

RATIONAL APPROXIMATIONS TO STIELTJES TRANSFORMS

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1.

Rational sums of the form $\sum a_i/(x + b_i)$ where a_i and b_i are positive can be expressed as Stieltjes transforms of discrete positive measures. The Stieltjes transforms of the measure $\alpha(t)$ is the function

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t}.$$

Rational approximations that interpolate such functions on positive intervals are particularly amenable to analysis because all of the poles of these approximations lie on the negative real axis [1]. Furthermore, if $g(x)$ is the Stieltjes transform of $\beta(t)$, $f(x)$ is the transform of $\alpha(t)$ and if α, β and $\alpha - \beta$ are all positive measures then g can be approximated more closely than f by rational functions on any positive interval (see Theorem 1). We will exploit these two observations to analyse the rate of rational approximation to certain functions of the form $\sum a_i/(x + b_i)$.

Let Π_n denote the real algebraic polynomials of degree at most n . Let $R_{n,m}$ denote the rational functions with numerators in Π_n and denominators in Π_m . Let

$$r_{n,m}(f: [a, b]) = \inf_{r \in R_{n,m}} \|f(x) - r(x)\|_{[a, b]}$$

where $\|\cdot\|_{[a, b]}$ denotes the supremum norm on $[a, b]$.

In a seminal paper ([6], see also [7]) Gončar shows that if f is the Stieltjes transform of a positive measure α with support in the interval $[a, b]$, if $\alpha > 0$ almost everywhere on $[a, b]$ and if $c > -a$ then

$$\lim_{n \rightarrow \infty} r_{n-1, n}(f: [c, d])^{1/n} = \frac{1}{q^2} < 1$$

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where ρ depends only on $a, b, c,$ and d . These results have been extended by Ganelius [5] who shows that under slightly more restrictive conditions

$$k_1 \leq r_{n-1,n}(f: [c, d])\rho^{2n} \leq k_2 .$$

Ganelius [4] also shows that for non-integral positive $\delta,$

$$b_\delta |\sin \pi \delta| \leq r_{n-1,n}(x^\delta: [0, 1])e^{2\pi/\delta n} \leq C_\delta e^{c_\delta n^{1/4}}$$

where $b_\delta, c_\delta,$ and C_δ depend only on δ . This settles the conjecture of Gončar that

$$(1) \quad \lim n^{-1/2} \ln r_{n-1,n}(x^\delta: [0, 1]) = -2\pi/\sqrt{\delta} .$$

As a corollary to Theorem 1 we deduce that

$$\lim_{n \rightarrow \infty} n^{-1/2} \ln r_{n-1,n}(x \ln x: [0, 1]) = -2\pi .$$

This amounts in some sense to the $\delta=1$ case of Gončar's conjecture.

In contrast to the above situation we will also consider functions which arise as transforms of discrete measures, that is, functions of the form $\sum_{i=1}^\infty a_i/(x+b_i), a_i, b_i \geq 0$. We will, for example, obtain results of the following nature:

$$(a) \quad r_{n-1,n}\left(\sum_{i=1}^{n+1} \frac{1}{x+i}: [0, 1]\right) = \frac{a_n}{16^n(n!)^2} \quad \text{where } \lim_{n \rightarrow \infty} a_n^{1/2n} = .278 \dots$$

and

$$(b) \quad r_{n-1,n}\left(\sum_{i=1}^{n+1} \frac{1}{x+i^2}: [0, 1]\right) = \frac{b_n}{16^n(n!)^4} \quad \text{where } \lim_{n \rightarrow \infty} b_n^{1/2n} = .439 \dots$$

The convergence problem for Padé approximants to functions of the form $\sum a_i/(x+b_i)$ is treated by Franzen in [3].

2. A comparison theorem.

A particularly useful theorem in polynomial approximation theory due to Bernstein states that if $|g^{(n+1)}(x)| \leq f^{(n+1)}(x)$ on $[a, b]$, then the error in best uniform polynomial approximation of degree n to g is no greater than the corresponding error in approximating f . Our first result is a modest extension of this to the case of rational approximations to Stieltjes transforms.

