$$\begin{array}{ccccc} V_i \times c\Sigma_i \to & \mathrm{PL}/\mathrm{O} \times \mathrm{PL}/C_{s+1} & \to & \mathrm{PL}/C_{s+1} & \\ & \downarrow & & \downarrow & \\ & \mathrm{PL}/\mathrm{O} \times G/\mathrm{O} & \longrightarrow & G/\mathrm{O} & \to G/\mathrm{PL} \times B\mathrm{O} \end{array}$$

This concludes the construction of η_M. We want to construct a natural transformation $[-, BC] \to [-, G/\mathrm{PL} \times B\mathrm{O}]$. Assume that X is a smooth manifold with tangent bundle τ_X. Let $\sigma: X \to M$ be a C-thickening of X. Then put

$$\eta_1 = \sigma(\eta_M)_1 : X \to G/\mathrm{PL} ,$$

$$\eta_2 = (\sigma(\eta_M)_2 - \tau_X) : X \to B\mathrm{O}$$

and finally

$$\eta = (\eta_1, \eta_2) : X \to G/\mathrm{PL} \times B\mathrm{O} .$$

In particular, let M be the disc bundle $D(\xi)$ of some vector bundle ξ on X. Then η_1 is trivial, and η_2 represents the vector bundle $(\tau_X \oplus \xi) - (\tau_X) = \xi$.

It is easy to see that the homotopy class of $\eta \circ \sigma$ only depends on the stable class of the thickening M. This defines a map

$$\eta: [X, BC] \to [X, G/\mathrm{PL} \times B\mathrm{O}] .$$

LEMMA 5.1. *The map* $[-, BC] \to [-, G/\mathrm{PL} \times B\mathrm{O}]$ *is a natural transformation in the category of smooth manifolds and homotopy classes of maps.*

PROOF. Let X, Y be smooth manifolds, $\sigma: Y \to BC$ represent a homotopy class, and let $f: X \to Y$ be a continuous map. As in the proof of lemma 2.1, we consider two cases.

CASE 1: $f\colon X \to Y = D(\xi)$ is the inclusion of X as the zero section in a disc bundle. Let $\bar{\sigma}\colon Y \to M$ be a thickening classified by σ. Then $\bar{\sigma}$ is a simple homotopy equivalence. Choose a vector bundle θ over M such that $D(\theta \oplus \bar{\sigma}^{-1}(\xi)) = M \times D^n$. Then there is a pullback of thickenings

$$\begin{array}{ccc} D(\theta) & \longrightarrow & M \times D^n \\ \uparrow \overline{\sigma f} & & \uparrow \bar{\sigma} \\ X & \longrightarrow & D(\xi) = Y \end{array}$$

It is clear that $\overline{\sigma f}$ is a thickening representing σf. From the definition of η we see that

$$(\eta_{D(\theta)}(\overline{\sigma f})) \sim (\eta_M \bar{\sigma}) f\,.$$

CASE 2. Now we can assume that X is a submanifold of codimension zero in Y. Any stable thickening of Y can be represented by a C-structure on a PL-disc bundle ξ on Y. Take the restriction of ξ to X. The total space of this bundle is a manifold, which has an induced C-structure. The inclusion $i_X\colon X \to D(\xi_X)$ is a thickening of X. It is clear that $(\eta_{D(\xi|X)}) i_X \sim (\eta_{D(\xi)}) i_Y f$.

The natural transformation of lemma 5.1 defines a map of classifying spaces

$$\eta\colon BC \to G/\mathrm{PL} \times BO\,.$$

If we approximate PL/C with a framed manifold M, and use that $\eta_M\colon M \to G/\mathrm{PL} \times BO$ is equal to the composite $M \to \mathrm{PL}/C \to G/\mathrm{O} \to G/\mathrm{PL} \times BO$ we get a commutative diagram

$$\begin{array}{ccc} \mathrm{PL}/C & \xrightarrow{\chi} & G/\mathrm{O} \\ \downarrow & & \downarrow {\scriptstyle (j,p)} \\ BC & \xrightarrow{\eta} & G/\mathrm{PL} \times BO \end{array}$$

Recall the classes $k^{2^r-2} \in H^{2^r-2}(G/\mathrm{PL}; \mathbf{Z}/2)$. Let Φ_{PL} be the fibre of the map

$$G/\mathrm{PL} \to \prod_{n \in A} K(\mathbf{Z}/2, n)\,,$$

where A is the set of dimensions congruent to 2 modulo 4, for which $bP_n = 0$. If we replace χ in the argument above with the map $\tilde{\chi}\colon \mathrm{PL}/C \to \Phi_0$ defined in section 4, we get a corresponding map $\tilde{\eta}\colon BC \to \Phi_{\mathrm{PL}} \times BO$.

