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UNIQUENESS OF HAHN–BANACH EXTENSIONS AND LIFTINGS OF LINEAR DEPENDENCES

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Abstract.

We study intersection properties of balls for a subspace M of a Banach space E which ensures either that each linear functional on M has a unique norm-preserving extension to E or that if $f_1,\ldots,f_n \in M^*$ are such that $\sum_{i=1}^n f_i=0$, then every f_i has a norm-preserving extension $g_i \in E^*$ such that $\sum_{i=1}^n g_i=0$. We relate these properties to the existence of norm-1 projection in E^* with kernel $M^\perp$.

1. Introduction.

Let E be a real Banach space and let M be a closed subspace. The dual space of E is denoted E^* and the annihilator of M in E^* is denoted $M^\perp$. $B(x,r)$ denotes the closed ball in E with center x and radius r. The closure of a set S is denoted $\bar{S}$, its convex hull conv (S) and the distance from y to S by $d(y,S)$. The unit ball of E is written E_1, and the set of extreme points of a set S is denoted $\delta_e S$.

We shall study extensions of linear functionals from M to E and we write for $f \in E^*$, $\|f\|_M$ for the norm of the restriction $f \mid M$ of f to M. By $M^\#$ we mean

$$M^\# = \{f \in E^* : \|f\| = \|f\|_M\}$$

$L(E,F)$ (respectively $K(E,F)$) denotes the space of bounded (respectively compact) linear operators from E into F.

M-ideals were first studied by Alfsen and Effros in [1]. They called M an M-ideal if there exists a projection P in E^* such that $P(E^*)=M^\perp$ and for all $f \in E^*$.

$$\|f\| = \|Pf\| + \|f-Pf\| .$$

One characterization of M-ideals is as follows:

M is an M-ideal in E if and only of whenever $\{B(a_i,r_i)\}_{i=1}^n$ is a finite family of balls in E such that

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$$\bigcap_{i=1}^{n} B(a_i, r_i) \neq \emptyset \quad \text{and} \quad M \cap B(a_i, r_i) \neq \emptyset \quad \text{for all } i,$$

then $M \cap \bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. [1], [10].

For example, c_0 is an M-ideal in l_∞. In this paper we are looking for weaker intersection properties which characterize those subspaces M of E such that $M^\perp$ is the kernel of a norm -1 projection in E^*.

One direction of weakening the intersection properties is to start with the characterization of semi M-ideals as defined in [10]. This leads us to characterizations of subspaces M such that if $f \in M^*$, then f has a unique norm-preserving extension to E. An example of this is Theorem 2.2 which says that if M is a closed subspace of E, then we have:

Every $f \in M^*$ which attains its norm on M_1 has a unique norm-preserving extension to E if and only if whenever $x \in M$, $y \in E$ with $\|x\| = \|y\| = 1$ and $\varepsilon > 0$, there exists $r \geqq 1$ such that

$$M \cap B(y + rx, r + \varepsilon) \cap B(y - rx, r + \varepsilon) \neq \emptyset .$$

This intersection property characterize semi M-ideals if we can take $r = 1$.

In the other direction we generalize the intersection property characterizing M-ideals in that we require that the centers of the balls are in M. Then we get a result that ensure that we can obtain simultaneous norm-preserving extensions of several linear functionals. For instance, Theorem 3.1 implies that the following statements are equivalent:

(i) If $\{B(a_i, r_i)\}_{i=1}^{n}$ are balls with centers in M and $\bigcap_{i=1}^{n} B(a_i, r_i) \neq \emptyset$ in E, then $M \cap \bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$.

(ii) If $f_1, \ldots, f_n \in M^*$ are such that $f_1 + \ldots + f_n = 0$, then there exist norm-preserving extensions g_i of f_i such that $g_1 + \ldots + g_n = 0$.

As shown in Corollary 4.9, if E is a smooth Banach space, then (i) with $n = 3$ is equivalent to $M^\perp$ being the kernel of a norm-1 projection in E^*.

In the course of these investigations, we also get characterizations of HB-subspaces. HB subspaces were defined by Hennefeld in [6]. He said that M is an HB subspace of E if $M^\perp$ is complemented by a subspace M_* in E^* such that whenever $f_* \in M_*$ and $f^\perp \in M^\perp \setminus \{0\}$, then $\|f_* + f^\perp\| \geqq \|f^\perp\|$ and $\|f_* + f^\perp\| > \|f_*\|$. In Theorem 4.1 we show that M is an HB-subspace of E if and only if M has property (i) above and every $f \in M^*$ has a unique norm-preserving extension to E.

We follow Sulivan [17] and say that M is (weakly) Hahn–Banach smooth in E if every $f \in M^*$ (which attains its norm on M_1) has a unique norm-preserving extension to E. By Phelps [14] and others, this has been called property U. With our notation we get that E is smooth if and only if every subspace of E is weakly Hahn–Banach smooth, and E^* is strictly convex if and only if every subspace of E is Hahn–Banach smooth. We call the intersection property in (i) the *n.E.* intersection property (*n.E.*I.P.) This resemble the *n.k.* intersection property as defined in [12].

2. Uniqueness of Hahn-Banach extensions.

We shall say that a subspace M of E is *Hahn–Banach smooth in* E if every functional on M has a unique norm-preserving extension to E. Moreover, M is *weakly Hahn–Banach smooth in* E if every functional on M which attains its norm on the unit ball of M has a unique norm-preserving extension to E.