THEOREM 1. *Let*

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t} \quad \text{and} \quad g(x) = \int_0^\infty \frac{d\beta(t)}{x+t} .$$

Suppose that α, β and $\alpha - \beta$ are all non-negative measures. Suppose that for $k \geq 0$

$$\frac{q_{n+k-1}(\zeta_i)}{p_n(\zeta_i)} - f(\zeta_i) = 0 = \frac{q_{n+k-1}^*(\zeta_i)}{p_n^*(\zeta_i)} - g(\zeta_i)$$

at $2n+k$ points $0 \leq \zeta_1 \leq \zeta_2 \dots \leq \zeta_{2n+k}$, where

$$q_{n+k-1}/p_n, q_{n+k-1}^*/p_n^* \in R_{n+k-1, n}.$$

Then, for $x > 0$,

$$\left| g(x) - \frac{q_{n+k-1}^*(x)}{p_n^*(x)} \right| \leq \left| f(x) - \frac{q_{n+k-1}(x)}{p_n(x)} \right|.$$

PROOF. If k of the ζ_i coincide, then we are assuming that $f - q_{n+k-1}/p_n$ and $g - q_{n+k-1}^*/p_n^*$ have zeros of multiplicity k at those ζ_i .

We may suppose that the ζ_i are distinct and that

$$f(x) = \sum_{i=1}^{\infty} \frac{\gamma_i}{x + \alpha_i} \quad \text{and} \quad g(x) = \sum_{i=1}^{\infty} \frac{\delta_i}{x + \alpha_i}$$

where for all i ,

$$0 \leq \alpha_i < \alpha_{i+1} \quad \text{and} \quad 0 \leq \delta_i \leq \gamma_i.$$

(The general argument is completed by taking limits.) Let I_f be the index of the first non-zero γ_i , and let I_g be the index of the first non-zero δ_i . Then, if $\beta = \alpha_{I_f}$ and $\beta^* = \alpha_{I_g}$, it follows from results in [1] that

$$\frac{q_{n+k-1}}{p_n} = \bar{q}_{k-1} + \sum_{k=1}^n \frac{c_i}{x + d_i} \quad d_i > \beta, c_i > 0$$

and

$$\frac{q_{n+k-1}^*}{p_n^*} = \bar{q}_{k-1}^* + \sum_{i=1}^n \frac{e_i}{x + h_i} \quad h_i > \beta^*, e_i > 0$$

where $\bar{q}_{k-1}, \bar{q}_{k-1}^* \in \pi_{k-1}$. ($\pi_{-1} \equiv 0$.)

Furthermore,

$$F(x) := \frac{q_{n+k-1}}{p_n} - f \quad \text{and} \quad G(x) := \frac{q_{n+k-1}^*}{p_n^*} - g$$

have exactly $2n+k$ simple zeros on

$$[-\beta, \infty) \quad \text{and} \quad [-\beta^*, \infty)$$

respectively. Also,

$$\lim_{x \rightarrow -\beta^+} F(x) = \lim_{x \rightarrow (-\beta^*)^+} G(x) = -\infty.$$

It follows that

$$\operatorname{sgn} F(x) = \operatorname{sgn} G(x) \quad \text{for } x \in [0, \infty) .$$

If there exists $x_0 > 0$, $x_0 \notin \{\zeta_1, \dots, \zeta_{2n+k}\}$ so that

$$(2) \quad |F(x_0)| \leq |G(x_0)|$$

then there exists $c > 1$ so that on $[0, \infty)$

$$cF(x) - G(x) \quad \text{has } 2n+k+1 \text{ zeros .}$$

Thus,

$$\frac{cq_{n+k-1}}{p_n} = cf - g + \frac{q_{n+k-1}^*}{p_n^*}$$

has $2n+k+1$ non-negative solutions.

If we differentiate the above k times we see that

$$c \sum_{i=1}^n \frac{c_i}{(x+d_i)^{k+1}} = \sum_{i=1}^{\infty} \frac{c\gamma_i - \delta_i}{(x+\alpha_i)^{k+1}} + \sum_{i=1}^n \frac{e_i}{(x+h_i)^{k+1}}$$

has $2n+1$ negative solutions. Since $c\gamma_i - \delta_i \geq 0$, this violates Descartes rule of signs (see [1] for further details). Thus, assumption (2) is not possible and the proof is complete.

The interpolation condition in Theorem 1 is satisfiable for all choices of non-negative ζ_i (see [1]). This observation yields the following corollaries.

