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ON THE POSTAGE STAMP PROBLEM WITH THREE STAMP DENOMINATIONS, II

ERNST S. SELMER and ARNE RÖDNE

The present paper is an immediate continuation of Selmer [8]. With one exception for Theorems 10.4–5 below, all references to theorems and formulas from section 1–10 are automatically to [8].—Incidentally, note also the completed references [4] and [7].

In section 4, we left the open question whether we always have

$$n_h(k) > g_h(k), \quad h \geqq 2,\ k \geqq 3 .$$

This has now been *proved* by Rossbach [5]. We mention this here since [5] is not easily accessible.

10. Some inequalities for $n_h(A_3)$ (continued).

For use below, we shall need other upper bounds for η and ϑ of (10.9). With $f=1$, it follows from (10.16) that

$$\eta \leqq a_3 - r - 1 = a_2 - 1 ,$$

with equality for instance in the case (10.10). For general f, we can only prove the slightly weaker

THEOREM 10.4. *For a non-pleasant basis A_3, we always have*

$$\eta \leqq a_2 - f + 1 . \tag{10.17}$$

We note that the bound $a_2 - f + 1 \geqq r + 2 \geqq 3$. There is equality in (10.17) for instance if h_0 is even, $a_2 = \frac{1}{2}h_0 + 2$, $f = \frac{1}{2}h_0$, $r = 1$, when $\eta = 3$ by (2.28).

The proof runs exactly as for Theorem 10.2. In both cases 1 and 2, we need the inequality

$$Q_{v+1} \geqq Q_f = (f-1)q_f - (f-2) \geqq 3(f-1) - (f-2) = 2f - 1 ,$$

cf. the comments to (7.14-15).

We can also give another upper bound for ϑ:

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THEOREM 10.5. *We always have*

$$\vartheta \leqq a_2+1 . \tag{10.18}$$

If $f>1$, we even have

$$\vartheta \leqq a_2 , \tag{10.19}$$

except in the pleasant cases with $s=1$.

In the proof of Theorem 10.2, it was pointed out that $\vartheta_1 \leqq a_2+1$, while the possibility $\vartheta_2 = a_2+2$ was considered separately. However, this can be excluded immediately by the contradiction

$$\vartheta_2 = a_2+2 \Rightarrow Q_{v+1} = s_{v+1} \leqq Q_v .$$

We then turn to the inequality $\vartheta \leqq a_2$. If A_3 is pleasant, $\vartheta = r+2$ by (2.11), and the only exception clearly occurs for $r=a_2-1$. We may therefore assume A_3 non-pleasant.

Since $s_{v+1}<s_v$ and $Q_{v+1}>Q_v$, $\vartheta_1 \leqq a_2$ follows immediately. To prove that $\vartheta_2 \leqq a_2$, we must show that $s_{v+1} \leqq Q_{v+1}-2$. The only exception to this would be for

$$s_{v+1} = Q_v, \quad Q_{v+1} = Q_v+1 . \tag{10.20}$$

Again, we use that $v \geqq f-1$ and $q_f>2$. We have $Q_1=Q_0+1$, even if $q_1>2$. For $f>1$, however, it follows from the recursion $Q_{i+1}=q_{i+1}Q_i-Q_{i-1}$ that

$$q_f > 2 \Rightarrow Q_f > Q_{f-1}+1 \Rightarrow Q_{i+1} > Q_i+1, \quad i \geqq f-1 ,$$

showing that (10.20) is impossible.

We can characterize equality in (10.18) by

$$f = 1 \ : \ \vartheta = a_2+1 \Leftrightarrow (\varrho+1) \mid (r+1) . \tag{10.21}$$

As in (8.4), we have put $h_0=\tau r+\varrho$, $0 \leqq \varrho < r$.