M. Smith and F. Sullivan studied in [16] spaces E which are Hahn–Banach smooth or weakly Hahn–Banach smooth in E^{**}. They showed that if a space E is weakly Hahn–Banach smooth in E^{**}, then E^* has the Radon–Nikodym property.

A. E. Taylor [18] and S. R. Foguel [4] have shown that every subspace of E is Hahn–Banach smooth in E if and only if E^* is strictly convex.

From R. R. Phelps [14], we get the following theorem. We use the notation

$$M^{\#} = \{f \in E^* : \|f\| = \|f\|_M\} .$$

THEOREM 2.1. *Let M be a closed subspace of E. The following statements are equivalent*:

1) *M is Hahn–Banach smooth in E.*
2) *$M^\perp$ is a Haar-subspace of E^*,* i.e. *if $x \in E^*$, then there exists a unique $y \in M^\perp$ such that $\|x-y\| = d(x, M^\perp)$.*
3) *If $x, y \in M^{\#}$ and $x+y \in M^\perp$, then $x+y=0$.*
4) *Every element in E^* can be written in a unique way as a sum of elements from $M^{\#}$ and $M^\perp$.*

It is known that semi M-ideals are Hahn–Banach smooth [10]. Recall from [10] that M is a semi M-ideal in E if and only if whenever $x \in M$, $y \in E$ with $\|x\| = 1 = \|y\|$ and $\varepsilon > 0$, then there exists $z \in M$ such that $\max_{\pm} \|x \pm (y-z)\| \leqq 1+\varepsilon$. We can generalize this result as follows.

THEOREM 2.2. *The following statements are equivalent for a closed subspace M of E.*

1) *M is weakly Hahn–Banach smooth in E.*
2) *If $x \in M$, $y \in E$ with $\|x\| = 1 = \|y\|$ and $\varepsilon > 0$, then there exist $r \geqq 1$ and $z \in M$ such that*

$$\max_{\pm} \|rx \pm (y-z)\| \leqq r+\varepsilon .$$

PROOF. 2) $\Rightarrow$ 1). Let $f \in M^*$ be such that $\|f\| = f(x)$ for some $x \in M$ with $\|x\| = 1$. Let $g, h \in E^*$ be norm-preserving extensions of f and let $\varepsilon > 0$. Then it suffices to show that $(g-h)(y) \leqq \varepsilon 2\|f\|$ for each $y \in E$ with $\|y\| = 1$.

Let $r \geqq 1$ and let $z \in M$ be such that $\max_{\pm} \|rx \pm (y-z)\| \leqq r+\varepsilon$. Then we have

$$\begin{aligned}(g-h)(y)+2r\|f\| &= (g-h)(y)+2rf(x)\\ &= (g-h)(y)+g(rx)+h(rx)\\ &= g(rx+y-z)+h(rx-y+z)\\ &\leqq \|g\|\cdot\|rx+y-z\|+\|h\|\cdot\|rx-y+z\|\\ &\leqq 2\|f\|(r+\varepsilon) .\end{aligned}$$

Hence $(g-h)(y) \leqq \varepsilon 2\|f\|$.

1) $\Rightarrow$ 2). Assume 2) is false. Then there exist $x \in M$, $y \in E$ with $\|x\| = 1 = \|y\|$ and $\varepsilon > 0$ such that

$$M \cap B(y+rx, r+\varepsilon) \cap B(y-rx, r+\varepsilon) = \emptyset \qquad \text{for all } r \geqq 1 .$$

Let

$$A = \bigcup_{r \geqq 1} B(y+rx, r) \qquad \text{and} \qquad B = \bigcup_{r \geqq 1} B(y-rx, r) .$$

Let $\Delta_M = \{(x,x) \in M \times M\}$. A and B are convex, and

$$\Delta_M \cap [(A \times B) + B(0, \varepsilon)] = \emptyset$$

in $E \oplus_\infty E$. By the Hahn–Banach theorem, there exist $\lambda \in \mathrm{R}$ and $g_1, g_2 \in E^*$ such that

$$\sup_{x \in M} (g_1+g_2)(x) < \lambda < \inf_{(u,v) \in A \times B} (g_1(u)+g_2(v)) .$$

Since M is a subspace, we get $g_1+g_2 \in M^\perp$. If $(u,v) \in A \times B$, then we have

$$\|u-(y+rx)\| \leqq r \qquad \text{and} \qquad \|v-(y-rx)\| \leqq r$$

for all sufficiently large r. Hence

$$0 < \lambda < \inf_{r \geqq 1} [g_1(y+rx)+g_2(y-rx)-r\|g_1\|-r\|g_2\|] .$$

From this we get

$$r\|g_1\|+r\|g_2\|+\lambda < g_1(y+rx)+g_2(y-rx) .$$

We divide by r and let $r \to \infty$. Hence

$$\|g_1\| + \|g_2\| \leqq g_1(x) + g_2(-x) \leqq \|g_1\| + \|g_2\| .$$

Thus g_1 and $-g_2$ are norm-preserving extensions of $f = g_1|_M$. Moreover,

$$r\|g_1\| + r\|g_2\| + \lambda \leqq g_1(y) + r\|g_1\| + g_2(y) + r\|g_2\|$$

so that

$$0 < \lambda \leqq g_1(y) + g_2(y) .$$

Thus $g_1 \neq -g_2$.