COROLLARY 1. *Let*

$$f(x) = \int_0^\infty \frac{d\alpha(t)}{x+t} \quad \text{and} \quad g(x) = \int_0^\infty \frac{d\beta(t)}{x+t} .$$

Suppose that α , β and $\alpha - \beta$ are all non-negative measures. Then, for any $n, k, a, b \geq 0$,

$$r_{n+k-1, n}(g; [a, b]) \leq r_{n+k-1, n}(f; [a, b]) .$$

COROLLARY 2. *Let f and g be as above. Let $p_{n+k-1, n}(f; x)$ be the $(n+k-1, n)$ Padé approximant to f concentrated at the point $a \geq 0$. Then, for $x \geq 0$*

$$|g(x) - p_{n+k-1, n}(g; x)| \leq |f(x) - p_{n+k-1, n}(f; x)| .$$

As an application of Theorem 1 we have

COROLLARY 3.

$$\lim_{n \rightarrow \infty} n^{-1/2} \ln r_{n,n}(x \ln x: [0, 1]) = -2\pi .$$

PROOF. FOR $\delta \in (0, 1)$, $x \geq 0$

$$x^{\delta-1} = \frac{\sin(\delta\pi)}{\pi} \int_0^\infty \frac{t^{\delta-1} dt}{t+x} .$$

Let $s_{n-1,n} \in R_{n-1,n}$ interpolate $x^{\delta-1}$ at any $2n$ points in $(0, 1]$ and let $t_{n-1,n} \in R_{n-1,n}$ interpolate

$$\frac{\sin(\delta\pi)}{\pi} (\ln(x+1) - \ln x) = \frac{\sin(\delta\pi)}{\pi} \int_0^1 \frac{dt}{t+x}$$

at the same points. By Theorem 1, for $x > 0$,

$$\left| t_{n-1,n}(x) - \frac{\sin(\delta\pi)}{\pi} (\ln(x+1) - \ln x) \right| \leq |s_{n-1,n}(x) - x^{\delta-1}|$$

and for suitably chosen interpolation points,

$$\begin{aligned} \left| x t_{n-1,n}(x) - \frac{\sin(\delta\pi)}{\pi} (x \ln(x+1) - x \ln x) \right| &\leq |x s_{n-1,n}(x) - x^\delta| \\ &\leq 2r_{n,n}(x^\delta: [0, 1]) \\ &\leq C_\delta e^{c_\delta n^{1/4}} / e^{2\pi\sqrt{\delta n}} \end{aligned}$$

where the latter inequality, due to Ganelius, was mentioned in the introduction. Since $x \ln(x+1)$ is analytic in a region containing $[0, 1]$ there exists $\varrho < 1$ so that

$$r_{n,n}(x \ln(x+1): [0, 1]) < \varrho^n$$

and hence

$$r_{n,n}(x \ln x: [0, 1]) \leq \left(\frac{C_\delta e^{c_\delta (n-n^{2/3})^{1/4}}}{e^{2\pi\sqrt{\delta(n-n^{2/3})}} + \varrho^{n^{2/3}} \right) \left(\frac{\pi}{\sin(\delta\pi)} \right) .$$

Taking logarithms and letting δ tend to 1 yields

$$\overline{\lim} n^{-1/2} \ln r_{n,n}(x \ln x: [0, 1]) \leq -2\pi .$$

The lower bound is achieved by observing that

$$\int_0^1 \frac{t^\delta dt}{t+x} = k_\delta x^\delta + f_\delta(x) \quad \delta \in (0, 1)$$

where $f_\delta(x)$ is analytic in $\{|z-1/2|<1\}$. We observe that by Theorem 1 (applied to the above and $\ln(x+1)-\ln x$) we have

$$2r_{n,n}(x \ln(x+1) - x \ln(x) : [0, 1]) \geq r_{n,n}(k_\delta x^{\delta+1} + x f_\delta(x) : [0, 1])$$

and the lower bound is now completed in a similar fashion to the upper bound.

