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THE MINIMAL RANGE OF ADDITIVE h-BASES

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1. Introduction.

For any two sets A and B of non-negative integers, the sum $A+B$ is defined by

$$A+B = \{a+b \mid a \in A \text{ and } b \in B\} .$$

Further, for $h \geqq 1$, $hA = A+A+\ldots+A$ (h times), that is

$$hA = \{a^{(1)}+a^{(2)}+\ldots+a^{(h)} \mid a^{(i)} \in A \text{ for } i=1,2,\ldots,h\} .$$

For integers n_1 and n_2,

$$[n_1,n_2] = \{m \mid m \text{ integer and } n_1 \leqq m \leqq n_2\} .$$

All sets considered in this paper will be assumed to contain 0 and 1. We say that A is an h-base for n if $[0,n] \subset hA$. The largest n such that A is an h-base for n is called the *h-range* of A and is denoted by $n(h,A)$. Let $A=\{a_0,a_1,a_2,\ldots,a_k\}$ where $a_0=0$, $a_1=1<a_2<\ldots<a_k$. If $n(h,A)<a_k$, then a_k does not play any part in representing $m \in [0,n(h,A)]$ as $m=\sum_{j=1}^{h} a_{i_j}$, and so $n(h,\{a_0,a_1,\ldots,a_{k-1}\})=n(h,A)$. In this paper our object of study is the h-range. Therefore, it is a natural restriction to consider only bases A such that $n(h,A) \geqq \max A$. Such bases are called *h-admissible*. We denote by $\mathscr{A}(h,k)$ the set of h-admissible bases which contain the integer 0 and k positive integers. Let

$$n(h,k) = \max\{n(h,A) \mid A \in \mathscr{A}(h,k)\} ,$$

$$\nu(h,k) = \min\{n(h,A) \mid A \in \mathscr{A}(h,k)\} .$$

The first of these, $n(h,k)$, has been studied extensively, see e.g. [1], [4], [6], [8] and references given in those papers. In particular $n(h,2)$ and $n(h,3)$ are known for all h. The minimum, $\nu(h,k)$, has been studied only recently. In an unpublished note [5], Selmer describes some computations by Mossige. Mossige computed $\nu(h,k)$ for $h=2$, $k \leqq 10$; $h=3$, $k \leqq 18$, $h \leqq 10$, $k \in [4,8]$; $h \leqq 8$, $k \in [9,13]$; $h \leqq 6$, $k \in [14,15]$. He also computed all bases A in $\mathscr{A}(h,k)$ with $n(h,A)=\nu(h,k)$. The computations by Mossige were prompted by a conjecture by Rødseth (unpublished) that $\nu(h,k)=hk$ for all h and k. The conjecture has

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been referred to by Selmer [5] and [7] as the *minimum-conjecture*. The computations by Mossige showed that $\nu(h,k)=hk$ for the range of values he computed. A simple argument proves that $\nu(2,k)=2k$ for all k. The proof was given in [2] and referred to in [5]. It may easily be generalized to prove that $\nu(h,k)\geqq 2k+(h-2)$ for all $h\geqq 2$ and all k, a proof is given below in section 2. Rødne [3] has proved that $\nu(h,k)=hk$ for $k\leqq 7$ and all h. Despite this mounting evidence for the truth of the minimum-conjecture, Rødseth himself has expressed doubt about its general validity. A main result in this paper is that for each $h\geqq 2$ and each $\varepsilon>0$ there exists a base A in $\mathscr{A}(h,k)$, for some k, such that $n(h,A)<(2+\varepsilon)k$. In particular, this proves that the minimum-conjecture is far from true in general. In section 3 we study a class of bases with small range and in section 4 we show that this class contains bases with the property $n(h,A)<(2+\varepsilon)k$.

The computations by Mossige showed that the number of bases in $\mathscr{A}(2,k)$ with range $\nu(2,k)$ is increasing rapidly with k for $k\leqq 10$. In section 5 we give a characterization of the bases in $\mathscr{A}(2,k)$ with range $\nu(2,k)$. Further we give some results on the number of such bases. A preliminary version of the results on 2-bases appeared in the unpublished notes [2].

