MATH. SCAND. 53 (1983), 125–144

CONVEXITY OF MEASURES IN CERTAIN CONVEX CONES IN VECTOR SPACE σ-ALGEBRAS

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1. Introduction.

The Brunn–Minkowski theory on vector spaces deals with all types of connections between set functions and linear combination of sets. Below we will treat a special situation when the sets are restricted to a certain convex cone in the underlying σ-algebra.

Let $0<\theta<1$ and $-\infty \leqq a \leqq +\infty$ be fixed. For any $0<s,t\leqq +\infty$ define the mean

$$\begin{aligned} M_a^\theta(s,t) &= (\theta s^a + (1-\theta)t^a)^{1/a}, \quad a \in \mathbf{R}\setminus\{0\};\\ &= \min(s,t), \quad a = -\infty;\\ &= s^\theta t^{1-\theta}, \quad a=0;\\ &= \max(s,t), \quad a=+\infty\,. \end{aligned}$$

Here $0^\alpha = +\infty$, if $-\infty<\alpha<0$. Finally, for arbitrary $0\leqq s,t\leqq+\infty$,

$$M_a^\theta(s,t) = 0, \quad \text{if } s=0 \text{ or } t=0\,.$$

Throughout E denotes a real, locally convex Hausdorff vector space and $C \ni 0$ stands for a fixed closed convex cone in E. Set

$$\langle C\rangle = \{K-C;\ E \supseteqq K \text{ compact}\}\,.$$

Clearly,

$$s,t \geqq 0, \quad A,B \in \langle C\rangle \Rightarrow sA+tB \in \langle C\rangle\,.$$

In addtion, each set $A \in \langle C\rangle$ is C-invariant, that is, $A-C=A$. Given $-\infty\leqq\alpha <+\infty$, we shall write $\mu \in \mathscr{M}_\alpha(E;\,C)$, if μ is a finite positive Radon measure on E (abbreviated $\mu \in \mathscr{R}(E)$) and

$$\mu(\theta A + (1-\theta)B) \geqq M_\alpha^\theta(\mu(A),\mu(B))$$

for all $A,B \in \langle C\rangle$ and every $0<\theta<1$. A measure satisfying these assumptions

Received February 16, 1982; in revised form June 3, 1982.

is said to be :α:-concave in $\langle C\rangle$. For brevity, $\mathscr{M}_\alpha(E; \{0\})$ is written $\mathscr{M}_\alpha(E)$ and an :α:-concave measure in $\langle\{0\}\rangle$ is simply called :α:-concave. Below, by abuse of language, ":α:-concave" is sometimes shortened to "α-concave" ("α-convex") if $\alpha \geqq 0$ $(\alpha \leqq 0)$.

The interest in $1/n$-concave measures originates from Brunn and Minkowski who show that the uniform distribution in an arbitrary convex body in $\mathbf{R}^n$ is $1/n$-concave (restricted to convex bodies). The main features of 0-concave measures on $\mathbf{R}^n$ are due to Davidovič, Korenbljum, and Hacet [11], Prekopa ([21], [22], [23]) and the author [3]. In ([4], [5], [6], [7]) we continue this program introducing α-convex measures on possibly infinite-dimensional spaces. During the past few years this subject has been enriched on the foundational level, mainly by Brascamp and Lieb ([9], [10]), Dubuc ([12], [13]), and Hoffmann-Jørgensen [17].

The present paper is devoted to a study of :α:-concave measures in convex cones of the type $\langle C\rangle$ introduced above. One motivation for this is the following. Let $X=(X_1,\ldots,X_n)$ be a random vector in $\mathbf{R}^n$ with probability distribution function $F_X(x_1,\ldots,x_n)=P[X_1\leqq x_1,\ldots,X_n\leqq x_n]$. In a variety of different contexts it may be useful to know that F_X is :α:-concave, that is, to know that the inequality $F_X(\theta x+(1-\theta)y)\geqq M_\alpha^\theta(F_X(x),F_X(y))$ is true for all $x,y\in\mathbf{R}^n$ and each $0<\theta<1$ (see e.g. Barlow and Proschan (reliability theory) [1], Berwald (convexity) [2], Hoffmann-Jørgensen, Shepp, and Dudley (absolute continuity of semi-norms) [18], Prekopa (stochastic programming) [24], and Rinott (statistics) [27]). Here two remarks are in order. Firstly, in almost all cases of interest, it is a non-trivial problem to decide whether F_X is :α:-concave or not. Secondly, it seems to be an almost hopeless task to develope a convex analysis based on :α:-concave distribution functions in $\mathbf{R}^n(n>1)$. In this context :α:-concave measures in $\langle\mathbf{R}^n_+\rangle$ have some advantages as will be seen below.

A second reason for this paper is to deepen the Bruun–Minkowski approach to measures on linear spaces. Among other things, we prove zero-one laws and integrability of appropriate semi-norms.

ACKNOWLEDGMENT. The author is grateful to the referee for pointing out an error in an earlier version of this paper.

2. The basic results for the class $\mathscr{M}_\alpha(\mathbf{R}^n)$.

Throughout, $-\infty\leqq\alpha<+\infty$, $-\infty\leqq\beta\leqq+\infty$, and $0<\theta<1$ are assumed to be fixed if not otherwise stated. Given $\mu,\nu\in\mathscr{R}(E)$, we let

$$\mathscr{M}_\alpha^\theta(\mu,\nu;\,C) = \{\tau\in\mathscr{R}(E);\ \tau(\theta A+(1-\theta)B) \geqq M_\alpha^\theta(\mu(A),\nu(B)),\ A,B\in\langle C\rangle\}$$

and set $\mathscr{M}_\alpha^\theta(\mu,\nu;\,\{0\})=\mathscr{M}^\theta(\mu,\nu)$. If $V\subseteq E$ is universally Borel measurable and convex, then the notation $h \in \mathscr{F}^{\theta}_{\beta}(f, g \mid V)$ will mean that $f, g, h\colon V \to [0, +\infty]$ are universally Borel measurable functions satisfying the inequality $h(\theta x + (1-\theta)y) \geqq M^{\theta}_{\beta}(f(x), g(y))$ for all $x, y \in V$. The members in the class

$$\mathscr{F}_{\beta}(V) = \{f\colon\ V \to [0, +\infty]; f \in \mathscr{F}^{\theta}_{\beta}(f, f \mid V),\ 0<\theta<1\}$$

are called $:\beta:$-concave functions in V.

THEOREM 2.1. ([3, Th. 3.1]) *If* $f, g, h \in L_1^+(m_n)$, $-\infty \leqq \alpha \leqq 1/n$, *and* $h \in \mathscr{F}^{\theta}_{\alpha/(1-\alpha n)}(f, g \mid \mathsf{R}^n)$, *then* $hm_n \in \mathscr{M}^{\theta}_{\alpha}(fm_n, gm_n)$.

Here m_n denotes Lebesgue measure in R^n and $\alpha/(1-\alpha n) = -1/n$, $\alpha = -\infty$, $= +\infty$, $\alpha = 1/n$.

THEOREM 2.2. ([3, Th. 3.2]) a) *Let* $-\infty \leqq \alpha \leqq 1/n$ *and suppose* $\mu \in \mathscr{M}_{\alpha}(\mathsf{R}^n)$. *If the convex set* $\operatorname{supp}\mu$ *is n-dimensional, then* μ *is absolutely continuous with respect to* m_n *and a suitable version of* $d\mu/dm_n$ *is* $:\alpha/(1-\alpha n):$*-concave in* R^n. (b) *If* $\alpha > 1/n$ *and* $\mu \in \mathscr{M}_{\alpha}(\mathsf{R}^n)$, *then* $\dim \operatorname{supp} \mu < n$.