From the above proof, it follows that we can have $\vartheta = a_2+1$ only if (10.20) holds. Now,

$$Q_{v+1} = Q_v+1 \Rightarrow q_i = 2, \quad i \in \{2,3,\ldots,v+1\} \text{ (empty if } v=0) .$$

Then $Q_i=i$ for $i \leqq v+1$, and the division algorithm (2.20) gives

$$s_{v+1} = (v+1)s_1 - vs_0 . \tag{10.22}$$

We are applying Theorem 6.2, hence

$$a = a_3-1 = a_2+r-1 = h_0+r = (\tau+1)r+\varrho, \quad s_0 = a_3-a_2 = r .$$

If $\varrho=0$, then $(\varrho+1) \mid (r+1)$. If $\varrho>0$, we can write

$$a = (\tau+2)r-(r-\varrho) \Rightarrow q_1 = \tau+2, \quad s_1 = r-\varrho .$$

Substituting $s_0=r$, $s_1=r-\varrho$ in (10.22), and using $s_{v+1}=Q_v=v$, we get the relation $r+1=(v+1)(\varrho+1)$, and $\Rightarrow$ in (10.21) follows. To show $\Leftarrow$, we note that $(\varrho+1)\mid(r+1)$ implies (8.2), and $\vartheta=a_2+1$ is then an easy consequence of (8.3).

11. The sequence of h_0-ranges.

Let the positive integer n have the *regular* representation

$$n \doteq e_3a_3+e_2a_2+e_1; \quad e_1 \leqq a_2-1,\ e_1+e_2a_2 \leqq a_3-1 . \tag{11.1}$$

We may in some cases *reduce the coefficient sum* Σ by using what we will call "*a_3-transfers*", a technique closely related to the "*s-Stellen*" of Hofmeister [2]. Assuming A_3 non-pleasant, hence $q \leqq s$, we may replace one a_3 in (11.1) by qa_2-s:

$$n = (e_3-1)a_3+(e_2+q)a_2+e_1-s . \tag{11.2}$$

If $e_3 \geqq 1$ and $e_1 \geqq s$, this is a (non-regular) representation where Σ has been reduced with at least one unit compared to (11.1).

If $e_3 \geqq 2$, the process can be repeated. To recover a non-negative constant term, it may sometimes be necessary to use substitutions $a_3=fa_2+r$ instead of $a_3=qa_2-s$.

To determine $n_{h_0}(A_3)$, we must construct a consequtive string of integers n with $\Sigma \leqq h_0=a_2+f-2$. We begin with $e_3=0$ in (11.1):

$$n = e_2a_2+e_1; \quad e_1 \leqq a_2-1,\ e_2 \leqq f-1$$

$$n = fa_2+e_1; \quad e_1 \leqq r-1 \leqq a_2-2 .$$

With $n=fa_2+r=a_3$, we start a new sequence with $e_3=1$:

$$n = a_3+e_2a_2+1_1; \quad e_1 \leqq a_2-1,\ e_2 \leqq f-2$$

$$n = a_3+(f-1)a_2+e_1; \quad e_1 \leqq a_2-2 .$$

If A_3 is *pleasant*, we cannot get further, since an a_3-transfer does not reduce Σ. In this case, the h_0-range is consequently given by (2.11):

$$n_{h_0}(A_3) = a_3+(f-1)a_2+a_2-2 = 2a_3-(r+2) . \tag{11.3}$$

If A_3 is *non-pleasant*, however, the a_3-transfer of (11.2) works for the next n:

$$n = a_3+(f-1)a_2+a_2-1 = 2fa_2+r-1 ,$$

where $\Sigma \leqq h_0$ since

$$(11.4) \qquad 1 \leqq r \leqq a_2 - f - 1 \leqq a_2 - 2$$

by (4.3). Using (2.9) and (2.13), we have thus given a very simple proof of the inequality (2.12).