Consider the homotopy commutative diagram

$$\begin{array}{ccccc} \mathrm{PL} & \longrightarrow & \mathrm{PL} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ \Phi_0 & \longrightarrow & G/\mathrm{O} & \longrightarrow & \prod_{n \in A} K(\mathbf{Z}/2, n) \\ \downarrow & & \downarrow & & \downarrow \\ \Phi_{\mathrm{PL}} \times BO & \to & G/\mathrm{PL} \times BO & \to & \prod_{n \in A} K(\mathbf{Z}/2, n) \end{array}$$

Since all other columns and rows are fibrations, the left column is a fibration.

Recall that $\mathrm{PL} \to G/\mathrm{O}$ factors as $\mathrm{PL} \to \mathrm{PL}/\mathrm{O} \to \mathrm{PL}/C \to G/\mathrm{O}$. This shows that the following diagram is homotopy commutative

$$\begin{array}{ccccc} \mathrm{PL} & \to & \mathrm{PL}/C & \to & BC \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \tilde{\chi}} & & \downarrow{\scriptstyle \tilde{\eta}} \\ \mathrm{PL} & \longrightarrow & \Phi_0 & \longrightarrow & \Phi_{\mathrm{PL}} \times BO \end{array}$$

Since both rows are fibrations, and the maps id and $\tilde{\chi}$ are homotopy equivalences, the map $\tilde{\eta}$ is also an equivalence. We have proved

THEOREM 5.3. *There is a fibration*

$$BC \xrightarrow{\eta} G/\mathrm{PL} \times BO \xrightarrow{k} \prod_{n \in A} K(\mathbf{Z}/2, n)\,,$$

where k *is given by the classes* $k^n \in H^n(G/\mathrm{PL}; \mathbf{Z}/2)$, *and* A *is the set of dimensions of form* $4m-2$ *for which* $bP_{4m-2} = 0$.

If X is a loop space, and $\mathrm{PL}/\mathrm{O} \to X$ is a loop map we can construct other categories of manifolds, similar to C.

Assume for example that X is the localisation of Coker J at odd primes.

Recall that the localisation of G/O at odd primes splits as a product of loop spaces

$$G/\mathrm{O}[\tfrac{1}{2}] \xrightarrow[\sim]{\pi + e} (\mathrm{Coker}\, J)[\tfrac{1}{2}] \times BSO_{\oplus}[\tfrac{1}{2}]\,.$$

Let

$$\lambda_0 : \mathrm{PL}/\mathrm{O} \to (\mathrm{Coker}\ J)[\tfrac{1}{2}]$$

be the map

$$\mathrm{PL}/\mathrm{O} \xrightarrow{\chi_0} G/\mathrm{O} \to G/\mathrm{O}[\tfrac{1}{2}] \xrightarrow{\pi} (\mathrm{Coker}\ J)[\tfrac{1}{2}]\,.$$

Since (Coker χ_{0*}) $\in \pi_*(G/\mathrm{O})$ contains no odd torsion, and π_*(Coker J) only contains torsion we know that

$$\lambda_{0*} : \pi_* \mathrm{PL}/\mathrm{O} \to \pi_*(\mathrm{Coker}\ J)[\tfrac{1}{2}]$$

is an epimorphism on homotopy groups. We define the $\bar{C}_1$-category as the category with singularities $c\Sigma$, where Σ are representatives of the concordance classes satisfying $\lambda_0(\Sigma) = 0$.

Following the procedure in section 2 we can construct a map

$$\lambda_1 : \mathrm{PL}/\bar{C}_1 \to (\mathrm{Coker}\ J)[\tfrac{1}{2}]$$

given by a natural transformation

$$\lambda_1 : [-, \mathrm{PL}/\bar{C}_1] \to [-, G/\mathrm{O}] .$$

More precisely, let M be a smooth manifold, and let $\sigma \in [M, \mathrm{PL}/\bar{C}_1]$ be represented by a $\bar{C}$-structure

$$\bar{\sigma} : M_0 \cup \bigcup_i (V_i \times c\Sigma_i) \to M .$$

We get a set of classifying maps

$$\begin{aligned} g_0 &: M_0 \to \mathrm{PL/O} \\ g_{V_i} &: V_i \to \mathrm{PL/O} \\ g_{\Sigma_i} &: \Sigma_i \to \mathrm{PL/O} \end{aligned}$$

related by

$$g_0 |_{V_i \times \Sigma_i} = \mu_{\mathrm{PL/O}}(g_{V_i} \times g_{\Sigma_i}) .$$