From Taylor [18], Foguel [4] and Phelps [14], we have

THEOREM 2.3. *The following statements are equivalent*:

1) *E^* is strictly convex.*
2) *Every closed subspace of E is Hahn–Banach smooth in E.*
3) *Every closed hyperplane through* 0 *in E is Hahn–Banach smooth in E.*

The following theorem in easy.

THEOREM 2.4. *The following statements are equivalent*:

1) *E is smooth.*
2) *Every one dimensional subspace of E is weakly Hahn–Banach smooth in E.*
3) *Every closed subspace of E is weakly Hahn–Banach smooth in E.*
4) *Every closed hyperplane through* 0 *in E is weakly Hahn–Banach smooth in E.*

PROOF. 1) $\Rightarrow$ 3) $\Rightarrow$ 4) and 3) $\Rightarrow$ 2) $\Rightarrow$ 1) are trivial.

4) $\Rightarrow$ 1). Assume 1) is false. Then there exist $x \in E$, $\|x\| = 1$ and $f, g \in E^*$ with $f \neq g$ such that $\|f\| = f(x) = 1 = g(x) = \|g\|$. Let $M = (\ker f \cap \ker g) + \mathbf{R} \cdot \{x\}$. Then M is a closed hyperplane such that $f = g$ on M and $\|f\| = \|f\|_M$. Thus f and g are norm-preserving extensions of $f|_M$ and M is not weakly Hahn–Banach smooth.

Lima and Uttersrud [20] have given a characterization of smooth Banach spaces as follows: E is smooth if and only if $\overline{\bigcup_{n=1}^{\infty} B(nx, n)}$ is a half-space whenever $\|x\| = 1$.

This is related to Vlasov's theorem characterizing preduals of strictly convex spaces [19]. Taking Vlasov's theorem as a starting point, we can find a characterization of Hahn–Banach smooth subspaces of E similar to Theorem 2.2.

We shall use this result in the proof of Theorem 4.5. There we prove that if $K(E)$ is Hahn–Banach smooth in $L(E)$, then E is Hahn–Banach smooth in E^{**}.

THEOREM 2.5. *Let M be a closed subspace of E. The following statements are equivalent:*

1) *M is Hahn–Banach smooth in E.*
2) *If $\varepsilon \geqq 0$, $y \in E \setminus M$ and $(a_n)_{n=1}^{\infty}$ is a sequence in M such that $\|a_1\| \leqq 1+\varepsilon$ and*

$$\|a_{n+1}-a_n\| \leqq 1+\frac{\varepsilon}{2^{n+1}} \quad \text{for all } n \geqq 1 ,$$

then $M \cap A_1 \cap A_2 \neq \emptyset$, where

$$A_i = \bigcup_{n=1}^{\infty} B\left(y+(-1)^i a_n, n+2\varepsilon-\frac{\varepsilon}{2^n}\right); \qquad i=1,2 .$$

PROOF. 1) $\Rightarrow$ 2). Assume there exist $\varepsilon \geqq 0$, $y \in E \setminus M$ and a sequence $(a_n)_{n=1}^{\infty}$ as in 2) such that $M \cap A_1 \cap A_2 = \emptyset$. Define

$$B_i = \bigcup_{n=1}^{\infty} B\left(y+(-1)^i a_n, n+\frac{3}{2}\varepsilon-\frac{\varepsilon}{2^n}\right) .$$

Then $A_i = B_i + B(0, \varepsilon/2)$. Since $\|a_1\| \leqq 1+\varepsilon$ and $\|a_{n+1}-a_n\| \leqq 1+\varepsilon/2^{n+1}$, we get

$$y \in B\left(y+(-1)^i a_n, n+\frac{3}{2}\varepsilon-\frac{\varepsilon}{2^n}\right) \subseteq B\left(y+(-1)^i a_{n+1}, n+1+\frac{3}{2}\varepsilon-\frac{\varepsilon}{2^{n+1}}\right) .$$

Thus B_i is convex for $i=1,2$.

Let Δ_M be as in the proof of Theorem 2.2. Then Δ_M and $B_1 \times B_2$ can be strongly separated. Thus as in the proof of Theorem 2.2 there exist $g, h \in E^*$ and $\lambda > 0$ such that $g+h \in M^{\perp}$ and

$$\lambda \leqq \inf_{b_i \in B_i} (g(b_1)+h(b_2)) .$$

Thus we get

$$\lambda \leqq \inf_n \left(g(y-a_n)+h(y+a_n) - (\|g\|+\|h\|)\left(n+\frac{3}{2}\varepsilon-\frac{\varepsilon}{2^n}\right)\right) .$$

Since $\|a_n\| \leqq n+\frac{3}{2}\varepsilon-\varepsilon/2^n$, we find by dividing by n and then letting $n \to \infty$, that

$$\|g\|+\|h\| \leqq \lim_{n\to\infty} (h-g)\left(\frac{a_n}{n}\right) \leqq \|g-h\| .$$

Thus $\|g\|+\|h\| = \|g-h\|$. Hence it follows that

$$(\|g\|+\|h\|)\left(n+\frac{3}{2}\varepsilon-\frac{\varepsilon}{2^n}\right)+\lambda \leqq (g+h)(y)+(h-g)(a_n)$$

$$\leqq (g+h)(y)+\|h-g\|\cdot\left(n+\frac{3}{2}\varepsilon-\frac{\varepsilon}{2^n}\right),$$

and

$$0<\lambda \leqq (g+h)(y).$$

Thus shows that M is not Hahn–Banach smooth in E.