In what follows, we assume that A_3 is non-pleasant, hence (11.4) satisfied. The string of h_0-representable integers n then continues with

$$n = a_3 + fa_2 + e_1; \qquad e_1 \leqq r-1$$

$$n = 2a_3 + e_2 a_2 + e_1; \qquad e_1 \leqq a_2 - 1,\ e_2 \leqq f-3$$

$$n = 2a_3 + (f-2)a_2 + e_1; \qquad e_1 \leqq a_2 - 2\,.$$

The next n, with $e_1 = a_2 - 1$, can be "saved" with one a_3-transfer (11.2), and we enter a new sequence

$$n = 2a_3 + (f-1)a_2 + e_1; \qquad e_1 \leqq a_2 - 3\,.$$

In particular, we have shown that

$$(11.5) \qquad n_{h_0}(A_3) \geqq 2a_3 + (f-1)a_2 + a_2 - 3 = 3a_3 - (r+3)$$

for non-pleasant A_3.

The next n, with $e_1 = a_2 - 2$, cannot be saved by (11.2) if $s = a_2 - 1$, hence $r = 1$. But using one more a_3-transfer, now in the form $a_3 = fa_2 + 1$, we get $n = 3fa_2$ with $\Sigma = 3f \leqq h_0$ if $a_2 \geqq 2q$. Hence (11.5) holds with equality if and only if $r = 1$, $a_2 < 2q$.

It is clear how these arguments may be continued. Details are found in Rödne [6], from which we quote the following

THEOREM 11.1. *Let A_3 be non-pleasant. We then have the following sequence of h_0-ranges:*

$$(11.6) \qquad n_{h_0}(A_3) = 3a_3 - (r+3) \Leftrightarrow r = 1,\ q > \tfrac{1}{2}a_2$$

$$(11.7) \qquad n_{h_0}(A_3) = 3a_3 - (r+2) \Leftrightarrow \tfrac{1}{2}(a_2 - 1) < s = q < a_2 - 1$$

$$(11.8) \qquad n_{h_0}(A_3) = 4a_3 - (r+4) \Leftrightarrow r = 2,\ \tfrac{1}{2}(a_2 - 2) < q < a_2 - 2 \ \text{or}$$

$$r = 1,\ \tfrac{1}{3}(a_2 + 1) < q < \tfrac{1}{2}(a_2 + 1)$$

$$(11.9) \qquad n_{h_0}(A_3) = 4a_3 - (r+3) \Leftrightarrow s = q = \tfrac{1}{2}(a_2 - 1)$$

$$(11.10) \qquad n_{h_0}(A_3) = 4a_3 - (r+2) \Leftrightarrow \tfrac{1}{2}(a_2 - 1) < s = q+1 < a_2 - 2 \ \text{or}$$

$$\tfrac{1}{3}(a_2 - 1) < s = q < \tfrac{1}{2}(a_2 - 1)\,.$$

It is apparant that the number of possible forms of $n_{h_0}(A_3)$ is restricted. This fact is expressed by the following

THEOREM 11.2. *The regular representation of* $n_{h_0}(A_3)$ *has the form*

$$(11.11) \quad n_{h_0}(A_3) = e_3a_3+(f-1)a_2+e_1; \qquad a_2-e_3-1 \leqq e_1 \leqq a_2-2 .$$

This holds for pleasant A_3 by (11.3). To prove it for A_3 non-pleasant, we need the deeper theorems of section 10.

We see at once that (11.11) holds in the case (10.10). In the remaining cases, we write (10.9) as

$$n_{h_0}(A_3) = \eta a_3 - \vartheta = (\eta-1)a_3+(f-1)a_2+a_2+r-\vartheta .$$

By (10.13) and (10.18), this is the regular representation. Since $e_3=\eta-1$, the bounds for $e_1=a_2+r-\vartheta$ in (11.11) follow immediately from (10.16) and (10.13). The lower bound is non-negative by (10.17).

The form of $n_{h_0}(A_3)$ was studied in detail by Windecker [9]. In particular, his Lemma 4.2 states that $e_2=f-1$, as in (11.11).