2. General bounds on $\nu(h,k)$.

In this section we give upper and lower bounds on $\nu(h,k)$ valid for all $h\geqq 2$ and all k.

THEOREM 1. *For all $h\geqq 2$ and all $k\geqq 1$, $2k+(h-2)\leqq\nu(h,k)\leqq hk$.*

PROOF. The upper bound is well known and there are a number of bases that prove it, see [3] or [5]. The simplest is $A=[0,k]$ which belongs to $\mathscr{A}(h,k)$ for all h and has h-range $n(h,A)=hk$.

Next, let $A=\{a_0=0,\ a_1=1,a_2,\ldots,a_k\}$ be any base in $\mathscr{A}(h,k)$ and let $a=a_k=\max A$. By definition of $\mathscr{A}(h,k)$, $[0,a]\subset hA$. Suppose $a\geqq 2k$. Then $n(h,A)\geqq a+(h-1)\geqq 2k+h-1$ since $a+l=a+l\cdot 1+(h-1-l)\cdot 0\in hA$ for $1\leqq l\leqq h-1$.

Next, suppose $a<2k$. Let $m>a$ be an integer such that $m\notin 2A$. Consider the set of pairs $P=\{(j,m-j)\mid 0\leqq j\leqq m\}$. Each pair contains at most one element from A since $m=j+(m-j)$ and $m\notin 2A$. On the other hand, each element $a_i\in A$ appears in exactly two pairs in P, namely $(a_i,m-a_i)$ and $(m-a_i,a_i)$. Hence

$$m+1 \;=\; \#P \;\geqq\; 2\#A \;=\; 2(k+1)$$

and so $m\geqq 2k+1$. Hence $[a+1,2k]\subset 2A$. In particular, $2k\in 2A$, and so $2k+l=2k+l\cdot 1+(h-2-l)\cdot 0\in hA$ for $1\leqq l\leqq h-2$. This proves the lower bound.

COROLLARY 2. *For* $k \geqq 1$, $\nu(2,k) = 2k$.

3. A class of bases with small range.

For a base A and integers h and K consider the following condition which may or may not be true,

$$(*) \qquad (\forall d \in [0, K+1])(d \in (h-1)A \Leftrightarrow K+1-d \notin A).$$

Our reason for considering (*) is that all known h-bases A of minimal range satisfy (*) with $K = n(h, A)$. Moreover, for $h = 2$, this characterizes the bases with minimal range; we prove this in section 5. On the other hand, for $h \geqq 3$ there exist bases A which satisfy (*) with $K = n(h, A)$ but which do not have minimal range. We shall use the following notations.

NOTATIONS. (i) For B a base such that $B \subset [0, \varkappa]$ and $n(h, B) \geqq \varkappa$, let

$$\bar{B}_\varkappa = \{\varkappa - d \mid d \in [0, \varkappa] \text{ and } d \notin (h-1)B\}.$$

$$\hat{B}_\varkappa = B \cup \{\varkappa + 1 + \beta \mid \beta \in \bar{B}_\varkappa\}.$$

(ii) A base B is called *restricted* for h if $n(h, B) = h \cdot \max B$; the set of such bases is denoted by $\mathscr{R}_h$.

(iii) For $B \in \mathscr{R}_h$, let $n_B = n(h, B)$, $N_B = n(h, B) + 1$.

(iv) Let

$$\mathscr{R}_h = \{B \in \mathscr{R}_h \mid \hat{B}_{n_B} \text{ satisfies } (*) \text{ with } K = 2n_B\}.$$

(v) For $B, C \in \mathscr{R}_h$, let

$$B \circ C = \{cN_B + b \mid c \in C \text{ and } b \in B\}.$$

REMARKS. (i) The sets $\bar{B}_\varkappa$ and $\hat{B}_\varkappa$ depend on B and $\varkappa$. In this section we write just $\bar{B}$ and $\hat{B}$ when $\varkappa = n_B$.