2) ⇒ 1). Assume for contradiction that there exists $f \in M^*$, $\|f\|=1$ such that f has two different norm-preserving extensions $g, h \in E^*$. Let $y \in E \setminus M$ be such that $g(y) \neq h(y)$. Without loss of generality, we may assume $E=M \oplus \mathsf{R}\cdot\{y\}$. Define $N=\ker g \cap \ker h \subseteqq E$. Clearly $N \subseteqq M$ and $\dim E/N=2$. Now $g, h \in N^{\perp}$ and $\|g\|+\|h\|=\|g+h\|=2$. Choose $c \in E/N$ such that $1=\|c\|=g(c)=h(c)$. Notice that if $z \in B(c,1)$, then $g(z) \geqq 0$ and $h(z) \geqq 0$.

We now follow Vlasov's reasoning:

Put $c_n=nc$. Let Q be the quotient map onto E/N. Let $C_n=Q^{-1}(c_n)$. Since $g=h$ exactly on M, we get that $C_n \subseteqq M$. Let $r_n=n-\varepsilon/2^n$. First choose $a_1 \in C_1$ with $\|a_1\| \leqq 1+\varepsilon$. Next assume that $a_1,\ldots,a_n$ has been found such that $a_k \in C_k$ and $\|a_{k+1}-a_k\| \leqq r_{k+1}-r_k$, for $k=1,2,\ldots,n-1$. Since $a_n \in C_n$, we have

$$d(a_n, C_{n+1}) = \|c_{n+1}-c_n\| = 1<r_{n+1}-r_n .$$

Thus we can find $a_{n+1} \in C_{n+1}$ such that $\|a_{n+1}-a_n\| \leqq r_{n+1}-r_n$. Since $r_{n+1}-r_n =1+\varepsilon/2^{n+1}$, we have found a sequence in M as in 2). By 2) there exist $z \in M$ and n such that for $i=1,2$.

$$\|y+(-1)^i a_n-z\| \leqq n+2\varepsilon-\frac{\varepsilon}{2^n} \leqq n+2\varepsilon .$$

This can be written as

$$\max_{\pm} \|a_n \pm (y-z)\| \leqq n+2\varepsilon .$$

Hence

$$\begin{aligned} n+2\varepsilon &\geqq \max_{\pm} |g(a_n) \pm g(y-z)| \\ &= \max_{\pm} |g(c_n) \pm g(y-z)| \\ &= \max_{\pm} |n \pm (y-z)| \\ &= n+|g(y-z)| . \end{aligned}$$

Thus $2\varepsilon \geqq |g(y-z)|$.

Similarly $2\varepsilon \geqq |h(y-z)|$.

Since $z \in M$, we have $g(z)=h(z)=f(z)$. Thus

$$|g(y)-h(y)| \leqq 4\varepsilon .$$

Starting with a sufficiently small $\varepsilon>0$, we obtain a contradiction.

We shall use Theorem 2.5 in section 4. But first we need some results about another intersection property.

3. Liftings and intersections of balls.

We shall assume M is a closed subspace of E. Let $f_1,\ldots,f_n \in M^*$ with $f_1+\ldots+f_n=0$. We shall find conditions on M which ensure the existence of norm-preserving extensions $\hat{f}_i$ such that $\hat{f}_1+\ldots+\hat{f}_n=0$ in E^*.

DEFINITION. Let $n \geqq 3$ be a natural number. We shall say that M has the *n.E. intersection property* (*n.E.*I.P.) if whenever $\{B(a_i,r_i)\}_{i=1}^n$ are n closed balls in M with $\bigcap_{i=1}^n B(a_1,r_i) \neq \emptyset$ in E, then $M \cap \bigcap_{i=1}^n B(a_i,r_i+\varepsilon) \neq \emptyset$ for all $\varepsilon>0$.

The following result is the main theorem.

THEOREM 3.1. *Let $n \geqq 3$. The following statements are equivalent:*

1) *M has the n.E.*I.P.
2) *$M^{\perp\perp}$ has the n.E^{**}.*I.P.
3) *If $f_1,\ldots,f_n \in M^*$ are such that $f_1+\ldots+f_n=0$, then there exist norm-preserving extensions $\hat{f}_i$ to E such that $\hat{f}_1+\ldots+\hat{f}_n=0$.*
4) *If $f_1,\ldots,f_n \in E^*$ with $f_1+\ldots+f_n=f \in M^\perp$ and $r_i=d(f_i,M^\perp)$, then there exist $h_i \in M^\perp \cap B(f_i,r_i)$ such that $h_1+\ldots+h_n=f$.*

PROOF. 2) $\Rightarrow$ 1) follows from the "principle of local reflexivity" [13] since we can identify $M^{\perp\perp}$ with M^{**}.

3) $\Rightarrow$ 1). Let $\{B(a_i,r_i)\}_{i=1}^n$ be n balls in M such that $\bigcap_{i=1}^n B(a_i,r_i) \neq \emptyset$ in E. Let $f_1,\ldots,f_n \in M^*$ be such that $f_1+\ldots+f_n=0$. By 3) there exist norm-preserving extensions $\hat{f}_i$ such that $\hat{f}_1+\ldots+\hat{f}_n=0$.

Let $a \in \bigcap_{i=1}^n B(a_i,r_i)$.