We may also formulate Theorem 11.2 as

$$(11.12) \qquad n_{h_0}(A_3) = (e_3+1)a_3-(r+t), \quad 2 \leqq t \leqq e_3+1 .$$

In this form, we recognize the h_0-ranges as given in Theorem 11.1.

For use in a different context, we shall also mention representations with $\Sigma \leqq h_0-1$. The smallest number where this fails is given by

$$(11.13) \qquad n_0 = (f-1)a_2+a_2-1 = a_3-r-1 .$$

For non-pleasant A_3, the next case is $g_{h_0}(A_3)$ of (11.3), which cannot be saved by an a_3-transfer if $s=a_2-1$. We formulate this as

PROPOSITION 11.1. *If* A_3 *is non-pleasant, all integers in the interval* $[a_3-r, 2a_3-(r+3)]$ *have representations with at most* h_0-1 *addends.*

12. On the minimal value of $n_h(A_3)$.

For given h, the *extremal bases* A_3^*, with the largest possible *extremal h-range* $n_h(3)=n_h(A_3^*)$, were determined by Hofmeister [2, Satz 2], cf. the comments to (4.7). He also considered the *extremal* h_0*-ranges* $l_h(3)$ [2, Satz 3–4], where we only consider those bases A_3 which for given h have $h_0=h$.

In the literature, "extremal" in this connection always means "maximal". It is not unnatural, however, to ask also for *minimal* h-ranges and h_0-ranges. With $k=3$, it turns out that the search should be made over pleasant and non-pleasant A_3 *separately.* Of the four combinations thus arising, three of them have a trivial solution. It is not difficult to prove the following results for given h_0 or h:

For A_3 pleasant, the minimal h_0-range occurs in the case

$$n_{h_0}(1, h_0+1, 2h_0+1) = 3h_0 , \tag{12.1}$$

with one additional possibility for $h_0=2$:

$$n_2(1,2,4) = 6 = 3\cdot 2 . \tag{12.2}$$

For A_3 pleasant, the minimal h-range occurs in the cases

$$n_h(1, h+1, 2h+1) = n_h(1,2,3) = 3h ,$$

with the one additional possibility (12.2).

For A_3 non-pleasant, the minimal h-range occurs in the case

$$n_h(1,3,4) = 4h .$$

The only non-trivial result is expressed by the following

THEOREM 12.1. *Over non-pleasant bases A_3, the minimal h_0-range for $h_0 \neq 4$ is given by*

$$h_0 \equiv 0 \pmod 3) : n_{h_0}\left(1, \frac{2h_0+6}{3}, \frac{2h_0^2+9h_0+9}{9}\right) = \frac{2h_0^2+8h_0}{3} \tag{12.3}$$

$$h_0 \equiv 1 \pmod 3) : n_{h_0}\left(1, \frac{2h_0+4}{3}, \frac{2h_0^2+8h_0+17}{9}\right) = \frac{2h_0^2+8h_0+5}{3} \tag{12.4}$$

$$h_0 \equiv 2 \pmod 3) : n_{h_0}\left(1, \frac{2h_0+5}{3}, \frac{2h_0^2+7h_0+14}{9}\right) = \frac{2h_0^2+7h_0+2}{3} . \tag{12.5}$$

The bases A_3 were first conjectured by inspection, and the result was verified by Mossige for $h_0 \leqq 100$. The proof is simplified by assuming $h_0 \geqq 24$.

The cases (12.4–5) have $r=1$, and (12.3) has $s=q$. The expressions for $n_{h_0}(A_3)$ then follow easily from (2.28–29).