If f and g are Stieltjes transforms of non-negative measures then an immediate consequence of Corollary 1 is that for $a, b \geq 0$,

$$r_{n+k-1,n}(f+g : [a, b]) \geq \max(r_{n+k-1,n}(f : [a, b]), r_{n+k-1,n}(g : [a, b])).$$

Another application of Theorem 1 is

COROLLARY 4. *Suppose that $0 < \gamma_1 < \gamma_2 < \dots < \gamma_m < 1$ and suppose that $c_1, c_2, \dots, c_m > 0$. Then*

$$\lim_{n \rightarrow \infty} n^{-1/2} \ln \left(r_{n,n} \left(\sum_{i=1}^m c_i x^{\gamma_i} : [0, 1] \right) \right) = -2\pi \sqrt{\gamma_1}.$$

PROOF. That the limit exceeds $-2\pi \sqrt{\gamma_1}$ is apparent from the comment preceding the Corollary and (1). To derive an upper bound on the limit we observe once again that

$$c_i x^{\gamma_i} = d_i \int_0^1 \frac{t^{\gamma_i}}{x+t} + h_{\gamma_i}(x)$$

where h_{γ_i} is analytic on $\{|z-1/2|<1\}$. Thus,

$$\sum_{i=1}^m c_i x^{\gamma_i} = \int_0^1 \frac{\sum_{i=1}^m d_i t^{\gamma_i}}{x+t} + h^*(x)$$

where h^* is analytic on $\{|z-1/2|<1\}$. We note that for $t \in [0, 1]$,

$$0 < \sum_{i=1}^n d_i t^{\gamma_i} \leq \left(\sum_{i=1}^n d_i \right) t^{\gamma_1}.$$

We may now compare, as in the proof of Corollary 3, the rational approximation to $\sum_{i=1}^n c_i x^{\gamma_i}$, and the known rational approximation to x^{γ_1} .

3. Approximating rational sums.

We begin by examining rational approximations with n poles to certain rational sums with $n+1$ poles.

THEOREM 2. *Fix $k \geq 0$. Suppose that $\gamma_i > 0$ and $\beta_{i+1} > \beta_i \geq 0$. Let*

$$f(x) = \sum_{i=1}^{n+1} \frac{\gamma_i}{x + \beta_i}.$$

If $\zeta_1, \dots, \zeta_{2n+k}$ are $2n+k$ (not necessarily distinct) non-negative points then there exists $p_{n+k-1}/q_n \in R_{n+k,n}$ that interpolates f at each of the ζ_i . Furthermore,

a)
$$\frac{p_{n+k-1}(x)}{q_n(x)} = p_{k-1}(x) + \sum_{k=1}^n \frac{\delta_i}{x + \alpha_i}$$

where $p_{k-1} \in \pi_{k-1}$, $\delta_i > 0$ and $\beta_i < \alpha_i < \beta_{i+1}$ for $i=1, \dots, n$.

Also,

b)
$$f(x) - \frac{p_{n+k-1}(x)}{q_n(x)} = \frac{a_n \prod_{i=1}^{2n+k} (x - \zeta_i)}{\prod_{i=1}^n (x + \alpha_i) \prod_{i=1}^{n+1} (x + \beta_i)}$$

where, for all j ,

$$|a_n| = \frac{\left| \prod_{\substack{i=1 \\ i \neq j}}^{n+1} (\beta_i - \beta_j) \prod_{i=1}^n (\alpha_i - \beta_j) \right| \gamma_j}{\left| \prod_{i=1}^{2n+k} (\beta_j + \zeta_i) \right|}.$$

Furthermore, if $k=0$, then $|a_n| \leq \gamma_{n+1}$.

PROOF. Part a) can be found in [1]. Part b) is straightforward since $f - p_{n+k-1}/q_n$ is an element of $R_{2n+k, 2n+1}$ with $2n+k$ zeros at the ζ_i and $2n+1$ poles at $-\alpha_i$ and $-\beta_i$. The bound on a_n is obtained by observing that

$$\lim_{x \rightarrow -\beta_j} (x + \beta_j) \left(f - \frac{p_{n+k-1}}{q_n} \right) = \gamma_j = \frac{a_n \prod (\beta_j + \zeta_i) (-1)^k}{\prod_{i \neq j} (\beta_i - \beta_j) \prod (\alpha_i - \beta_j)}.$$

When $k=0$ the right hand side of the above equation has absolute value greater than a_n for $j=n+1$.

For certain choices of β_i we can be more precise.