(ii) For a discussion of restricted bases, see [8].

LEMMA 3. *If* $\hat{B}_\varkappa$ *satisfies* (*) *with* $K = 2\varkappa$, *then* $n(h, \hat{B}_\varkappa) = 2\varkappa$.

PROOF. First we note that if $m \in [0, \varkappa]$, then $m \leqq n(h, B)$ and so $m \in hB \subset h\hat{B}_\varkappa$. Next we show by induction on m that $\varkappa + m \in h\hat{B}_\varkappa$ for $0 \leqq m \leqq \varkappa$. For $m = 0$ this was shown above.

Let $0 < m \leqq \varkappa$. Then $0 < 2m - 1 \leqq \varkappa + m - 1 < \varkappa + m$. By the induction hypothesis, $2m - 1 = a + b$ where $a \in \hat{B}_\varkappa$ and $b \in (h-1)\hat{B}_\varkappa$. If $\varkappa + m - a \in (h-1)\hat{B}_\varkappa$, then $\varkappa + m = a + (\varkappa + m - a) \in h\hat{B}_\varkappa$ and the induction step is complete.

On the other hand, if $\varkappa+m-a \notin (h-1)\hat{B}_\varkappa$, then by (*), $2\varkappa+1-(\varkappa+m-a) \in \hat{B}_\varkappa$. But

$$\begin{aligned} 2+1-(\varkappa+m-a) &= \varkappa+m-(2m-1-a) \\ &= \varkappa+m-b\,. \end{aligned}$$

Hence $\varkappa+m=(\varkappa+m-b)+b \in h\hat{B}_\varkappa$ which completes the induction step.

The induction proves that $[0,2\varkappa]\subset h\hat{B}_\varkappa$. Finally, by (*), if $b \in \hat{B}_\varkappa$, then $2\varkappa+1-b \notin (h-1)\hat{B}_\varkappa$. Hence $2\varkappa+1 \notin h\hat{B}_\varkappa$ and so $n(h,\hat{B}_\varkappa)=2\varkappa$.

Corollary 4. *If $B \in \hat{\mathscr{R}}_h$, then $n(h,\hat{B})=2n(h,B)$.*

Next we give a couple of properties of restricted bases.

Lemma 5. *Let $B \in \mathscr{R}_h$. Then*

(i) $\#\hat{B}=\#B+\#\bar{B}$,
(ii) $[0,(\max B)-1]\subset\bar{B}$.

Proof. (i) follows directly from the definition of $\hat{B}$. (ii) Let $b=\max B$. Clearly $(h-1)B\subset[0,(h-1)b]$. Hence if $d \in [0,b-1]$, then $hb-d \notin (h-1)B$ and so $d \in \bar{B}$.

Lemma 6. *Let $B, C \in \mathscr{R}_h$. Then*

(i) $B\circ C \in \mathscr{R}_h$,
(ii) $\#(B\circ C)=\#B\cdot\#C$,
(iii) $n(h,B\circ C)+1=(n(h,B)+1)(n(h,C)+1)$.

Proof. In this and later proofs in this section we use the notations

$$\begin{aligned} B &= \{b_0=0,b_1,\ldots,b_k\}, \qquad b_k=\max B\,, \\ C &= \{c_0=0,c_1,\ldots,c_l\}, \qquad c_l=\max C\,, \\ D &= B\circ C\,. \end{aligned}$$

Note that $\max D=c_lN_B+b_k=hc_lb_k+c_l+b_k$. Further

$$\begin{aligned} hD &= \{yN_B+x \mid y \in hC \text{ and } x \in hB\} \\ &= \{yN_B+x \mid y \in [0,hc_l] \text{ and } x \in [0,hb_k]\} \\ &= [0,hc_lN_B+hb_k] = [0,h\cdot\max D]\,. \end{aligned}$$

Hence $D \in \mathscr{R}_h$ and

$$N_D = hc_l N_B + hb_k + 1 = N_B \cdot N_C ,$$

which proves (i) and (iii). Finally, (ii) follows directly from the definition of $B \circ C$.