3. Some simple construction methods of :α:-concave measures in conves cones.

To begin with, note that

$$\mathscr{M}_{\alpha_1}(E;\, C) \supseteqq \mathscr{M}_{\alpha_2}(E;\, C), \qquad \alpha_1 \leqq \alpha_2\,,$$

$$\mathscr{M}_{\alpha}(E;\, C_1) \subseteqq \mathscr{M}_{\alpha}(E;\, C_2), \qquad C_1 \subseteqq C_2\,,$$

$$\mathscr{M}_{\alpha}(E;\, E) = \mathscr{R}(E)$$

and

$$\mathscr{M}_{-\infty}(E;\, H) = \mathscr{R}(E), \qquad H \text{ closed half space}\,.$$

Also, by the Zorn lemma, any $\mu \in \mathscr{M}_{\alpha}(\mathrm{E}; C)$ belongs to at least one class $\mathscr{M}_{\alpha}(E; C_{\alpha}(\mu))$, where $C_{\alpha}(\mu)$ is minimal.

The one-dimensional case $E = \mathsf{R}$ is especially simple to treat since there only are four closed convex cones in R. Recall that a smooth positive $:\beta:$-concave function ($\beta \in \mathsf{R}$) f on a subinterval of R is characterized by the differential inequality $ff'' + (\beta-1)f'^2 \leqq 0$. Often, this enables us to construct measures on R which are $:\alpha:$-concave in the cones in question. However, there are lots of interesting exceptional cases and, in such a case, Theorem 2.1 may sometimes be helpful.

EXAMPLE 3.1. We claim that each stable probability measure μ on R with topological support $\mathbf{R}_+$ is 0-concave in $\langle \mathbf{R}_+ \rangle$. In fact, due to a representation formula of Zolotarev [31] there exist $\delta > 0$ and $0 < \alpha < 1$ such that

$$\mu(]-\infty, \delta x]) = \frac{1}{\pi} \int_0^\pi \exp(-v_\alpha(x,t))\, dt, \qquad x > 0 ,$$

where for all $x > 0$ and $0 < t < \pi$,

$$v_\alpha(x,t) = x^{\alpha/(\alpha-1)} \left(\frac{\sin \alpha t}{\sin t} \right)^{\alpha/(1-\alpha)} \frac{\sin (1-\alpha)t}{\sin t} .$$

Thus the claim above follows if we prove that v_α is convex. To see this we write

$$v_\alpha(x,t) = x^{\alpha/(\alpha-1)} \left[\left(\frac{\sin \alpha t}{\sin t} \right)^{\alpha} \left(\frac{\sin (1-\alpha)t}{\sin t} \right)^{1-\alpha} \right]^{1-\alpha/(\alpha-1)}$$

and note that the function $(\xi, \eta) \curvearrowright \xi^a \eta^{1-a}$, $\xi, \eta > 0$, is convex for each $a < 0$. Consequently, v_α is convex if the function

$$\left(\frac{\sin \alpha t}{\sin t} \right)^{\alpha} \left(\frac{\sin (1-\alpha)t}{\sin t} \right)^{1-\alpha}, \qquad 0 < t < \pi ,$$

is convex, which is obvious as

$$\frac{d^2}{dt^2} ln \frac{\sin \alpha t}{\sin t} = \frac{\sin^2 \alpha t - \alpha^2 \sin^2 t}{\sin^2 \alpha t \sin^2 t} > 0, \qquad 0 < t < \pi .$$

It is well-known from the early Brunn–Minkowski theory that each concave function, defined on a convex body in $\mathbf{R}^n$, induces a distribution measure which is $1/n$-concave in $\langle \mathbf{R}_- \rangle$. Before pushing this into a more general framework we introduce some new definitions.

Under the conditions on E and C stated in the Introduction, the ordered pair $(E; C)$ is called a semi-ordered, locally convex Hausdorff space over $\mathbf{R}$. For all $x, y \in (E; C)$, the shorthand notation $x \prec y$ means that $y - x \in C$. Suppose $(F; D)$ is another semi-ordered, locally convex Hausdorff space over $\mathbf{R}$ and let u be a mapping of a convex subset V of $(E; C)$ into $(F; D)$. Then u is said to be increasing if $[x, y \in V,\ x \prec y \ \Rightarrow\ u(x) \prec u(y)]$ and convex if

$$[x, y \in V,\ 0 < \theta < 1 \ \Rightarrow\ u(\theta x + (1-\theta) y)] \prec \theta u(x) + (1-\theta) u(y)] .$$

THEOREM 3.1. *Let $\tau \in \mathcal{M}_\alpha^0(\mu, \nu; C)$, let $u\colon (E; C) \to (F; D)$ be Lusin μ-, ν-, and τ-measurable, and suppose there exists a C-invariant convex support V of the measure $\mu + \nu$. If $u|_V$ is increasing and convex, then $u(\tau) \in \mathcal{M}_\alpha^0(u(\mu), u(\nu); D)$.*

PROOF. Let $A, B \subseteq F$ be D-invariant. It is readily seen that

$$u^{-1}(\theta A+(1-\theta)B) \supseteqq \theta(u^{-1}(A) \cap V)+(1-\theta)(u^{-1}(B) \cap V)$$

where the sets $u^{-1}(A) \cap V$ and $u^{-1}(B) \cap V$ are C-invariant. Finally, using that

$$m_{\text{inner measure}}(u^{-1}(\cdot)) = (u(m))_{\text{inner measure}}, \qquad m=\mu, \nu, \tau ,$$

(see e.g. Schwartz [28, p. 25]) we are done.

EXAMPLE 3.2. Let $E \neq \{0\}$ be a Banach space and suppose $\mu \in \mathscr{M}_\alpha(E)$ ($\alpha > -\infty$) has topological support E. Then each sphere in E is a μ-null set. In the Gaussian case the same result is due to Gross [16]. The interest of such a message has been further emphasized by Topsøe [30], who studies uniform weak convergence of measures in restricted Banach spaces.

A combination of Theorems 2.1 and 3.1 yields

COROLLARY 3.1. *Suppose $\mu = fm_n \in \mathscr{R}(\mathbf{R}^n)$ and let u: $\mathbf{R}^n \to (\mathbf{R}^n; C)$ be a C^2 mapping. Moreover, assume there exists an open convex set $V \subseteqq \mathbf{R}^n$ such that $u(V)$ supports μ and such that $u|_V$ is injective and convex. Then $\mu \in \mathscr{M}_\alpha(\mathbf{R}^n; C)$ ($-\infty \leqq \alpha \leqq 1/n$), if $(f \circ u)|Ju|$ is :$\alpha/(1-\alpha n)$:-concave in V, where Ju denotes the Jacobian of u.*

EXAMPLE 3.3. Let $X_1, \ldots, X_n, Y$ be stochastically independent $N(0; 1)$-distributed random variables and set $Z = (X_1^2/Y^2, \ldots, X_n^2/Y^2)$. The density function f_Z of Z vanishes off $\mathbf{R}_+^n$ and

$$f_Z(z) = \text{const.}\, z_1^{-\frac{1}{2}} \ldots z_n^{-\frac{1}{2}}(1+z_1+\ldots+z_n)^{-(n+1)/2}, \qquad z>0 .$$

Introducing $u(\xi) = (\xi_1^2, \ldots, \xi_n^2)$, $\xi \in \mathbf{R}^n$, and applying Corollary 3.1 we now conclude that $P_Z \in \mathscr{M}_{-1}(\mathbf{R}^n; \mathbf{R}_+^n)$. From the proof it also follows that $P_{(|X_1/Y|, \ldots, |X_n/Y|)} \in \mathscr{M}_{-1}(\mathbf{R}^n)$.