The formulas (12.3–5) may also be written in the more concentrated form:

$$a_2 = \left[\frac{2h_0}{3}\right] + 2$$

$$\begin{aligned} h_0 &\equiv 0 \pmod 3) : a_3 = \tfrac{1}{2}a_2(a_2-1), && n_{h_0}(A_3) = 3a_3 - \tfrac{1}{2}a_2 - 2 \\ h_0 &\equiv 1 \pmod 3) : a_3 = \tfrac{1}{2}a_2^2+1 && \\ h_0 &\equiv 2 \pmod 3) : a_3 = \tfrac{1}{2}a_2(a_2-1)+1 && \Big\} \; n_{h_0}(A_3) = 3a_3 - 4 . \end{aligned}$$

To prove Theorem 12.1, we note that the largest h_0-range, as a function of h_0, occurs in the case (12.4). We denote this range by

$$m_{h_0} = \tfrac{1}{3}(2h_0^2 + 8h_0 + 5) ,$$

and proceed to show that "most" bases A_3 have $n_{h_0}(A_3) > m_{h_0}$. For this purpose, we use (10.2):

$$n_{h_0}(A_3) > h_0 a_2 , \tag{12.6}$$

but must then consider the *Frobenius-dependent* bases separately. For these bases, we know by (5.5) that

$$n_{h_0}(A_3) \geqq (h_0+1)^2 - r(r-1) - 1 .$$

Since $r \mid h_0$ or $r \mid (h_0+1)$, and A_3 is non-pleasant, we have $r \leqq \frac{1}{2}(h_0+1)$. It then follows that $n_{h_0}(A_3) > m_{h_0}$ for $h_0 \geqq 10$.

Using (12.6), we may thus confine ourselves to bases A_3 with $a_2 < m_{h_0}/h_0 \leqq \frac{1}{3}(2h_0+9)$ for $h_0 \geqq 5$, hence $f = h_0+2-a_2 > \frac{1}{3}h_0 - 1$,

$$f \geqq \tfrac{1}{3}(h_0-2) . \tag{12.7}$$

On the other hand, it follows from (4.4) that

$$a_2 \geqq \tfrac{1}{2}(h_0+4) . \tag{12.8}$$

It turns out that it suffices to consider the bases of Theorem 11.1. By Theorem 11.2, the next value in the sequence of h_0-ranges is namely

$$n_{h_0}(A_3) = 4a_3 + (f-1)a_2 + a_2 - 5 \geqq 5fa_2 - 1 , \tag{12.9}$$

since $a_3 \geqq fa_2+1$. And by (12.7–8), $5fa_2 - 1 > m_{h_0}$ for $h_0 \geqq 12$.

We first consider the case (11.6), with $r=1$:

$$n_{h_0}(A_3) = 3a_3 - 4 = 3fa_2 - 1 = 3(h_0+2-a_2)a_2 - 1 , \tag{12.10}$$

where

$$q = f+1 = h_0+3-a_2 > \tfrac{1}{2}a_2 \Rightarrow a_2 < \tfrac{2}{3}h_0+2 .$$

The last expression in (12.10) attains its (formal) maximum for $a_2 = \frac{1}{2}(h_0+2)$, which is smaller than the bound (12.8). The minimal value of $n_{h_0}(A_3)$ in (12.10) thus occurs for the largest value of a_2 with $a_2 < \frac{2}{3}h_0 + 2$, which depends on the residue of h_0 (mod 3).

$$h_0 \equiv 0 \pmod 3 : a_2 = \tfrac{2}{3}h_0 + 1 \Rightarrow n_{h_0}(A_3) = \tfrac{2}{3}h_0^2 + 3h_0 + 2 > m_{h_0}$$

$$h_0 \equiv 1 \pmod 3 : a_2 = \frac{2h_0+4}{3} \Rightarrow (12.4)$$

$$h_0 \equiv 2 \pmod 3 : a_2 = \frac{2h_0+5}{3} \Rightarrow (12.5) .$$

We next consider the case (11.7), with $s=q$:

$$n_{h_0}(A_3) = 3(qa_2-q)-(a_2-q+2) = (h_0+3-a_2)(3a_2-2)-a_2-2\,,$$

where

$$q = h_0+3-a_2 > \tfrac{1}{2}(a_2-1) \Rightarrow a_2 < \tfrac{1}{3}(2h_0+7)\,.$$

Now $h_0 \equiv 0 \pmod 3) \Rightarrow$ (12.3), while $h_0 \equiv 1,2$ give $n_{h_0}(A_3) > m_{h_0}$ for $h_0 \geqq 5$.