EXAMPLE 1. Fix $c, k \geq 0$. Let $\sum_{i=1}^n \delta_i/(x + \alpha_i) + p_{k-1}(x)$, $p_{k-1} \in \Pi_{k-1}$, interpolate $\sum_{i=1}^{n+1} 1/(x + i)$ at $2n+k$ points $\zeta_1, \dots, \zeta_{2n+k} \in [0, c]$.

Then,

$$\left| \sum_{i=1}^{n+1} \frac{1}{x+i} - \left(\sum_{i=1}^n \frac{\delta_i}{x+\alpha_i} + p_{k-1}(x) \right) \right| = \frac{a_n \left| \prod_{i=1}^{2n+k} (x-\zeta_i) \right|}{\left| \prod_{i=1}^{n+1} (x+i) \prod_{i=1}^n (x+\alpha_i) \right|}$$

and independent of the choice of the ξ_i ,

$$\lim_{n \rightarrow \infty} |a_n|^{1/2n} = .27846 \dots = v$$

where v is the solution of $ve^{1+v} = 1$.

EXAMPLE 2. Fix $c, k \geq 0$. Let $\zeta_1, \dots, \zeta_{2n+k}$ be $2n+k$ points in $[0, c]$. Then there exists $\delta_i, \alpha_i > 0, p_{k-1}, i^2 < \alpha_i < (i+1)^2$ so that

$$\left| \sum_{i=1}^{n+1} \frac{1}{x+i^2} - \left(\sum_{i=1}^n \frac{\delta_i}{x+\alpha_i} + p_{k-1}(x) \right) \right| = \frac{a_n \left| \prod_{i=1}^{2n+k} (x-\zeta_i) \right|}{\left| \prod_{i=1}^{n+1} (x+i^2) \prod_{i=1}^n (x+\alpha_i) \right|}$$

where, independent of the choice of the ζ_i ,

$$\lim_{n \rightarrow \infty} |a_n|^{1/2n} = .439 \dots = 1/\eta^2 - 1$$

where $\eta = .8335 \dots$ is the solution of $(1+\eta)/(1-\eta) = e^{2/\eta}$.

Both examples are consequences of Theorem 2. It is essentially just a calculus exercise to estimate the size of a_n . The two results (a) and (b) of the introduction follow from these examples by choosing the ζ_i to be the roots of the Čebyšev polynomial of degree $2n$ shifted to the interval $[0, 1]$.

COMMENT. For the circle $C = \{|z|=1\}$ we observe the following. Suppose that

$$f(z) = \sum_{i=1}^{n+1} \frac{1}{z+\alpha_i} \quad \alpha_{i+1} \geq \alpha_i + 1 \geq 2$$

and suppose that $p_{n+k-1}/q_n \in R_{n+k-1, n}$ interpolates $f(z)$ at $2n+k$ points $\zeta_1, \dots, \zeta_{2n+k} \in [-1, 1]$. Then, by Theorem 2,

$$\frac{\min_{z \in C} |f(z) - p_{n+k-1}(z)/q_n(z)|}{\max_{z \in C} |f(z) - p_{n+k-1}(z)/q_n(z)|} \geq \frac{\min_{z \in C} \left| \prod_{i=1}^{2n+k} (z-\zeta_i) \right|}{(\alpha_{n+1})^4 \max_{z \in C} \left| \prod_{i=1}^{2n+k} (z-\zeta_i) \right|}$$

It follows from Rouché's theorem (see [2] for details) that if we choose

p_{n+k-1}^*/q_n^* to be the $(n+k-1, n)$ Padé approximant (i.e. $\zeta_i \equiv 0$) then p_{n+k-1}^*/q_n^* is, up to a multiple of $1/(\alpha_{n+1})^4$, as efficient as a best rational approximation of corresponding degree in the sense that

$$\|f - p_{n+k-1}^*/q_n^*\|_C \geq r_{n+k-1, n}(f; C) \geq \frac{1}{(\alpha_{n+1})^4} \|f - p_{n+k-1}^*/q_n^*\|_C.$$

We can use the previous results to get upper estimates for approximations to $\sum_{i=1}^{\infty} \delta_i/(x + \alpha_i)$.