Next we will discuss a quite different construction method which only makes sense for $C \neq \{0\}$.

Let $C \neq \{0\}$ be a closed convex cone in E and suppose $\alpha \geqq 1$ is fixed. We now choose a non-empty Borel set $C_0 \subseteqq C \setminus \{0\}$ such that $(x\mathbf{R}_+) \cap C_0 = \{x\}$, $x \in C_0$, and a bounded Borel function f: $E \to \mathbf{R}_+$ possessing the following properties;

(i) the measure ν_x: $A \curvearrowright \int_0^{+\infty} f(rx) 1_A(rx)\, dr$ is α-concave in $\langle C \rangle$ for each $x \in C_0$,

(ii) $0 \in \operatorname{supp} \nu_x$, $\quad x \in C_0$.

Let $\tau \in \mathscr{R}(E)$ be supported on C_0. We claim that the Radon measure

$$\mu = \int \nu_x(\cdot)\, d\tau(x)$$

is an α-concave measure in $\langle C \rangle$. To see this, assume $A, B \in \langle C \rangle$ are both of positive μ-measure and note that

$$\nu_x(\theta A + (1-\theta)B) \geqq (\theta \nu_x^\alpha(A) + (1-\theta)\nu_x^\alpha(B))^{1/\alpha}, \qquad x \in C_0 ,$$

because $0 \in A \cap B$. Finally, using the Minkowski inequality it follows that $\mu \in \mathscr{M}_\alpha(E; C)$.

The above construction shows the necessity in the following

THEOREM 3.2. *Let $\alpha > 0$. Each $\mu \in \mathscr{M}_\alpha(E; C)$ is concentrated on a finite-dimensional subspace of E if and only if C is finite-dimensional.*

PROOF. Suppose C is finite-dimensional and represent E as a topological direct sum of $C - C$ and a complementary supspace F of E. Let u: $(E; C) \to (F; \{0\})$ be the canonical map and note that u is increasing and convex. Thus, for any $\mu \in \mathscr{M}_\alpha(E; C)$, $u(\mu) \in \mathscr{M}_\alpha(F)$ and Theorem 2.2 implies that $u(\mu)$ is concentrated on a finite-dimensional subspace of F. Consequently, μ is concentrated on a finite-dimensional subspace of E.

4. Finite-dimensional projections.

In the sequel, E' denotes the topological dual of E and $C^+ = \{\xi \in E';\ \xi|_C \geqq 0\}$. If $\tau \in \mathscr{M}_\alpha^\theta(\mu, \nu; C)$ and $\xi_1, \ldots, \xi_n \in C^+$, then, by Theorem 3.1, $u(\tau) \in \mathscr{M}_\alpha^\theta(u(\mu), u(\nu);\ \mathbf{R}_+^n)$, where $u = (\xi_1, \ldots, \xi_n)$. To begin with in this section we shall prove the following converse result.

THEOREM 4.1. *Assume that the cone $C^\# \subseteq C^+$ strictly separates C and points belonging to the complement of C. If $\mu, \nu, \tau \in \mathscr{R}(E)$ and $u(\tau) \in \mathscr{M}_\alpha^\theta(u(\mu), u(\nu);\ \mathbf{R}_+^n)$ for all $u = (\xi_1, \ldots, \xi_n)$ such that $\xi_1, \ldots, \xi_n \in C^\#$, $n \in \mathbf{N}_+$, then $\tau \in \mathscr{M}_\alpha^\theta(\mu, \nu; C)$.*

PROOF. Let $A, B \subseteq E$ be compact. We shall prove the following inequality

$$\tau(\theta A + (1-\theta)B - C) \geqq M_\alpha^\theta(\mu(A-C), \nu(A-C)) . \tag{4.1}$$

To this end, first note that

$$\theta A + (1-\theta)B - C =$$

$$\bigcap \{\theta A + (1-\theta)B - [\xi_1 \geqq -1, \ldots, \xi_n \geqq -1]:\ \xi_1, \ldots, \xi_n \in C^\#,\ n \in \mathbf{N}_+\}$$

as $\theta A + (1-\theta)B$ is compact. Now let $\varepsilon > 0$ be fixed and choose

$$C_0 = [\xi_1 \geqq -1, \ldots, \xi_n \geqq -1] \qquad (\xi_1, \ldots, \xi_n \in C^\#)$$

satisfying the estimate

$$\tau(\theta A+(1-\theta)B-C)+\varepsilon \geqq \tau(\theta A+(1-\theta)B-C_0)\,.$$

Moreover, by compactness, we may pick $a_1,\ldots,a_p \in A$, $b_1,\ldots,b_q \in B$ such that $A \subseteqq \{a_1,\ldots,a_p\}-C_0$ and $B \subseteqq \{b_1,\ldots,b_q\}-C_0$. Then

$$\tau(\theta A+(1-\theta)B-C_0) \geqq \tau(\theta\{a_1,\ldots,a_p\}+(1-\theta)\{b_1,\ldots,b_q\}-C_0)$$

where the last member does not exceed

$$M_\alpha^\theta(\mu(\{a_1,\ldots,a_p\}-C_0),\nu(\{b_1,\ldots,b_q\}-C_0)) \geqq M_\alpha^\theta(\mu(A-C),\nu(B-C))\,.$$

Summing up, we have

$$\tau(\theta A+(1-\theta)B-C)+\varepsilon \geqq M_\alpha^\theta(\mu(A-C),\nu(B-C))$$

and (4.1) follows at once.

Theorem 4.1 raises the question how to characterize the classes $\mathcal{M}_\alpha(\mathsf{R}^n;\, \mathsf{R}^n_+)$ in a *simple* way, which, however, seems to be very complicated for each $n>1$. It should be remarked that an :α:-concave measure in $\langle \mathsf{R}^2_+ \rangle$ is not generally, a convex image of an :α:-concave measure on R^2 even if $\alpha \leqq \frac{1}{2}$.

EXAMPLE 4.1. Let $I_1, I_2, I_3 \subseteqq \{|x|=1,\, x \in \mathsf{R}^2_+\}$ be mutually disjoint closed arcs of positive lengths. Set $S_i=$convex hull$\{0\}\cup I_i)$, $i=1,2,3$, and introduce the measure $\mu(dx)=1_{S_1\cup S_2\cup S_3}(x)dx/|x|$. Of course, $\mu \ll m_2$ and from the previous section we know that μ is 1-concave in $\langle \mathsf{R}^2_+ \rangle$. However, there do not exist a $\nu \in \mathcal{M}_{-\infty}(\mathsf{R}^n)$ and a convex function $u\colon \operatorname{supp}\nu \to \operatorname{supp}\mu$ such that $u(\nu)=\mu$. In fact, assuming the converse, necessarily, $k=\dim\operatorname{supp}\nu>0$ and $\dim u^{-1}(\{0\}) \leqq k-1$. Consequently, there exists a continuous curve in $(\operatorname{supp}\mu)\setminus\{0\}$ connecting two of the three connected components of $\operatorname{int}\operatorname{supp}\mu$, which is absurd.

We must leave the above question unanswered here and shall next discuss some applications of Theorem 4.1.