We finally consider the cases (11.8–10). In analogy with (12.9), these give

(12.11) $$n_{h_0}(A_3) \geqq 4fa_2-1 = 4(h_0+2-a_2)a_2-1\,.$$

For $h_0 \geqq 10$, hence $a_2 \geqq 7$ by (12.8), the smallest bound for q in (11.8–10) is $\frac{1}{3}(a_2-1)$, and

$$q = h_0+3-a_2 > \tfrac{1}{3}(a_2-1) \Rightarrow a_2 < \tfrac{1}{4}(3h_0+10)\,.$$

The last expression (12.11) then shows that $n_{h_0}(A_3) > m_{h_0}$ for $h_0 \geqq 24$. This completes the proof of Theorem 12.1.

13. On minimal ranges in general.

For *regular* h-ranges, it is easily shown that for given h and k, the minimal regular h-range is attained in the one case

(13.1) $$g_h(A_k) = g_h(1,h+1,h+2,\ldots,h+k-1) = 2h+k-2\,.$$

Since A_k has $h_0 = h$, this also gives the minimal regular h_0-range. For $k \geqq 3$, A_k is non-pleasant, with $n_h(A_k) = ha_k$.

The problem of minimal *ordinary* ranges is much more difficult. We solved it completely for $k=3$ in the previous section, and shall mention briefly some theoretical and numerical results for $k>3$.

In what follows, we *disregard* the trivial case

(13.2) $$n_h(1,2,\ldots,k) = hk$$

(the only basis with $h_0 = 1$). All other bases A_k have $a_k > k$.

It was observed that for a large number of combinations of small h_0 and k, we always have

(13.3) $$n_{h_0}(A_k) \geqq h_0 k\,.$$

For a long time, we even denoted this inequality as the "minimum-conjecture", until it was recently disproved by Klöve [3]. His simplest counter-example (given by his Theorem 13) is

$$(13.4)\quad \begin{cases} h_0 = 3\,, \\ A_{44} = \{1,4,5,16,17,20,21,64\text{–}84,88,92,96\text{–}100,104\,, \\ \qquad 108,112\text{–}116,120,124\} \\ n_3(A_{44}) = 126 < 3\cdot 44\,. \end{cases}$$

If (13.3) holds, it follows from (2.13) and $a_k > k$ that

$$n_h(A_k) > hk, \qquad h > h_0\,.$$

We saw in the previous section that (13.3) holds for $k=3$, with equality for the *pleasant* bases (12.1–2). An immediate generalization of (12.1) is

$$(13.5)\qquad n_{h_0}(A_k) = n_{h_0}(1, h_0+1, 2h_0+1, \ldots, (k-1)h_0+1) = h_0 k\,,$$

with pleasant A_k. For $k>3$, however, there are also *non-pleasant* bases satisfying (13.3). This means that the non-trivial minimal bases of Theorem 12.1 have *no counterpart for* $k>3$.

Klöve [3] shows that (13.3) *always holds for* $h_0=2$. We shall indicate below a proof of

THEOREM 13.1. *We always have*

$$n_{h_0}(A_k) \geqq h_0 k \qquad \textit{for } k \leqq 7\,.$$

For this purpose, we need the following result which was suggested in a private communication by Rödseth (see also [7, p. 174]):

$$(13.6)\qquad a_i \leqq (i-1)h+1, \quad i = 2,3,\ldots,k \Rightarrow n_h(A_k) \geqq hk\,.$$

In particular, equality for all a_i implies equality for $n_h(A_k)$ by (13.5).