For each $g_{\Sigma_i} : \Sigma_i \to \mathrm{PL/O}$ satisfying that the composite

$$\Sigma_i \to \mathrm{PL/O} \to (\mathrm{Coker}\ J)[\tfrac{1}{2}]$$

is nullhomotopic, we can choose a particular nullhomotopy

$$g_{c\Sigma_i} : c\Sigma_i \to (\mathrm{Coker}\ J)[\tfrac{1}{2}] .$$

We can now define $\lambda_i(\sigma) \in [M, (\mathrm{Coker}\ J)[\tfrac{1}{2}]]$ by the following formulas:

i) On M_0 it is the composite $M_0 \xrightarrow{g_0} \mathrm{PL/O} \xrightarrow{\lambda_0} G/\mathrm{O}$.
ii) On $V_i \times c\Sigma_i$ it is given by the composite

$$V_i \times c\Sigma_i \xrightarrow{g_{V_i} \times g_{c\Sigma_i}} \mathrm{PL/O} \times (\mathrm{Coker}\ J)[\tfrac{1}{2}] \to$$
$$\to (\mathrm{Coker}\ J)[\tfrac{1}{2}] \times (\mathrm{Coker}\ J)[\tfrac{1}{2}] \xrightarrow{\mu} (\mathrm{Coker}\ J)[\tfrac{1}{2}] .$$

Following section 2 we can now prove that $\sigma \mapsto \lambda_1(\sigma)$ is a natural transformation. In particular it does define a map

$$\lambda_1 : \mathrm{PL}/C_1 \to (\mathrm{Coker}\ J)[\tfrac{1}{2}] .$$

By induction we, as in section 3-4 obtain a category of manifolds with singularities we denote the $\bar{C}$-category. We also obtain a classifying space $B\bar{C}$. Let $\mathrm{PL}/\bar{C}$ denote the fibre of the natural map $B\bar{C} \to B\mathrm{PL}$.

The arguments in sections 3-4 generalize to yield

THEOREM 5.4. *There is a category $\bar{C}$ of manifolds with singularities. This category depends on choices, but if $\bar{C}'$ is another category, constructed in the same way, there is a natural equivalence*

$$E : S_{\bar{C}}(-) \to S_{\bar{C}'}(-) .$$

Furthermore, there is a homotopy equivalence

$$\lambda : \mathrm{PL}/\bar{C} \to (\mathrm{Coker}\ J)[\tfrac{1}{2}]\ .$$

Recall that $B\mathrm{SPL}[\frac{1}{2}]$ splits as a product

$$B(\mathrm{Coker}\ J)[\tfrac{1}{2}] \times BSO[\tfrac{1}{2}] \xrightarrow[\sim]{j \times \gamma} B\mathrm{SPL}[\tfrac{1}{2}]\ .$$

The right inverse of γ is a map

$$\varrho : B\mathrm{SPL}[\tfrac{1}{2}] \longrightarrow BSO[\tfrac{1}{2}]\ .$$

The natural map SPL $\to$ Coker J factors as SPL $\to$ PL/O $\to$ G/O $\to$ Coker J. Also,

$$\mathrm{SPL} \to \mathrm{PL/O} \to \mathrm{PL}/\bar{C} \to (\mathrm{Coker}\, J)[\tfrac{1}{2}]$$

is the natural map, since λ is an extension of λ_0. We conclude that the following diagram is commutative

$$\begin{array}{ccc} & \nearrow & \pi_*(\mathrm{PL}/\bar{C}) \\ \pi_*(\mathrm{SPL}) & & \downarrow \approx \\ & \searrow & \pi_*(\mathrm{Coker}\ J)[\tfrac{1}{2}]\ . \end{array}$$

Consider the map $\bar{\varrho}$: $BS\bar{C} \to B\mathrm{SPL} \xrightarrow{\varrho} BSO$.

There is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \pi_*(BS\bar{C}) & \to & \pi_*(BSPL) & \to & \pi_{*-1}(\mathrm{PL}/\bar{C}) & \to & 0 \\ & & \downarrow \bar{\varrho}_* & & \downarrow \mathrm{id}_* & & \downarrow \lambda_* & & \\ 0 & \to & \pi_*(BSO) & \xrightarrow{\gamma} & \pi_*(B\mathrm{SPL}) & \to & \pi_{*-1}(\mathrm{Coker}\ J)[\tfrac{1}{2}] & \to & 0\ . \end{array}$$

THEOREM 5.5. *There is a homotopy equivalence*

$$\bar{\varrho} : BS\bar{C} \to BSO[\tfrac{1}{2}]\ .$$

Finally we note that we could have defined the C-manifolds with the additional restriction that the singular manifolds must be framed. This is the main case considered in Levitt [10]. We can still construct the map χ, and we get a classifying space BC homotopy equivalent to the one considered in this paper. The proofs carry over with minimal changes. In this case there is no obvious way to construct the action $BO \times BC \to BC$, and the statement $\mathscr{S}_C(M) \approx [M, \mathrm{PL}/C]$ is not obvious if M is not framed.

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