Below, if a net (μ_i) in $\mathscr{R}(E)$ converges weakly to $\mu \in \mathscr{R}(E)$, this fact is expressed $\mu_i \Rightarrow \mu$.

COROLLARY 4.1. *The map* $(\mu,\nu) \to \mathcal{M}_\alpha^\theta(\mu,\nu;\, C)$ *is weakly closed, that is, if* $\tau_i \in \mathcal{M}_\alpha^\theta(\mu_i,\nu_i;\, C)$ *and* $\mu_i \Rightarrow \mu$, $\nu_i \Rightarrow \nu$, $\tau_i \Rightarrow \tau$, *then* $\tau \in \mathcal{M}_\alpha^\theta(\mu,\nu;\, C)$.

PROOF. By Theorem 4.1 we may assume that $(E;\, C)=(\mathsf{R}^n;\, \mathsf{R}^n_+)$ and the result follows at once (compare [4, Th. 2.2]).

THEOREM 4.2. *If* $\mu,\nu \in \mathcal{M}_\alpha(E)$, *then* $\mu \wedge \nu \in \mathcal{M}_\alpha(E)$.

Theorem 4.2 does not extend to arbitrary $:\alpha:$-concave measures in convex cones. Note, however, that $\mu \wedge \nu \in \mathcal{M}_{\alpha \wedge 1}(\mathbf{R}; \mathbf{R}_+)$ if $\mu, \nu \in \mathcal{M}_\alpha(\mathbf{R}; \mathbf{R}_+)$, which follows by differentiation.

PROOF. The finite-dimensional case is a consequence of Theorems 2.1 and 2.2. In the general case we argue as follows.

Let $u: E \to \mathbf{R}^n$ be an arbitrary linear continuous mapping. It shall be proved that $u(\mu \wedge \nu)$ is $:\alpha:$-concave. To this end, suppose A, B are compact subsets of $\mathbf{R}^n$. Moreover, let G be a Borel set in $\mathbf{R}^p$ and choose an arbitrarily linear continuous map $f: E \to \mathbf{R}^p$. Then, setting $H = \mathbf{R}^p \setminus G$, we have

$$\begin{aligned}&\mu(u^{-1}(\theta A + (1-\theta)B) \cap f^{-1}(G)) + \nu(u^{-1}(\theta A + (1-\theta)B) \cap f^{-1}(H)) \\ &= \mu_{(u,f)}((\theta(A \times \mathbf{R}^p) + (1-\theta)(B \times \mathbf{R}^p)) \cap (\mathbf{R}^n \times G)) + \\ &+ \nu_{(u,f)}((\theta(A \times \mathbf{R}^p) + (1-\theta)(B \times \mathbf{R}^p) \cap (\mathbf{R}^n \times H))\end{aligned}$$

where the last expression does not exceed

$$\begin{aligned}&(\mu_{(u,f)} \wedge \nu_{(u,f)})(\theta(A \times \mathbf{R}^p) + (1-\theta)(B \times \mathbf{R}^p)) \\ &\geqq M_\alpha^\theta((\mu_{(u,f)} \wedge \nu_{(u,f)})(A \times \mathbf{R}^p), (\mu_{(u,f)} \wedge \nu_{(u,f)})(B \times \mathbf{R}^p)) .\end{aligned}$$

Finally, using the inequality $\mu_{(u,f)} \wedge \nu_{(u,f)} \geqq (\mu \wedge \nu)_{(u,f)}$, Theorem 4.2 follows at once.

5. Multiplication by densities.

For all $\alpha, \beta \in \mathbf{R}$ satisfying $\alpha + \beta \geqq 0$, we introduce half the harmonic mean

$$\varkappa(\alpha, \beta) = \begin{cases} (\alpha^{-1} + \beta^{-1})^{-1}, & \alpha + \beta > 0,\ \alpha \neq 0,\ \beta \neq 0\,, \\ -\infty \quad , & \alpha + \beta = 0,\ (\alpha, \beta) \neq (0,0)\,, \\ 0 \quad , & \alpha = \beta = 0\,. \end{cases}$$

THEOREM 5.1. *Suppose* $\tau \in \mathcal{M}_\alpha^\theta(\mu, \nu; C)$ $(\alpha \in \mathbf{R})$ *and let* $h \in \mathcal{F}_\beta^\theta(f, g \mid E)$ $(\beta \in \mathbf{R})$, *where* $\alpha + \beta \geqq 0$. *If* $f, g, h: (E, C) \to \mathbf{R}$ *are bounded and decreasing, then* $h\tau \in \mathcal{M}_{\varkappa(\alpha,\beta)}^\theta(f\mu, g\nu; C)$.

Here and throughout $\mathbf{R}$ is assumed to be endowed with its usual cone ordering if not otherwise stated.

The proof of Theorem 5.1 is based on the next

LEMMA 5.1. *Let* $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R} \setminus \{0\}$, *and suppose* $H \in \mathcal{F}_\alpha^\theta(F, G \mid \mathbf{R}_+)$.

a) *If* $\alpha > 0 > \beta$ *and* $\alpha + \beta \geqq 0$, *then*

$$(5.1)\quad \int_0^\infty x^{1/\beta-1}H(x)\,dx \geqq M^\theta_{\varkappa(\alpha,\beta)}\left(\int_0^\infty x^{1/\beta-1}F(x)\,dx,\ \int_0^\infty x^{1/\beta-1}G(x)\,dx\right).$$

b) *If* $\alpha+\beta\geqq 0$ *and* F, G, H *decrease, then* (5.1) *is true.*

PROOF. Recall that the function $\xi^a\eta^{1-a}$, ξ, $\eta>0$, is concave (convex) if $0<a<1$ ($a<0$ or $a>1$).

a) Since $x^{1/\beta-1}H(x)\in\mathscr{F}^\theta_{(\alpha^{-1}+\beta^{-1}-1)^{-1}}(x^{1/\beta-1}F(x), x^{1/\beta-1}G(x)\,|\,\mathbf{R}_+)$ the inequality (5.1) follows from Theorem 2.1.

b) STEP 1. $0<\alpha\leqq 1$, $\beta>0$.

PROOF OF STEP 1. Set $I_\alpha=I^\alpha$, $I=F,G,H$. Without loss of generality we may assume that $I(a(I))=0$ for a suitable $a(I)>0$ and that the function $I|_{[0,a(I)]}$ is strictly decreasing and $\mathscr{C}^1$. Then, by partial integration,

$$\int_0^\infty x^{1/\beta-1}I(x)\,dx = -\frac{\beta}{\alpha}\int_0^{a(I)} x^{1/\beta}I_\alpha^{1/\alpha-1}(x)I'_\alpha(x)\,dx$$

and if i_α denotes the inverse of the function $I_\alpha|_{[0,a(I)]}$, we have

$$\int_0^\infty x^{1/\beta-1}I(x)\,dx = \frac{\beta}{\alpha}\int_0^{i_\alpha(0)} i_\alpha^{1/\beta}(x)x^{1/\alpha-1}\,dx\,.$$

Moreover,

$$h_\alpha(\theta x+(1-\theta)y)\geqq \theta f_\alpha(x)+(1-\theta)g_\alpha(y),\qquad 0\leqq x\leqq f_\alpha(0),\ 0\leqq y\leqq g_\alpha(0)\,.$$

Thus, defining $i_\alpha(x)=0$, $x>i_\alpha(0)$, it follows that

$$h_\alpha^{1/\beta}(x)x^{1/\alpha-1}\in\mathscr{F}^\alpha_{(\alpha^{-1}+\beta^{-1}-1)^{-1}}(f_\alpha^{1/\beta}(x)x^{1/\alpha-1}, g_\beta^{1/\beta}(x)x^{1/\alpha-1}\,|\,\mathbf{R}_+)$$

and (5.1) is an immediate consequence of Theorem 2.1.