We now study $\hat{\mathscr{R}}_h$ in some detail. First we give two alternative characterizations which are easier to apply than the definition itself. Next we give some subclasses of $\hat{\mathscr{R}}_h$. Finally we show that $\hat{\mathscr{R}}_h$ is closed under $\circ$.

LEMMA 7. *The following three conditions are equivalent for* $B \in \mathscr{R}_h$.

(i) $B \in \hat{\mathscr{R}}_h$

(ii) $(\forall x \in [0, n_B])(x \in \bar{B} + (h-2)B \Leftrightarrow n_B - x \notin B)$

(iii) $(\forall x \in [(h-1)b, hb])(x \in \{\beta \in \bar{B} \mid \beta \geqq b\} + (h-2)B \Leftrightarrow n_B - x \notin B)$, *where* $b = \max B$.

PROOF. First we show that (i) implies (ii). Let $B \in \hat{\mathscr{R}}_h$ and let $x \in [0, n_B]$. Then

$$\begin{aligned} & x \in \bar{B} + (h-2)B \\ \Leftrightarrow\ & n_B + 1 + x \in \{n_B + 1 + \beta \mid \beta \in \bar{B}\} + (h-2)B \\ \Leftrightarrow\ & n_B + 1 + x \in (h-1)\hat{B} \\ \Leftrightarrow\ & n_B - x = (2n_B + 1) - (n_B + 1 + x) \notin \hat{B} \\ \Leftrightarrow\ & n_B - x \notin B . \end{aligned}$$

Next we show that (ii) implies (i). Let $B \in \mathscr{R}_h$ satisfy (ii). If $d \in [0, n_B]$, then

$$\begin{aligned} & d \in (h-1)\hat{B} \\ \Leftrightarrow\ & d \in (h-1)B \\ \Leftrightarrow\ & n_B - d \notin \bar{B} \\ \Leftrightarrow\ & 2n_B + 1 - d = (n_B + 1) + (n_B - d) \notin \hat{B} . \end{aligned}$$

If $d \in [n_B + 1,\ 2n_B + 1]$, then

$$\begin{aligned} & d \in (h-1)\hat{B} \\ \Leftrightarrow\ & d \in \{n_B + 1 + \beta \mid \beta \in \bar{B}\} + (h-2)B \\ \Leftrightarrow\ & d - n_B - 1 \in \bar{B} + (h-2)B \\ \Leftrightarrow\ & 2n_B + 1 - d = n_B - (d - n_B - 1) \notin B \\ \Leftrightarrow\ & 2n_B + 1 - d \notin \hat{B} \end{aligned}$$

since $2n_B+1-d<n_B+1$. Hence $\hat{B}$ satisfies (*) and so $B\in\hat{\mathscr{R}}_h$. Further we show that (ii) and (iii) are equivalent. Let $B\in\mathscr{R}_h$. By Lemma 5 (ii), $\bar{B}=[0,b-1]\cup E$, where $E=\{\beta\in\bar{B}\mid \beta\geqq b\}$. Consider $x\in[0,(h-1)b-1]$. On the one hand, $hb-x>b$ and so $hb-x\notin B$. On the other hand, $x=qb+r$ where $q\in[0,h-2]$, $r\in[0,b-1]$. Hence $x\in qB+\bar{B}\subset\bar{B}+(h-2)B$. Therefore

$$(\forall x\in[0,hb])(x\in\bar{B}+(h-2)B \Leftrightarrow n_B-x\notin B)$$
$$\Leftrightarrow (\forall x\in[(h-1)b,hb])(x\in\bar{B}+(h-2)B \Leftrightarrow n_B-x\notin B)$$
$$\Leftrightarrow (\forall x\in[(h-1)b,hb])(x\in E+(h-2)B \Leftrightarrow n_B-x\notin B)$$

since $[0,b-1]+(h-2)B\subset[0,(h-1)b-1]$.