Then we have

$$\left|\sum_{i=1}^{n} f_i(a_i)\right| = \left|\sum_{i=1}^{n} \hat{f}_i(a_i)\right|$$
$$= \left|\sum_{i=1}^{n} \hat{f}_i(a_i - a)\right|$$
$$\leqq \sum_{i=1}^{n} r_i \|\hat{f}_i\|$$
$$= \sum_{i=1}^{n} r_i \|f_i\| .$$

By Theorem 1.1 in [10], we get that

$$M \cap \bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) \neq \emptyset \quad \text{for all } \varepsilon > 0 .$$

1) $\Rightarrow$ 3). We introduce sets $A \subseteq (M^* \oplus \ldots \oplus M^*)_{l_1^n}$ and $B \subseteq (E^* \oplus \ldots \oplus E^*)_{l_1^n}$ as follows:

$$A = \left\{(f_1, \ldots, f_n) : \sum_{i=1}^{n} f_i = 0 \text{ and } \sum_{i=1}^{n} \|f_i\| \leqq 1\right\}$$

and

$$B = \left\{(g_1, \ldots, g_n) : \sum_{i=1}^{n} g_i = 0 \text{ and } \sum_{i=1}^{n} \|g_i\| \leqq 1\right\} .$$

Let Q: $(E^* \oplus \ldots \oplus E^*)_{l_1^n} \to (M^* \oplus \ldots \oplus M^*)_{l_1^n}$ be defined by

$$Q(g_1, \ldots, g_n) = (g_1|_M, \ldots, g_n|_M) .$$

$Q(B)$ is a convex w^*-compact subset of A. Clearly it suffices to show that $Q(B) = A$. Assume for contradiction that there exists $(f_1, \ldots, f_n) \in A \setminus Q(B)$. By the Hahn–Banach theorem there exist $a_1, \ldots, a_n \in M$ such that

$$\sum_{i=1}^{n} f_i(a_i) > 1 = \sup_{(g_1, \ldots, g_n) \in B} \sum_{i=1}^{n} g_i(a_i) .$$

By Theorem 1.1 in [10], we have $\bigcap_{i=1}^{n} B(a_i, 1+\varepsilon) \neq \emptyset$ in E for all $\varepsilon > 0$, and

$$M \cap \bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) = \emptyset \quad \text{for some } \varepsilon > 0.$$

3) $\Leftrightarrow$ 4) is trivial.

4) $\Rightarrow$ 2) follows by using Theorem 1.2 in [10].

Note that it follows from the proof of 1) $\Rightarrow$ 3) that we can take all $r_i = 1$ in the definition of the $n.E.$I.P. This also follows from Theorem 4.3 in [12].

REMARKS.

a) Let $E = C[0,1]$ and let M be a subspace of E isometric to l_1. Since l_1 has the 3.2.I.P. but not the 4.2.I.P., it follows that M has the 3.E.I.P. but not the 4.E.I.P.
b) Let $E = 1^3_\infty$ and let $M = \{(x, y, z) \in E : x + y + z = 0\}$. It is easy to see that M does not have the 3.E.I.P.
c) From the "principle of local reflexivity", it easily follows that every Banach space M has the $n.M^{**}$.I.P. for all n.

From Theorem 3.1 and the proof of Theorem 5.9 in [12] we get the following result.

PROPOSITION 3.2. *The statements below are related as follows* 1) ⇒ 2) ⇒ 3) ⇔ 4):

1) *There exists a norm* 1 *projection in E with range M.*
2) *There exists a norm* 1 *projection in E^* with kernel $M^\perp$.*
3) *M has the $n.E$.I.P. for all n.*
4) *For each Banach space Y such that $M^{\perp\perp} \subseteq Y \subseteq E^{**}$ and* $\dim Y/M^{\perp\perp} = 1$, *there is a norm* 1 *projection from Y onto $M^{\perp\perp}$.*

REMARKS.

a) Clearly 2) ⇏ 1) in Proposition 3.2, but we do not know if 3) ⇒ 2).
b) We do not know if there exists a number $k \geqq 4$ such that if M has the $k.E$.I.P., then M has the $n.E$.I.P. for all $n \geqq k$.
c) Using Helly's theorem [5], we get that if $\dim M = k < \infty$ and M has the $(k+1)$. E.I.P., then M has the $n.E$.I.P. for all n.
d) From Proposition 3.2 and [8], we get that E is isometric to a Hilbert space if and only if every two-dimensional subspace of E has the 3.E.I.P. This result was first proved by Comfort and Gordon in [2].

We refer to [10] for the definition of M-ideals and semi M-ideals. An easy corollary of Theorem 3.1 is the following result.

COROLLARY 3.3. *Assume M is a semi M-ideal in E. Then the following statements are equivalent*:

1) *M is an M-ideal in E.*
2) *M has the $n.E$.I.P. for all n.*
3) *M has the* 3.E.I.P.

An easy corollary of this result and the Remarks above, is the following result of Saatkamp [15].

COROLLARY 3.4. *If M is a semi M-ideal in M^{**}, then M is an M-ideal in M^{**}.*

From a result of J. Johnson [7], we get:

PROPOSITION 3.5. *If F or E^* has the metric approximation property, then $K(E,F)$ has the n.$L(E,F)$.I.P. for all n. Moreover, if also $K(E,F)$ is a semi M-ideal in $L(E,F)$, then $K(E,F)$ is an M-ideal.*

We shall end this section by considering which subspaces of $L_1(\mu)$-spaces and predual $L_1(\mu)$-spaces have the *n.E.*I.P.