STEP 2. $1<\alpha<+\infty$, $\beta>0$.

PROOF OF STEP 2. Set $I(\cdot,\xi)=\xi I$, $\xi>0$, $I=F,G,H$, and note that for all fixed $\xi,\eta>0$,

$$H(\cdot,\theta\xi+(1-\theta)\eta)\in\mathscr{F}^\theta_{\alpha/(1+\alpha)}(F(\cdot,\xi),G(\cdot,\eta)\,|\,\mathbf{R}_+)\,.$$

Now using the previous step, we have

$$(\theta\xi+(1-\theta)\eta)\int_0^\infty x^{1/\beta-1}H(x)\,dx$$
$$\geqq M^\theta_{((\alpha/(1+\alpha))^{-1}+\beta^{-1})^{-1}}\left(\xi\int_0^\infty x^{1/\beta-1}F(x)\,dx,\ \eta\int_0^\infty x^{1/\beta-1}G(x)\,dx\right).$$

If $F=0$ or $G=0$ a.s. $[m_1]$ there is nothing to prove. If not, we set

$$\xi = \left(\int_0^\infty x^{1/\beta-1}F(x)\,dx\right)^{\varkappa(\alpha,\beta)}$$

and

$$\eta = \left(\int_0^\infty x^{1/\beta-1}G(x)\,dx\right)^{\varkappa(\alpha,\beta)}$$

and a simple computation gives (5.1).

STEP 3. $\alpha<0$, $\beta>0$.

PROOF OF STEP 3. By making some minor changes in the proof of Step 1, the result follows at once. We omit the details here.

STEP 4. $\alpha=0$, $\beta>0$.

PROOF OF STEP 4. The inequality (5.1) results from the previous step using an obvious limit argument.

This concludes the proof of Lemma 5.1.

PROOF OF THEOREM 5.1. For each $A\in\langle C\rangle$ the indicator function 1_A: $(E,C)\to\mathbf{R}$ is non-negative and decreasing and, hence, it is enough to prove that

$$\int h\,d\tau \geqq M^\theta_{\varkappa(\alpha,\beta)}\left(\int f\,d\mu, \int g\,dv\right).$$

To this end, first suppose $\beta\neq0$. Then, if $s,t>0$,

$$[h\geqq(\theta s+(1-\theta)t)^{1/\beta}] \geqq \theta[f\geqq s^{1/\beta}]+(1-\theta)[g\geqq t^{1/\beta}]$$

where all the involved sets are C-invariant. Accordingly,

$$\tau(h\geqq(\theta s+(1-\theta)t)^{1/\beta}) \geqq M^\theta_\alpha(\mu(f\geqq s^{1/\beta}), v(g\geqq t^{1/\beta}))$$

and the desired inequality is obious from Lemma 5.1.

Finally, the case $\beta=0$, $\alpha>0$ follows from the case already proved and the case $\alpha=\beta=0$ is a direct consequence of Theorem 2.1.

EXAMPLE 5.1. Suppose $\mu\in\mathscr{M}_\alpha(E;C)$ $(\alpha\geqq0)$ is concentrated on $-C$ and let $c(\alpha,p)=1$, $\alpha=0$; $=\Gamma(\alpha^{-1}+p+1)$, $\alpha>0$. If φ: $-C\to\mathbf{R}_+$ is Borel measurable, concave, and decreasing, then the function

$$p \curvearrowright \frac{c(\alpha, p)}{\Gamma(p+1)} \int_0^{+\infty} \varphi^p \, d\mu, \qquad p>0,$$

is 0-concave.

To prove this assertion there is no loss of generality assuming $\mu \in \mathscr{M}_\alpha(\mathsf{R}; \mathsf{R}_-)$, $\varphi(x)=x \in \mathsf{R}_+$ and the result follows exploiting the same line of reasoning as in the author's work [8], which treats the case $\alpha=1/n$, $n \in \mathsf{N}_+$. For the case $\alpha=0$, $p \geqq 1$, see also Marshall and Olkin [20, p. 494].

It is simple to settle variants of the above conclusion in the parameter interval $-\infty<\alpha<0$ to the cost of some beauty.

We shall next discuss some examples of convexity in potential theory.

EXAMPLE 5.2. Let $a_1, \ldots, a_n$ be non-zero vectors in Euclidean R^3 satisfying $\langle a_i, a_j \rangle \geqq 0$, $i, j=1, \ldots, n$. Suppose $\mu \in \mathscr{R}(\mathsf{R}^3)$ is concentrated on the union of the line segments $[0, a_i]$, $i=1, \ldots, n$, and assume μ reduces to a linear measure on each individual line segment. Of course, μ is 1-concave in $\langle C \rangle$, where C is the convex cone spanned by the a_i. From the above assumptions we conclude that the Newtonian potential of μ, that is $\int d\mu(y)/|x-y|$, is a $-\infty$-convex function of x in $-C^+$.

EXAMPLE 5.3. Let Γ be a closed convex cone in R^n and suppose f; $(\mathsf{R}^n; \Gamma) \to (\mathsf{R}^n; \Gamma)$ is an increasing, convex, and uniformly Lipschitz continuous function. Below we let X denote the Brownian motion in R^n with the drift vector f, that is

$$dX(t) = dB(t)+f(X(t))\,dt, \qquad t \geqq 0,$$

where $(B(t), t \geqq 0)$ stands for the standard Brownian motion in R^n. It is natural that X inherits suitable convexity properties from those of the drift vector and the Brownian motion. To explain this, let $\Omega=(\mathscr{C}(\mathsf{R}_+))^n$, $\Omega_\Gamma=\{\omega \in \Omega;\, \omega(t) \in \Gamma, t \geqq 0\}$, and $\mu_x=P_X[\,\cdot \mid X(0)=x]$, respectively. We claim that

$$\mu_{\theta x+(1-\theta)y} \in \mathscr{M}_0^\theta(\mu_x, \mu_y; \Omega_\Gamma).$$

This is evident if $f=0$. To prove the general case, suppose $\omega \in \Omega$ is fixed and define

$$\begin{cases} X_0(\omega, t) = \omega(t) \\ X_{k+1}(\omega, t) = \omega(t)+\displaystyle\int_0^t f(X_k(\omega, s)\,ds, \qquad t \geqq 0. \end{cases}$$

Here each map X_k: $(\Omega; \Omega_\Gamma) \to (\Omega; \Omega_\Gamma)$ is (increasing and) convex and applying Theorem 3.1, we have

$$X_k(\mu_{\theta x+(1-\theta)y}^{f=0}) \in \mathscr{M}_0^\theta(X_k(\mu_x^{f=0}), X_k(\mu_y^{f=0}); \Omega_\Gamma) .$$

Now using Corollary 4.1, the claim above follows by letting k tend to plus infinity.