PROPOSITION 8. *The following classes of bases belong to* $\hat{\mathscr{R}}_h$.

(i) $B=[0,d]$, $d>0$; $\#\bar{B}=d$.
(ii) $B=\{0,dh+2\}\cup\{jh+1\mid j\in[0,d]\}$, $d\geqq 0$; $\#\bar{B}=2dh-d+2$.
(iii) $B=\{j\Gamma+i\Delta\mid j\in[0,s]$ *and* $i\in[0,r]\}$
$\cup\{j\Gamma+i\Delta+1\mid j\in[0,s]$ *and* $i\in[0,r]\}$
$\cup\{j\Gamma+i\Delta+1\mid j\in[0,s-1]$ *and* $i\in[r+1,rh-1]\}$

where $r\geqq 1$, $s\geqq 1$, $\Delta=h+1$, $\Gamma=rh(h+1)+1$;

$$\#\bar{B} = 2h^2rs+hr(2r+s)-rs+s+1 .$$

PROOF. (i). Since $(h-1)B=[0,(h-1)d]$ and $n_B=hd$ we have $\bar{B}=[0,d-1]$. Hence $\bar{B}+(h-2)B=[0,(h-1)d-1]$. By Lemma 7, $B\in\hat{\mathscr{R}}_h$.

(ii) Let $\Delta=hd+1$. For $l\in[1,h]$, $i\in[0,l]$, $j\in[0,d-1]$, let

$$S(l,i) = [i\Delta,i\Delta+l] ,$$
$$T(l,i,j) = [i\Delta+jh+1,\ i\Delta+jh+l] .$$

First we prove by induction that, for $l\in[1,h]$,

$$lB = \bigcup_{i=0}^{l} S(l,i)\cup\bigcup_{i=0}^{l-1}\bigcup_{j=1}^{d-1} T(l,i,j) .$$

It is true for $l=1$. Let $l\in[1,h-1]$. Then for $i\in[0,l]$, $i'\in[0,1]$, $j,j'\in[1,d-1]$

$$S(l,i)+S(1,i') = S(l+1,i+i') ,$$
$$T(l,i,j)+S(1,i') = S(l,i)+T(1,i',j) = T(l+1,i+i',j) ,$$

$$T(l,i,j)+T(1,i',j') \subset \begin{cases} T(l+1,i+i',j+j') & \text{for } 2\leqq j+j'\leqq d-1 ,\\ S(l+1,i+i'+1) & \text{for } j+j'=d \\ T(l+1,i+i'+1,j+j'-d) & \text{for } d<j+j'\leqq 2d-2 .\end{cases}$$

This proves the induction step.

In particular $hB=[0,h(dh+2)]$ and so $B \in \mathscr{R}_h$. Further,

$$(h-1)B = \bigcup_{i=0}^{h-1} [i\Delta, i\Delta+h-1] \cup \bigcup_{i=0}^{h-2} \bigcup_{j=1}^{d-1} [i\Delta+jh+1, i\Delta+jh+h-1] .$$

Hence

$$[0,n_B] \setminus (h-1)B = \{i\Delta+jh \mid i \in [0,h-2], j \in [1,d]\} \cup [(h-1)\Delta+h,\ n_B]$$

and so, since $n_B=h(\Delta+1)=(h-1)\Delta+(d+1)h+1$,

$$\begin{aligned}\bar{B} &= [0,\Delta] \cup \{(h-1-i)\Delta+(d+1-j)h+1 \mid i \in [0,h-2],\ j \in [1,d]\} \\ &= [0,\Delta] \cup \{y\Delta+xh+1 \mid y \in [1,h-1],\ x \in [1,d]\} .\end{aligned}$$

In particular $\#\bar{B}=\Delta+1+(h-1)d=2hd-d+2$. Next we will show that B satisfy the condition of Lemma 7 (iii). Let $U(y,x)=[y\Delta+xh+1,\ y\Delta+xh+h-1]$. Then

$$(y\Delta+xh+1)+S(h-2,i) = U(y+i,x)$$

and

$$(y\Delta+xh+1)+T(h-2,i,j) \subset \begin{cases} U(y+i,x+j) & \text{for } x+j \leqq d , \\ U(y+i+1,x+j-d) & \text{for } x+j>d . \end{cases}$$