PROPOSITION 3.6. *Let $E = L_1(\mu)$ and let M be a closed subspace of M. Then M has the* 3.*E.*I.P. *if and only if M is the range of a norm-*1 *projection in E.*

PROOF. One way is trivial.

Assume M has the 3.*E.*I.P. Then M has the 3.2.I.P. By Theorem 4.3, Theorem 3.12, and Corollary 3.3 in [10], it follows that M is isometric to an $L_1(\nu)$-space. By Theorem 6.3 in [9] it follows that M is the range of a norm-1 projection in E.

PROPOSITION 3.7. *Assume $E^* = L_1(\mu)$ and that M is a subspace of E. Then M has the* 4.*E.*I.P. *if and only if $M^\perp$ is the kernel of a norm-*1 *projection in E^*.*

PROOF. Use proposition 3.8 and Theorem 2.17 in [10].

4. HB-subspaces.

Hennefeld [6] call a subspace M of E a HB-subspace if $M^\perp$ is complemented by a subspace M_* such that whenever $f_* \in M_*$ and $f^\perp \in M^\perp \setminus \{0\}$, then $\|f_* + f^\perp\| \geqq \|f^\perp\|$ and $\|f_* + f^\perp\| > \|f_*\|$.

We use the notation $M^\# = \{f \in E^* : \|f\| = \|f\|_M\}$.

THEOREM 4.1. *The following statements are equivalent*:

1) *M is a* HB-*subspace of E.*
2) *$M^\#$ is a linear subspace.*
3) *If $f_1, f_2, f_3 \in M^\#$ with $f_1 + f_2 + f_3 \in M^\perp$, then $f_1 + f_2 + f_3 = 0$.*
4) *M is Hahn–Banach smooth in E and has the* 3.*E.*I.P.
5) *M is Hahn–Banach smooth in E and has the n.E.*I.P. *for all $n \geqq 3$.*

PROOF. 5) $\Rightarrow$ 4) is trivial.

4) $\Rightarrow$ 3) follows from Theorem 3.1 since $f \in M^{\#}$ implies that f is a norm-preserving extension of $f|_M$.

3) $\Rightarrow$ 2). Let $f_1, f_2 \in M^{\#}$. Then we can write $f_1 + f_2 = -f_3 + f$ where $f_3 \in M^{\#}$ and $f \in M^{\perp}$.

By 3) $f_1 + f_2 = -f_3 \in M^{\#}$.

2) $\Rightarrow$ 5). Let $f \in M^*$ and let $g, h \in E^*$ be norm-preserving extensions of f. Then $g, h \in M^{\#}$ and $g - h \in M^{\perp}$. Thus $g - h = 0$ and M is Hahn–Banach smooth in E. Let P be the projection in E^* with range $M^{\#}$ and kernel $M^{\perp}$. Then $\|P\| = 1$ and M has the $n.E.$I.P. by Proposition 3.2.

1) $\Rightarrow$ 5) follows from Lemma 1.2 and 1.3 in [6].

5) $\Rightarrow$ 1). Define $M_* = M^{\#}$. Clearly if $f \in M^{\#}$ and $g \in M^{\perp} \setminus \{0\}$, then $\|f+g\| \geqq \|f+g\|_M = \|f\|$ and $\|f+g\| > \|f\|$ since M is Hahn–Banach smooth.

COROLLARY 4.2. *M is a* HB-*subspace of M^{**} if and only if M is Hahn–Banach smooth in M^{**}.*

COROLLARY 4.3. *If M is a* HB-*subspace of M^{**}, then M^* has the Radon–Nikodym property.*

PROOF. It follows from [16] and Corollary 4.2.

COROLLARY 4.4. *If E^* or F has the metric approximation property, then $K(E,F)$ is a* HB-*subspace of $L(E,F)$ if and only if $K(E,F)$ is Hahn–Banach smooth in $L(E,F)$.*

In [11], we proved that if $K(E)$ is an M-ideal in $L(E)$, then E is an M-ideal in E^{**}. A similar result is true for HB-subspaces.

THEOREM 4.5. *Assume $K(E)$ is a* HB-*subspace of $L(E)$. Then E is a* HB-*subspace of E^{**}. In particular E^* has the Radon–Nikodym property.*

Note that similar results are true if we replace the word HB-subspace by Hahn–Banach smooth or by weakly Hahn–Banach smooth.

PROOF. By Proposition 3.6 and Theorem 4.1, it suffices to show that E is Hahn–Banach smooth in E^{**}. To this end we use Theorem 2.5.

Let $\varepsilon > 0$ and let $y \in E^{**} \setminus E$. Clearly we may assume that $\|y\| = y(f)$ for some $f \in E^*$ with $\|f\| = 1$. (We use the Bishop–Phelps theorem.) Let $(a_n)_{n=1}^{\infty}$ be a sequence in E such that $\|a_1\| \leqq 1+\varepsilon$ and $\|a_{n+1}-a_n\| \leqq 1+\varepsilon/2^{n+1}$. Define $S_n \in K(E)$ by

$$S_n(u) = f(u)a_n .$$

Then $\|S_1\| \leqq 1+\varepsilon$ and $\|S_{n+1}-S_n\| \leqq 1+\varepsilon/2^{n+1}$.