Suppose g: $(\mathsf{R}^n; \Gamma) \to \mathsf{R}$ is bounded from below, increasing, and convex and let $A \in \langle \Gamma \rangle$ be convex. As is well-known the physical solution of the initial-value problem

$$\begin{cases} \frac{1}{2}\Delta u + f\cdot\nabla u - gu = \partial u/\partial t, & t>0 \\ u(\cdot,0)=1_A \end{cases}$$

is given by the Feynman–Kac formula

$$u(x,t) = \int_{\omega(t)\in A} \exp\left(-\int_0^t g(\omega(s)ds)\right) d\mu_x(\omega) .$$

Consequently, $u(\cdot,t)$ is 0-concave for each fixed $t>0$ and, of course, the same function decreases as a mapping of $(\mathsf{R}^n; \Gamma)$ into R.

THEOREM 5.2. *For each $i \in \{1,2\}$, let $(E_i; C_i)$ be semi-ordered, locally convex Hausdorff spaces over* R *and suppose $\tau_i \in \mathscr{M}_{\alpha_i}^\theta(\mu_i, \nu_i; C_i)$, where $\alpha_i \in \mathsf{R}$ and $\alpha_1+\alpha_2 \geqq 0$. Then $\tau_1 \otimes \tau_2 \in \mathscr{M}_{\varkappa(\alpha_1,\alpha_2)}^\theta(\mu_1\otimes\mu_2, \nu_1\otimes\nu_2; C_1\times C_2)$. In particular, if $E_1=E_2=E$, then $\tau_1 * \tau_2 \in \mathscr{M}_{\varkappa(\alpha_1,\alpha_2)}^\theta(\mu_1*\mu_2, \nu_1*\nu_2; \overline{C_1+C_2})$.*

PROOF. For every $M \subseteqq E_1 \times E_2$ and $x_1 \in E_1$, set

$$M(x_1) = \{x_2 \in E_2;\ (x_1,x_2) \in M\} .$$

Now choose $A,B \in \langle C_1 \times C_2 \rangle$ arbitrarily but fixed and note that for all $x_1, y_1 \in E_1$,

$$(\theta A + (1-\theta)B)(\theta x_1 + (1-\theta)y_1) \geqq \theta A(x_1) + (1-\theta)B(y_1)$$

where each individual set is C_2-invariant. Hence

$$\tau_2((\theta A + (1-\theta)B)(\theta x_1 + (1-\theta)y_1)) \geqq M_{\alpha_2}^\theta(\mu_2(A(x_1)), \nu_2(B(y_1)))$$

and since for each $c_1 \in C_1$, $A(x_1-c_1) \supseteqq A(x_1)$, and $B(y_1-c_1) \supseteqq B(y_1)$, the Fubini theorem and Theorem 5.1 imply that

$$(\tau_1\otimes\tau_2)(\theta A + (1-\theta)B) \geqq M_{\varkappa(\alpha_1,\alpha_2)}^\theta((\mu_1\otimes\mu_2)(A), (\nu_1\otimes\nu_2)(B)) .$$

Finally, the last statement in Theorem 5.2 follows by combining Theorem 3.1 and the first part of Theorem 5.2.

COROLLARY 5.1. *If $\alpha, \beta \in \mathsf{R}$ and $\alpha+\beta \geqq 0$, then*

$$\mathscr{M}_\alpha(E; C) * \mathscr{M}_\beta(E; C) \subseteqq \mathscr{M}_{\varkappa(\alpha,\beta)}(E; C) .$$

Corollary 5.1 is known in at least one special case for which $C \dotplus E$ is a proper cone. In fact, the inclusion

$$\mathcal{M}_0(\mathsf{R};\mathsf{R}_-) * \mathcal{M}_0(\mathsf{R};\mathsf{R}_-) \subseteqq \mathcal{M}_0(\mathsf{R};\mathsf{R}_-)$$

is frequently used in the theory of reliability [1].

We will end this section by proving some complements of the results obtained so far. Below X is a real-valued random variable and $X_1, \ldots, X_n$ stand for stochastically independent copies of X.

First note that

$$P_X \in \mathcal{M}_\alpha(\mathsf{R};\mathsf{R}_+) \Rightarrow P_{\max\limits_{1 \leqq k \leqq n} X_k} \in \mathcal{M}_{\alpha/n}(\mathsf{R};\mathsf{R}_+)$$

for each $-\infty \leqq \alpha < +\infty$. Here the special case $0 \leqq \alpha < +\infty$, in fact, is included in Theorem 5.2. More interesting, we have

THEOREM 5.3. *Assume $-\infty < \alpha < +\infty$ and let $\beta = \beta(\alpha)$ be the largest member $-\infty \leqq \beta < +\infty$ having the following property*:

$$(\forall n \in \mathsf{N}_+)(P_X \in \mathcal{M}_\alpha(\mathsf{R};\mathsf{R}_-) \Rightarrow P_{\max\limits_{1 \leqq k \leqq n} X_k} \in \mathcal{M}_\beta(\mathsf{R};\mathsf{R}_-)).$$

Then $\beta(\alpha) > -\infty$. Moreover, $\beta(\alpha) \leqq \alpha$, where equality occurs if and only if $\alpha \geqq -1$.

Theorem 5.3 is well-known if $\alpha = 0$ [1, p. 38]. The general case follows at once from the next

LEMMA 5.2. *Suppose $n \in \mathsf{N}_+$, $\alpha, \beta \in \mathsf{R} \setminus \{0\}$, $\alpha\beta > 0$, and $f(x) = (1-(1-x^{1/\alpha})^n)^\beta$, $x > 0$, $x^{1/\alpha} < 1$. Then for any $\alpha > 0$ $[-1 \leqq \alpha < 0]$ the largest β such that f is concave [convex] equals α. If $\alpha < -1$, then there exists a β, independent of n, such that f is convex for every $n \geqq 1$. The largest β with this property is strictly smaller than α.*

PROOF. The second derivative of $f(x)$ equals α times a strictly positive function times

$$g(y) = 1-n+(n-\alpha)y+(n\beta-1)y^n+(\alpha-n\beta)y^{n+1}, \qquad y = 1-x^{1/\alpha}.$$

Since $g(1-)=0$ and $g'(1-)=n(\alpha-\beta)$, necessarily, $\beta \leqq \alpha$ if $g \leqq 0$. Moreover, note that g'' has at most one change of sign and that $g(0+) \leqq 0$. Also, if $\alpha = \beta$, then $g''(1-) < 0$ (respectively >0) if and only if $(n-1)(\alpha+1) > 0$ (respectively <0). Consequently,

$$\alpha = \beta \geqq -1 \Rightarrow g \leqq 0$$

and

$$\alpha = \beta < -1 \Rightarrow \neg(g \leqq 0, \text{ all } n) .$$

In the following we suppose that the parameter α is strictly smaller than -1.

If $\beta(\alpha)$ has the same meaning as in Theorem 5.3, then

$$-\beta(\alpha) = \sup[(1-n+(n-\alpha)y-y^n+\alpha y^{n+1})/(n(y^n-y^{n+1})) : 0<y<1, n \in \mathsf{N}_+] .$$

Thus, $\beta(\alpha) > -\infty$ if and only if

$$\sup[(1-n+(n-\alpha)y+(\alpha-1)y^n)/(ny^n(1-y)) : 0<y<1, n \in \mathsf{N}_+] < +\infty .$$

Now setting

$$h_\alpha(z) = (1-\alpha-(n-\alpha)z+(\alpha-1)(1-z)^n)/(nz(1-z)^n), \quad 0<z<1 ,$$

and noting that $h_{\alpha|[(1-\alpha)/n, 1[} \leqq 0$, we conclude that $\beta(\alpha) > -\infty$ if and only if

$$\sup[h_\alpha(z): 0<z<(1-\alpha)/n, n \in \mathsf{N}_+] < +\infty .$$

This, however, follows at once from the formula

$$2h_\alpha(z) = (1-\alpha)h_{-1}(z)-(1-1/n)(\alpha+1)/(1-z)^n$$

and the already proved fact that the quantity $h_{-1}(z) = h_{-1}(z,n)$ is uniformly bounded from above. Lemma 5.2 is thereby completely proved.