Hence

$$\{\beta \in \bar{B} \mid \beta \geqq \Delta+1\}+(h-2)\bar{B} = \bigcup_{y=1}^{2h-3} \bigcup_{x=1}^{d} [y\Delta+xh+1, y\Delta+xh+(h-1)] .$$

Let $z \in [(h-1)(\Delta+1), h(\Delta+1)]$. Then

$$\begin{aligned} & z \in \{\beta \in \bar{B} \mid \beta \geqq \Delta+1\}+(h-2)\bar{B} \\ \Leftrightarrow\ & z \in [(h-1)\Delta+xh+1,\ (h-1)\Delta+xh+h-1] \quad \text{for some } x \in [1,d] \\ \Leftrightarrow\ & h(dh+2)-z \notin B .\end{aligned}$$

By Lemma 7, $B \in \hat{\mathscr{R}}_h$.

(iii). The proof of (iii) is similar to the proof of (ii) so we only sketch the proof. First we prove by induction on l that for $l \in [1,h]$ we have

$$\begin{aligned} lB = & \bigcup_{j=0}^{ls} \bigcup_{i=0}^{lr} [j\Gamma+i\Delta, j\Gamma+i\Delta+l] \\ & \cup \bigcup_{j=0}^{ls-1} \bigcup_{i=lr+1}^{rh-1} [j\Gamma+i\Delta+1, j\Gamma+i\Delta+l] .\end{aligned}$$

In particular $hB=[0,h(s\Gamma+r\Delta+1)]$ and so $B\in\mathscr{R}_h$.

Looking at $(h-1)B$ we get

$$\begin{aligned}\bar{B} = [0,s\Gamma+r\Delta] &\cup \{j\Gamma+i\Delta-1 \mid j\in[s+1,sh],\ i\in[1,r]\}\\ &\cup \{j\Gamma+i\Delta \mid j\in[s+1,sh],\ i\in[1,hr]\}\\ &\cup \{s\Gamma+i\Delta \mid i\in[r+1,hr]\}\ .\end{aligned}$$

Finally we determine $\{\beta\in\bar{B} \mid \beta\geqq b\}+(h-2)B$ and use Lemma 7 to conclude that $B\in\hat{\mathscr{R}}_h$. For this base,

$$\begin{aligned}\#\bar{B} &= 1+s\Gamma+r\Delta+(sh-s)(r+hr)+(hr-r)\\ &= 2h^2rs+hr(s+2r)-rs+s+1\ .\end{aligned}$$

To prove that $\hat{\mathscr{R}}_h$ is closed under $\circ$ we need another lemma.

LEMMA 9. *Let* $B,C\in\hat{\mathscr{R}}_h$. *Then*

(i) $\overline{B\circ C}=\{yN_B+x \mid y\in[0,n_C],\ x\in[0,n_B]$, *and* $(y\in\bar{C}$ *or* $x\in\bar{B})\}$,

(ii) $(\overline{B\circ C}+(h-2)(B\circ C))\cap[0,n_{B\circ C}]=$
$=\{yN_B+x \mid y\in[0,n_C],\ x\in[0,n_B]$ *and* $(y\in\bar{C}+(h-2)C$ *or* $x\in\bar{B}+(h-2B)\}$

(iii) $n_{B\circ C}+1-\#\overline{B\circ C}=(n_B+1-\#\bar{B})(n_C+1-\#\bar{C})$.