To prove (13.6), we use the well known theorem of Dyson, cf. Halberstam and Roth [1, Theorem 7, p. 17]. Let $\mathscr{A}$ be a finite set of non-negative integers, including 0, and define $A(m)$ as the number of positive integers $\leqq m$ in $\mathscr{A}$. The ratios $A(m)/m$ may then be considered as *densities* in sections of $\mathscr{A}$. In particular, $A(m)/m=1$ means that $\mathscr{A}$ contains *all* positive integers $\leqq m$.

As usual, we define

$$\mathscr{C} = h\mathscr{A} = \{\alpha_1+\alpha_2+\ldots+\alpha_h \mid \alpha_i \in \mathscr{A}\}\,,$$

and select $\mathscr{A} = A_k \cup \{0\}$. Then $C(m)/m = 1$ means that $n_h(A_k) \geqq m$.

It follows from Dyson's theorem that

$$\frac{A(m)}{m} \geqq \delta, \quad m = 1,2,\ldots,n \Rightarrow \frac{C(m)}{m} \geqq \min\{1, h\delta\}, \quad m = 1,2,\ldots,n\,.$$

With $\mathscr{A}=A_k \cup \{0\}$ and $\delta=1/h$, this implies

$$A(m) \geqq \frac{m}{h}, \quad m = 1,2,\ldots,n \Rightarrow n_h(A_k) \geqq n ,$$

where

$$m \in [a_{i-1}, a_i - 1] \Rightarrow A(m) = i-1, \quad i=2,3,\ldots,k ,$$
$$m \geqq a_k \Rightarrow A(m) = k .$$

Now (13.6) is an immediate consequence if we put $n=hk$.

The first inequality (13.6),

$$(13.7) \qquad a_2 \leqq h+1 ,$$

must *always* be satisfied for an admissible basis. For $i>1$, however, (13.6) puts severe restrictions on A_k. All the same, the result must be considered as "deep".

We note that

$$(13.8) \qquad a_k > (k-1)h \Rightarrow n_h(A_k) \geqq (h-1)\cdot 1 + 1\cdot a_k \geqq hk .$$

As a corollary to (13.6-8), we see at once that (13.3) holds for $k=3$. To study $k>3$, we examine the validity of the similar implication

$$(13.9) \qquad a_{k-j} > (k-j-1)h+1 \Rightarrow n_h(A_k) \geqq hk, \quad 1 \leqq j \leqq k-3 .$$

If this holds for $j \leqq j_0$, we conclude similarly that (13.3) holds for $k \geqq j_0+3$.

The proof for $j=j_0=1$ is simple, since

$$n_h(A_k) \geqq n_{h-1}(A_k)+a_k \geqq n_{h-1}(A_2)+a_k = (h+2-a_2)a_2-2+a_k .$$

If $a_2 \leqq h$, the product $(h+2-a_2)a_2$ attains its minimum $2h$ for $a_2=2$ and $a_2=h$. Since $a_k \geqq a_{k-1}+1>(k-2)h+2$, we even get $n_h(A_k)>hk$.

The case $a_2=h+1$ must be considered separately, using $a_{k-1}+h \leqq a_k+h-1 \leqq n_h(A_k)$. We can then also represent $a_{k-1}+h+1=a_{k-1}+a_2$ (assuming $h \geqq 2$), and further the succeeding integers up to $a_{k-1}+a_2+h-2>hk$.

The cases (13.9) with $j=2,3$ are treated in Rödne [6]. The proof for $j=2$ is moderately simple, but $j=3$ is rather complicated, with many alternatives and special cases.

No attempt has been made to examine (13.9) for $j=4$. To prove (13.3) also for the (final) value $k=7$, the following "shortcut" simplified matters considerably: By the preceding results, it sufficed to show that $n_h(A_7) \geqq 7h$ when $a_3>2h+1$ and $a_4 \leqq 3h+1$.