6. Examples of stochastic processes with increasing paths inducing 0-concave measures in suitable convex cones.

Throughout the present section I is assumed to be a fixed subinterval of the real line and R^I_π means R^I equipped with the topology of pointwise convergence.

As is well-known and easy to see each real-valued stochastic process $X = (X(t), t \in I)$ satisfying

$$P[X(s) \leqq X(t)] = 1, \quad s \leqq t ,$$

induces a Radon probability measure P_X on R^I_π such that the closed convex cone of all increasing functions on I supports P_X. For additional information, see e.g. Tjur [29, p. 170].

Now suppose Q: $\mathsf{R} \to]-\infty, +\infty]$ is a decreasing function such that $Q(x)\uparrow +\infty$, $x \downarrow -\infty$, and $Q(x) \downarrow 0$, $x \uparrow +\infty$. The extremal-Q process $X = (X(t), t>0)$, introduced by Dwass [14] and Lamperti [19], is a real-valued stochastic process characterized by the following equation

$$\begin{cases} P[X(t_1)\leqq x_1,.,X(t_n)\leqq x_n] = \exp\left[-\sum_{k=1}^{n} (t_k-t_{k-1})Q(x_k \wedge \ldots \wedge x_n)\right] \\ \text{all } 0=t_0<t_1<\ldots<t_n,\ x_1,\ldots,x_n \in \mathsf{R},\ n \in \mathsf{N}_+ . \end{cases}$$

If $0=t_0<t_1<\ldots<t_n$ and $U_1,\ldots,U_n$ are real-valued stochastically independent random variable with

$$P[U_k\leqq x] = \exp[-(t_k-t_{k-1})Q(x)], \quad k=1,\ldots,n,$$

then the random vectors $(X(t_1,\ldots,X(t_n))$ and $(U_1, U_1 \vee U_2,\ldots,U_1 \vee \ldots \vee U_n)$ obey the same probability law. Thus, combining Theorems 3.1 and 5.2, we have

THEOREM 6.1. *An extremal-Q process induces a 0-concave measure in* $\langle \mathsf{R}_+^{]0,+\infty[}\rangle$ *if and only if Q is convex.*

EXAMPLE 6.1. Consider a real-valued homogeneous Lévy process $X=(X(t), t>0)$, where

$$E[\exp(i\zeta X(1))] = \exp\left(\int_{-\infty}^{+\infty} (e^{i\theta x}-1-i\zeta \sin x)\, d\tau(x)\right)$$

and τ is a positive Borel measure on R such that $\{x \curvearrowright x^2\} \in L_{1,\mathrm{loc}}(\tau)$ and $\tau(\mathsf{R}\setminus[-x,x])<+\infty$, $x>0$. By a theorem of Dwass [15, p. 382], the stochastic process

$$Y(t) = \sup_{0<s\leqq t} (X(s+)-X(s-))^+, \quad t>0,$$

is an extremal-Q process with $Q(x)=+\infty$, $x<0$; $=\tau(]x,+\infty[)$, $x>0$ (see also Resnick and Rubinovitch [26, Th. 1]). In particular, if $0<\alpha<2$ and X is an α-stable, symmetric, and homogeneous Lévy process, then $\tau(]x,+\infty[) = \text{const.}\, x^{-\alpha}$, $x>0$, and, hence, P_Y is 0-concave in $\langle \mathsf{R}_+^{]0,+\infty[}\rangle$.

Recall that a real-valued stochastic process $X=(X(t), t \in I)$ is called additive if the increments $X(t_1), X(t_2)-X(t_1),\ldots,X(t_n)-X(t_{n-1})$ are stochastically independent for all points of time $t_1<\ldots t_n$, $n \in \mathsf{N}_+$. Below $D_+(I)$ denotes the set of all non-negative increasing functions on I.

THEOREM 6.2. *Any increasing and additive stochastic process* $X=(X(t), t \in I)$, *processing* $\mathscr{M}_0(\mathsf{R};\ \mathsf{R}_+)$ *distributed increments, induces a 0-concave measure in* $\langle D_+(I)\rangle$.

PROOF. Suppose $\xi_1,\ldots,\xi_m \in (D_+(I))^+$ and choose $t_1<\ldots<t_n$ such that each ξ_j only depends on the coordinates $x(t_1),\ldots,x(t_n)$. Then, from Theorem 5.2,

$$P_{(X(t_1), X(t_2)-X(t_1), \ldots, X(t_n)-X(t_{n-1}))} \in \mathcal{M}_0(\mathbf{R}^n; \mathbf{R}^n_+)$$

and using Theorem 3.1 we conclude that $P_{(X(t_1),\ldots,X(t_n))}$ is 0-concave in $\langle\{x \in \mathbf{R}^n;\ 0 \leqq x_1 \leqq \ldots \leqq x_n\}\rangle$. Hence

$$P_{(\xi_1(X),\ldots,\xi_m(X))} \in \mathcal{M}_0(\mathbf{R}^m, \mathbf{R}^m_+)$$

and the result follows from Theorem 4.1.

EXAMPLE 6.2. Let $X = (X(t),\ t \geqq 0)$ be an one-sided, stable, and homogeneous Lévy process. Remembering Example 3.1 we have that P_X is 0-concave in $\langle D_+(\mathbf{R}_+)\rangle$.

Now suppose B denotes a standard Brownian motion in $\mathbf{R}$ with $B(0)=0$ and let τ_x be the first time B hits $x \neq 0$. Since $(\tau_x)_{x>0}$ is a one-sided $\frac{1}{2}$-stable homogeneous Lévy process it follows that the probability

$$P\left[\max_{0 \leqq t \leqq t_k} B(t) \geqq x_k,\ k=1,\ldots,n\right]$$

is a 0-concave function of $(t_1,\ldots,t_n)>0$ for all fixed $x_1,\ldots,x_n>0$.

EXAMPLE 6.3. Consider an extremal-Q process $X = (X(t),\ t>0)$ such that $a = \inf\{x;\ Q(x) < +\infty\}$ and $b = \sup\{x;\ Q(x)>0\}$ do not coincide. Set $X^{-1}(x) = \inf\{t;\ X(t)>x\}$, $a<x<b$. From Resnick [25, Th. 1], we know that the stochastic process X^{-1} is increasing and additive. Moreover, for arbitrary $a<x<y<b$,

$$P[X^{-1}(x) \leqq t] = 1-\exp(-tQ(x)), \qquad t>0,$$

and

$$P[X^{-1}(y)-X^{-1}(x) \leqq t] = \theta + (1-\theta)(1-\exp(-tQ(y))), \qquad t>0,$$

for a suitable $0<\theta=\theta(x,y)<1$. Consequently, $P_{X^{-1}}$ is 0-concave in $\langle D_+(]a,b[)\rangle$.

7. A zero-one law.

A non-empty subset G of E is said to be an additive subgroup of E if $G-G = G$.

THEOREM 7.1. *Suppose $\mu \in \mathcal{M}_\alpha(E;\ C)$ and let G be a μ-measurable additive subgroup of E with strictly positive μ-measure.*

a) *If G is C-invariant, then μ is supported on G.*
b) *If $\alpha > -\infty$, then μ is supported on $C+G$.*

Here Part a) is a pure extension of the zero-one law for $-\infty$-convex measures [4].