THEOREM 3. *If*

$$f(x) = \sum_{i=1}^{\infty} \frac{\delta_i}{x + \alpha_i} \quad 0 \leq \delta_i \leq 1, \quad 1 \leq c \leq \alpha_i < \alpha_{i+1}$$

then

$$r_{n-1, n}(f; [0, 1]) \leq \left(\frac{c^2}{c^2 - 1} \right) \frac{2f(0)}{4^{2n-1} \left(\prod_{i=1}^n \alpha_i \right)^2}.$$

PROOF. Let $s_m \in R_{n+m-1, n+m}$ interpolate f at the $2n-1$ zeros of the $(2n-1)$ th Čebyšev polynomial T_{2n-1} shifted to $[0, 1]$ and also $2m+1$ times at zero. Note that, as in the proof of Theorem 1,

$$s_m = \sum_{i=1}^{n+m} \frac{\gamma_i^m}{x + \beta_i^m}$$

where $\beta_i^m > \alpha_i$ and

$$\sum_{i=1}^{n+m} \frac{\gamma_i^m}{\beta_i^m} = f(0).$$

In particular, since each $\gamma_i^m, \beta_i^m \geq 0$, we have for each m

$$\frac{\gamma_{n+m}^m}{\beta_{n+m}^m} \leq f(0).$$

From Theorem 2 we deduce that for $x \in [0, 1]$

$$\begin{aligned} |s_{m+1}(x) - s_m(x)| &\leq \frac{\gamma_{n+m+1}^{m+1} |T_{2n-1}(x)|}{\left(\prod_{i=1}^{n+m} (\beta_i^{m+1}) \right)^2 \beta_{n+m+1}^{m+1}} \\ &\leq \frac{f(0) |T_{2n-1}(x)|}{\prod_{i=1}^{n+m} \alpha_i^2}. \end{aligned}$$

$$\leq \frac{2f(0)}{4^{2n-1} \prod_{i=1}^{n+m} \alpha_i^2}.$$

We finish by observing that

$$\|f - s_0\|_{[0,1]} \leq \sum_{m=0}^{\infty} \|s_{i+1} - s_i\|_{[0,1]}.$$

We note that

$$e^z = 1 + \frac{2z}{2 - z + 2z^2 \sum_{n=1}^{\infty} \frac{1}{z^2 + (2\pi n)^2}}.$$

By Theorem 3, there exists C_1 so that

$$r_{2n-2, 2n} \left(\sum_{n=1}^{\infty} \frac{1}{z^2 + (2\pi n)^2} : [-1, 1] \right) \leq \frac{C_1}{4^{2n} (2\pi)^{4n} (n!)^4}$$

and hence, there exists C_2 so that

$$r_{2n+1, 2n+1} (e^z : [-1, 1]) \leq \frac{C_2}{4^{2n} (2\pi)^{4n} (n!)^4}.$$

This implies that

$$\overline{\lim} (n! n! r_{n,n} (e^z : [-1, 1]))^{1/n} \leq \frac{1}{(2\pi)^2} \leq \frac{1}{39.4}.$$

This should be compared to the “correct” result due to Németh [8]

$$\lim_{n \rightarrow \infty} (n! n! r_{n,n} (e^z : [-1, 1]))^{1/n} = \frac{1}{64}.$$

Thus, our method yields good but inexact upper bounds for $r_{n,n}$.

It is apparent from Theorem 1 that if $0 < c_1 \leq \gamma_i \leq c_2$ and $\alpha_i \geq 0$ then, on positive intervals,

$$c_1 r_{n+k,n} \left(\sum_{i=1}^{\infty} \frac{1}{x + \alpha_i} \right) \leq r_{n+k,n} \left(\sum_{i=1}^{\infty} \frac{\gamma_i}{x + \alpha_i} \right) \leq c_2 r_{n+k,n} \left(\sum_{i=1}^{\infty} \frac{1}{x + \alpha_i} \right)$$

and that

$$r_{n+k,n} \left(\sum_{i=1}^{n+1} \frac{1}{x + \alpha_i} \right) \leq r_{n+k,n} \left(\sum_{i=1}^{\infty} \frac{1}{x + \alpha_i} \right).$$

Lower bounds for rational approximation to $\sum_{i=1}^{n+1} 1/(x + \alpha_i)$ will depend

critically on the spacing of the α_i . However, the technique presented in this section can be extended to many more special cases.

ADDED IN PROOF. It has come to the author's attention that a version of Theorem 1 is derived by D. Braess in *Numer. Math.* 22 (1974), 219–232.

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