By Theorem 2.5 there exist $T \in K(E)$ and n such that

$$\max_{\pm} \|S_n \pm (I-T)\| \leqq n+2\varepsilon-\frac{\varepsilon}{2^n} .$$

Thus

$$\begin{aligned} n+2\varepsilon-\varepsilon/2^n &\geqq \max_{\pm} \|S_n^{**} \pm (I-T^{**})\| \\ &\geqq \max_{\pm} \|S_n^{**}y \pm (y-T^{**}y)\| \\ &= \max_{\pm} \|a_n \pm (y-T^{**}y)\| . \end{aligned}$$

Since T is compact, we have $T^{**}y \in E$. Thus E is Hahn–Banach smooth in E^{**} by Theorem 2.5.

THEOREM 4.6. *Assume M is a closed subspace of E and that E is smooth and reflexive. Then the following statements are equivalent:*

1) *M is the range of a norm* 1 *projection in E.*
2) *M has the n.E.I.P. for all $n \geqq 3$.*
3) *M has the* 3.E.I.P.
4) *M is a* HB-*subspace of E.*

PROOF. Since E is smooth and reflexive, it follows that M is Hahn–Banach smooth in E. The theorem now follows from Theorem 4.1 and Proposition 3.2.

From [9], we now get:

COROLLARY 4.7. *Let $E=L_p(\mu)$ for some measure μ and $1<p<\infty$. A subspace M of E has the* 3.E.I.P. *if and only if M is isometric to an $L_p(\nu)$ space.*

THEOREM 4.8. *Assume M has the* 3.E.I.P. *If M is weakly Hahn–Banach smooth in E, then $M^{\perp}$ is the kernel of a norm-*1 *projection in E^*.*

PROOF. For each $f \in M^*$, let $P(f)$ denote the non-empty convex and w^*-compact set of norm-preserving extensions of f. Clearly it suffices to find a linear selection of the map $f \dashrightarrow P(f)$.

If $f \in M^*$ attains its norm on M_1, let $\hat{f}$ be the unique norm-preserving extension of f. Then $P(f)=\{\hat{f}\}$.

Assume $f, g \in M^*$ both attains their norms on M_1. Then by Theorem 3.1, we get $\|f-g\|=\|\hat{f}-\hat{g}\|$. By the Bishop–Phelps theorem [3], the norm-attaining functionals in M^* are norm-dense. Hence we get that if $f \in M^*$, then there exists a unique $\tilde{f} \in P(f)$ such that if $f_\alpha \to f$ in norm and each f_α attain its norm, then $\hat{f}_\alpha \to \tilde{f}$ in norm. The selection $f \to \tilde{f}$ is linear.

The projection is $f \to (\tilde{f}|_M)\hat{}$.

COROLLARY 4.9. *Assume M is weakly Hahn–Banach smooth in E. Then M has the* 3.E.I.P. *if and only if* $M^{\perp\perp}$ *is the range of a norm-*1 *projection in* E^{**}.

COROLLARY 4.10. *Assume E is a smooth Banach space and that M is a closed subspace. If M has the* 3.E.I.P., *then M has the n.*E.I.P. *for all n, and* $M^\perp$ *is the kernel of a norm-*1 *projection in* E^*.

PROOF. Use Theorem 4.8, Proposition 3.2, and Theorem 2.4.

In [21] Belobrov studied Banach spaces which are Hahn–Banach smooth in their biduals.

He showed the following result under the stronger hypothesis that E is Hahn–Banach smooth (rather than weakly Hahn–Banach smooth).

THEOREM 4.11. *Assume E is weakly Hahn–Banach smooth in* E^{**}. *The following statements are true*:

1) *If M is a closed subspace of E, then M is weakly Hahn–Banach smooth in* M^{**}.
2) *If E is the range of a norm-*1 *projection in* E^{**}, *then E is reflexive.*

PROOF. 1). Let $f \in M^*$ and assume f attains its norm on M_1. Let f_1, f_2 be two norm-preserving extensions of f to E. By 1) each f_i has a unique norm-preserving extension $\hat{f}_i$ to E^{**} defined by $\hat{f}_i(y)=y(f_i)$. If $y \in M^{\perp\perp}=M^{**}$ and $(x_\alpha)_\alpha$ is a net in M converging weak* to y, then

$$\hat{f}_1(y) = y(f_1) = \lim_\alpha x_\alpha(f_1) = \lim_\alpha x_\alpha(f_2) = y(\hat{f}_2) = \hat{f}_2(y) .$$

Thus $\hat{f}_1=\hat{f}_2$ on $M^{\perp\perp}$.

Next let g, h be two norm-preserving extensions of f to $M^{\perp\perp}$. Then g and h have norm-preserving extensions $\tilde{g}$ and $\tilde{h}$ to E^{**}. Clearly $\tilde{g}=(\tilde{g}|_E)\hat{}$ and $\tilde{h}=(\tilde{h}|_E)\hat{}$ and by the first part of the proof, if $y \in M^{\perp\perp}$, then $g(y)=\tilde{g}(y)=\tilde{h}(y)=h(y)$. Thus f has a unique norm-preserving extension to $M^{\perp\perp}=M^{**}$.

2). Here we follow Belobrov's argument. Assume P is a norm-1 projection in E^{**} with range E. Assume there exists $x^{**} \in \ker P \setminus \{0\}$. Let $f \in E^*$ with $\|f\| = 1$ and $2x^{**}(f) > \|x^{**}\|$ and $x^{**}(f) \neq Px^{**}(f)$. By the Bishop–Phelps theorem we may assume f attains its norm on E_1. P^* is a norm-1 projection in E^{***} with kernel $E^{\perp}$. Let $\hat{f}$ be the unique norm-preserving extension of f to E^{**}. Thus $\hat{f} \in E^{***}$.