PROOF. We first choose a compact set $K=-K\subseteqq G$, with $\mu(K)>0$, and set

$$A = C+\bigcup [\underbrace{K+\ldots+K}_{n \text{ terms}} : n\in \mathbf{N}_+] .$$

Now, because $\mu(A\cap(K-C))>0$, there exists a compact set $L\subseteqq E\setminus[A\cup(K-C)]$ such that $\mu(E\setminus(A\cup L))<\mu(K-C)$. Moreover, for each $n\in\mathbf{N}_+$,

$$E\setminus(A\cup L) \supseteqq \frac{1}{n+1}[E\setminus\{A\cup(nK+(n+1)L+C\}]+\frac{n}{n+1}(K-C)$$

and as the complement of a $-C$-invariant set is C-invariant, we have

$$\mu(E\setminus(A\cup L)) \geqq \min\big(\mu(E\setminus\{A\cup(nK+(n+1)L+C)\}),\mu(K-C)\big) .$$

Thus

$$\mu(E\setminus(A\cup L)) \geqq \mu(E\setminus\{A\cup(nK+(n+1)L+C)\})$$

and, hence,

$$\mu(nK+(n+1)L+C) \geqq \mu(L), \qquad \text{all } n\in\mathbf{N}_+ .$$

However, for any fixed compact $M\subseteqq E$, $M\cap(nK+(n+1)L+C)=\emptyset$ for an appropriate $n\in\mathbf{N}_+$ and it follows that $\mu(L)=0$, which proves Part a).

To show Part b), first note that $\mu(E\setminus A)<\mu(K-C)$. If $\mu(E\setminus A)>0$, then we may use the relation

$$E\setminus A \supseteqq \tfrac{1}{2}(E\setminus A)+\tfrac{1}{2}(K-C)$$

and have

$$\mu^{\alpha\wedge(-1)}(E\setminus A) \leqq \tfrac{1}{2}\mu^{\alpha\wedge(-1)}(E\setminus A)+\tfrac{1}{2}\mu^{\alpha\wedge(-1)}(K-C)$$

that is, $\mu(E\setminus A)\geqq\mu(K-C)$, which is a contradiction. Thus $\mu(E\setminus A)=0$ and Part b) is proved, too.

COROLLARY 7.1. *Let $\mu\in\mathscr{M}_\alpha(E;C)$ $(\alpha>-\infty)$. If $a\in E$ is an atom of μ, then μ is concentrated on $a+C$.*

8. Integrability of sublinear functions.

A function $\varphi\colon E\to\mathbf{R}\cup\{+\infty\}$ is said to be an extended valued sublinear function if

$$\begin{cases} \varphi(x+y) \leqq \varphi(x)+\varphi(y), & x,y \in E\,, \\ \varphi(\lambda x) = \lambda\varphi(x), & \lambda>0,\ x \in E\,. \end{cases}$$

Below, for any $\varphi: E \to \mathbf{R} \cup \{+\infty\}$, we set $\varphi_-(x)=\varphi(-x)$, $x \in E$.

THEOREM 8.1. *Suppose $\mu \in \mathcal{M}_\alpha(E; C)$ $(\alpha>-\infty)$ and let φ and φ_- be μ-measurable extended valued sublinear functions such that $\varphi|_C<+\infty$ and $\mu(\varphi+\varphi_-<+\infty)>0$. Then $\varphi<+\infty$* a.s. $[\mu]$. *If $\varphi \geqq 0$, $\varphi|_C=0$, and*

(i) $-\infty<\alpha<0$, *then* $\varphi^p \in L_1(\mu)$, $0<p<-1/\alpha$,
(ii) $\alpha=0$, *then* $\exp(\varepsilon\varphi) \in L_1(\mu)$ *for some* $\varepsilon>0$,
(iii) $\alpha>0$, *then* $\varphi \in L_\infty(\mu)$.

In the special case $C=\{0\}$, Theorem 8.1 is well-known [4]. For connections with integrability of Gaussian semi-norms, se e.g. [17].

PROOF. The first part of Theorem 8.1 follows from Theorem 7.1. Now suppose $\varphi \geqq 0$ and $\varphi|_C=0$. Then, for all $s>0$ and $t>1$,

$$[\varphi>s] \supseteq \frac{2}{t+1}[\varphi \geqq st]+\frac{t-1}{t+1}[\varphi_-<s]$$

where the sets in the right-hand side are C-invariant. Consequently,

$$\mu(\varphi>s) \geqq M_\alpha^{2/(t+1)}(\mu(\varphi \geqq st), \mu(\varphi_-<s))\,.$$

CASE (i): First choose an $s>0$ satisfying the inequalities

$$\mu^\alpha(\varphi>s) > 2\mu^\alpha(\varphi_-<s) > 0\,.$$

Then $\mu(\varphi \geqq st)=O(t^{1/\alpha})$ as $t \to +\infty$ and thus $\varphi^p \in L_1(\mu)$ for each $0<p<-1/\alpha$.

CASE (ii) may be treated as Case (i).

CASE (iii): If $\varphi \notin L_\infty(\mu)$, then for all large $s>0$

$$\mu^\alpha(\varphi>s) \geqq \tfrac{2}{3}\mu^\alpha(\varphi \geqq 2s)+\tfrac{1}{3}\mu^\alpha(\varphi_-<s)$$

which implies the contradiction $0 \geqq (1/3)\mu^\alpha(\varphi_-<+\infty)$.
This completes the proof of Theorem 8.1.

Recall that a measure $\mu \in \mathcal{R}(E)$ has a barycentre at the point $e \in E$ if $E' \subseteq L_1(\mu)$ and $\xi(e)=\int \xi\, d\mu$, $\xi \in E'$. The next theorem is an example of an application of Theorem 8.1.

THEOREM 8.2. *Assume* $\mu \in \mathscr{M}_\alpha(E; C)$ $(\alpha > -1)$ *has a barycentre* $e \in E$. *Moreover, suppose G is an affine linear subspace of E such that* $\mu(K) > 0$ *for a suitable compact and convex* $K \subseteq G$. *Then* $e \in C + G$.

PROOF. Of course, there is no loss of generality to set $e = 0$. Now write $G = F - a$, where $a \in -G$ is fixed. If $0 \notin C + G$, that is, $a \notin C + F$, then we obtain a contraction as follows.

Suppose $L \subseteq F$ is a compact, convex, and symmetric set such that $\mu_a(L) = \mu(L - a) > 0$ and choose for each $n \in \mathbf{N}_+$ a $\xi_n \in E'$ such that $\xi_n(x) > \xi_n(a)$, $x \in C + nL$. Obviously, each $\xi_n \in C^+$ and without loss of generality we may assume that $\xi_n(a) = -1$. Set $\varphi = \sup_{n \in \mathbf{N}_+} \xi_n^-$. Then $\mu_a(\varphi + \varphi_- < +\infty) > 0$ and $\varphi|_C = 0$. Thus $\varphi \in L_1(\mu_a)$ by Theorem 8.1 and it follows that

$$\lim_{n \to +\infty} \int \xi_n^- \, d\mu_a = 0$$

since in view of Theorem 7.1, $\xi_n^- \to 0$ a.s. $[\mu_a]$ as $n \to +\infty$. But

$$\int \xi_n^- \, d\mu_a \geqq \left(\int \xi_n \, d\mu_a \right)^- = 1$$

and we have got a contradiction.

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