We shall also discuss the cases with *equality* in (13.3)—still disregarding (13.2). For $k=3$, all cases are given by (12.1–2). For $k=4$, it is easily shown (Rödne [6]) that all cases are given by (13.5) and by

$$n_{h_0}(1,2,2h_0+1,2h_0+2) = 4h_0 . \quad (13.10)$$

In addition, there are three bases with $h_0=2$:

$$A_4 = \{1,2,3,5\}, \{1,2,4,6\}, \{1,3,4,7\} . \quad (13.11)$$

For $k \geqq 5$, the numbers in Table 6 were computed by Mossige. Within the range of the table, there are no bases with $n_{h_0}(A_k) < h_0 k$.

Table 6. *The number of bases satisfying* $n_{h_0}(A_k) = h_0 k$.

h_0 \ k	5	6	7	8	9	10
2	9	19	36	72	138	274
3	2	4	1	4	6	7
4	1	4	2	3	2	4
5	1	3	2	4	2	3

It is quite striking that we get such large numbers for $h_0=2$, approximately proportional to 2^k. This case is discussed further in Klöve [3].

For $h_0>2$, a considerable number of the bases in Table 6 can be covered by general formulas. We first treat a generalization of (13.10) to all cases when k has a proper factorization $k=k_1k_2$. For instance, we have for $k=6$:

$$n_{h_0}(1,2,2h_0+1,2h_0+2,4h_0+1,4h_0+2) = 6h_0$$

$$n_{h_0}(1,2,3,3h_0+1,3h_0+2,3h_0+3) = 6h_0 .$$

It is not difficult to guess the general form:

$$A_k = A_{k_1k_2} = \{ik_2h_0+j \mid i=0,1,\ldots,k_1-1; j=1,2,\ldots,k_2\} .$$

A simple proof of (13.3) for these bases, by induction on k_1, is found in Rödne [6]. It is easily seen that all integers in $[0, h_0k]$ have *regular* representations by A_k with at most h_0 addends, so $g_{h_0}(A_k)=n_{h_0}(A_k)=h_0k$. All the same, A_k is *not pleasant* for $h_0>1$, since $a_k-a_{k-1}=1$, and (2.16) then implies $n_{h_1}(A_k)=h_1a_k > g_{h_1}(A_k)$.

Next, we note that the last basis (13.11) is of the form A_{h+2} of section 3, for which $n_h(A_{h+2})=h(h+2)=hk$. This observation may be generalized in several directions. As an illustration, we mention four simple possibilities.

Let $a_2=h_0+1$, $a_3=h_0+2$. The following bases then satisfy $n_{h_0}(A_k)=h_0k$:

$$\{1\} \cup \{(\delta i+1)a_2 \mid i=0,1,\ldots,t-1\}$$

$$\cup \{ja_2+a_3 \mid j=0,1,\ldots,th-1\}, \; k=th+t+1$$

$$\{1\} \cup \{\delta i a_2 + a_3 \mid i = 0, 1, \ldots, t-1\} \cup \{j a_2 \mid j = 1, 2, \ldots, th+1\},\ k = th + t + 2 .$$

In both cases, we may choose either $\delta = 1$ or $\delta = h_0$. The proofs (Rödne [6]) are simple, by induction on t. All the bases are *non-pleasant* for $h_0 > 1$. This is the case for the partial basis $A_3 = \{1, h_0+1, h_0+2\}$, and Zöllner [10] has shown in general that A_k pleasant $\Rightarrow$ A_3 pleasant.

We conclude with two natural questions regarding the "minimum-conjecture":

1) *What is the smallest* $k = K$ *such that* $n_{h_0}(A_K) < h_0 K$ *for some basis* A_K*?*
We have seen that $8 \leqq K \leqq 44$, which means quite a large gap in our knowledge.

2) *What is the smallest* $j = J$ *such that the implication* (13.9) *fails for some basis* A_k*?*
Since Klöve's basis (13.4) has $a_{k-j} \leqq (k-j-1)h + 1$ for $j \leqq 10$ but not for $j = 11$, we know that $4 \leqq J \leqq 11$.

Both questions are apparently very difficult.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BERGEN
N-5000 BERGEN
NORWAY