Then we have

$$P^*\hat{f}(x^{**}) = \hat{f}(Px^{**}) = (Px^{**})(f) \neq x^{**}(f) = \hat{f}(x^{**}) .$$

Moreover, if $y \in E$ with $\|y\| = 1$ and $f(y) = \|f\|$, then

$$P^*\hat{f}(y) = \hat{f}(Py) = \hat{f}(y) .$$

Thus $P^*\hat{f}$ and $\hat{f}$ are two different norm-preserving extensions of f to E^{**}. This is a contradiction. Hence $\ker P = (0)$.

5. More about liftings and intersections of balls.

We shall now dualize Theorem 3.1. We can prove the following result.

THEOREM 5.1. *Assume M is a closed subspace of E. Let $n \geqq 3$ be a natural number and let $\varphi: E \to E/M$ be the quotient map. The following statements are equivalent:*

1) *$M^{\perp}$ has the n.E^*.I.P.*
2) *If $x_1, \ldots, x_n \in E/M$ with $x_1 + \ldots + x_n = 0$ and $\varepsilon > 0$, then there exist $y_i \in E$ such that $\varphi(y_i) = x_i$, $y_1 + \ldots + y_n = 0$, and $\|y_i\| \leqq \|x_i\| + \varepsilon$ for all i.*
3) *If $y_1, \ldots, y_n \in E$ with $y = y_1 + \ldots + y_n \in M$ and $\varepsilon > 0$ and $r_i = d(y_i, M)$, then there exist $x_i \in M \cap B(y_i, r_i + \varepsilon)$ such that $y = x_1 + \ldots + x_n$.*

PROOF. 3) $\Rightarrow$ 2). Let $x_1, \ldots, x_n \in E/M$ with $x_1 + \ldots + x_n = 0$ and let $\varepsilon > 0$. Choose $z_i \in E$ with $\varphi(z_i) = x_i$ and $\|z_i\| \leqq \|x_i\| + \varepsilon$. Let $r_i = d(z_i, M)$ and let $z = z_1 + \ldots + z_n$. Then $z \in M$ and hence, there exist $y_i \in M \cap B(z_i, r_i + \varepsilon)$ such that $z = y_1 + \ldots + y_n$. Then we have $\varphi(z_i - y_i) = x_i$, $\|z_i - y_i\| \leqq \|x_i\| + \varepsilon$, and $(z_1 - y_1) + \ldots + (z_n - y_n) = 0$.

2) $\Rightarrow$ 3). Choose liftings z_i of $\varphi(y_i)$ as in 2) and let $x_i = y_i - z_i$.

2) $\Rightarrow$ 1). Let $\{B(f_i, r_i)\}_{i=1}^n$ be n balls in $M^{\perp}$ and assume there exists $f \in E^*$ such that $\|f - f_i\| \leqq r_i$ for all i. Let $x_1, \ldots, x_n \in E/M$ with $x_1 + \ldots + x_n = 0$ and let $\varepsilon > 0$. Let y_i be as in 2). Then we have, since $f_i \in M^{\perp}$,

$$\sum_{i=1}^{n} f_i(x_i) = \sum_{i=1}^{n} f_i(y_i)$$

$$= \sum_{i=1}^{n} (f_i - f)(y_i)$$

$$\leqq \sum_{i=1}^{n} r_i \|y_i\|$$

$$\leqq \sum_{i=1}^{n} r_i(\|x_i\| + \varepsilon) .$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\sum_{i=1}^{n} f_i(x_i) \leqq \sum_{i=1}^{n} r_i \|x_i\| .$$

By Theorem 1.2 in [10], it follows that $M^\perp \cap \bigcap_{i=1}^n B(f_i, r_i) \neq \emptyset$.

1) $\Rightarrow$ 2). This is similar to the proof of 1) $\Rightarrow$ 3) in Theorem 3.1.

Note that if M is proximinal, then we can take $\varepsilon = 0$ in Theorem 4.1.

If M is the kernel of a norm 1 projection or $M^\perp$ is the image of a norm 1 projection, then $M^\perp$ has the $n.E.^*$I.P. for all n.

PROPOSITION 5.2. *Assume M has the Haar property,* i.e. *for each $x \in E$, there is a unique $y \in M$ such that $\|x - y\| = d(x, M)$. Then $M^\perp$ has the* 3.$E.^*$I.P. *if and only if M is the kernel of a norm* 1 *projection.*

PROOF. Let P be a norm 1 projection in E with $\ker P = M$. Then P^* is a norm 1 projection in E^* with range $M^\perp$. Hence $M^\perp$ has the $n.E.^*$I.P. for all n.

Assume conversely that $M^\perp$ has the 3.$E.^*$I.P. Let

$$M^\ominus = \{x \in E : \|x\| = d(x, M)\} .$$

Clearly $M \cap M^\ominus = (0)$ and $M + M^\ominus = E$. It suffices to show that $M^\ominus$ is a linear subspace. Let $x_1, x_2 \in M^\ominus$. Then we can write $x_1 + x_2 = -x_3 + x$ where $x_3 \in M^\ominus$ and $x \in M$. Since we can take $\varepsilon = 0$ in Theorem 5.1 and $M \cap B(x_i, \|x_i\|) = \{0\}$, it follows from 3) in Theorem 5.1 that $x = 0$. Thus $M^\ominus$ is a linear subspace of E. The projection in E onto $M^\ominus$ with kernel M has norm 1.

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