PROOF. (i) By Lemma 6 (iii), $n_D=n_CN_B+n_B$. Further,

$$(h-1)D = \{yN_B+x \mid y\in(h-1)C \text{ and } x\in(h-1)B\}\ .$$

Hence

$$\begin{aligned}\bar{D} &= \{(n_C-y)N_B+(n_B-x) \mid y\in[0,n_C],\ x\in[0,n_B],\ \text{and}\\ &\qquad (y\notin(h-1)C \text{ or } x\notin(h-1)B)\}\\ &= \{yN_B+x \mid y\in[0,n_C],\ x\in[0,n_B],\ \text{and } (y\in\bar{C} \text{ or } x\in\bar{B})\}\ .\end{aligned}$$

(ii). Let $x\in[0,n_B]$. Then $z=yN_B+x\in[0,n_D]$ if and only if $y\in[0,n_C]$. We have to show that $z\in\bar{D}+(h-2)D$ if and only if $y\in\bar{C}+(h-2)C$ or $x\in\bar{B}+(h-2)B$. We first show the if-part. If $x\in\bar{B}+(h-2)B$, then $x=\beta+\sum_{j=1}^{h-2}b_{i_j}$ and so

$$z = (yN_B+\beta)+\sum_{j=1}^{h-2}(0\cdot N_B+b_{i_j})\in\bar{D}+(h-2)D;$$

similarly, if $y\in\bar{C}+(h-2)C$.

For the only-if-part, let $z \in \bar{D}+(h-2)D$. Then

$$z = (\gamma N_B+\beta)+\sum_{j=1}^{h-2} (c_{i_j}N_B+b_{i_j})$$

where $\gamma \in \bar{C}$ or $\beta \in \bar{B}$, and $c_{i_j} \in C$, $b_{i_j} \in B$. Define q and r by

$$\beta+\sum_{j=1}^{h-2} b_{i_j} = qN_B+r, \quad 0 \leqq r < N_B .$$

Then $r=x$. If $q=0$, then

$$y = \gamma+\sum_{j=1}^{h-2} c_{i_j} \quad \text{and} \quad x = \beta+\sum_{j=1}^{h-2} b_{i_j}$$

and so $y \in \bar{C}+(h-2)C$ or $x \in \bar{B}+(h-2)B$. Next, suppose that $q>0$. Let J be the least integer such that $\beta+\sum_{j=1}^{J} b_{i_j} \geqq qN_B$ and let

$$\beta' = \beta+\sum_{j=1}^{J} b_{i_j}-qN_B .$$

By definition of J. $\beta' \geqq 0$. On the other hand, $\beta+\sum_{j=1}^{J-1} b_{i_j} < N_B$ and so $\beta' < qN_B + b_{i_J} - qN_B \leqq b_k$. Hence, by Lemma 5 (ii), $\beta' \in \bar{B}$, and so

$$x = \beta'+\sum_{j=J+1}^{h-2} b_{i_j} \in \bar{B}+(h-2)B .$$

(iii) follows directly from (i) and Lemma 6 (iii).

PROPOSITION 10. *If $B, C \in \hat{\mathscr{R}}_h$, then $B \circ C \in \hat{\mathscr{R}}_h$.*

PROOF. We use Lemma 7 to prove the proposition

Let $z \in [(h-1)d, hd]$. Then $z=yN_B+x$ where $x \in [0, n_B]$. Then, by Lemma 9,

$$\begin{aligned} & z \in \{\delta \in \bar{D} \mid \delta \geqq d\}+(h-2)D \\ \Leftrightarrow\ & z \in \bar{D}+(h-2)D \\ \Leftrightarrow\ & y \in \bar{C}+(h-2)C \text{ or } x \in \bar{B}+(h-2)B \\ \Leftrightarrow\ & n_C-y \notin C \text{ or } n_B-x \notin B \\ \Leftrightarrow\ & n_D-z = (n_C-y)N_B+(n_B-x) \notin D . \end{aligned}$$

Hence, by Lemma 7, $D \in \hat{\mathscr{R}}_h$.

Using Propositions 8 and 10 we can generate a very large class of bases that belong to $\hat{\mathscr{R}}_h$. In this paper we will not use the full strength of this class. In fact we use only bases generated from Propositions 8 (i) and 10.

Finally, in this section we give a lemma which shows that besides $\hat{B}$ there are other bases of small range that may be constructed from B in $\hat{\mathscr{R}}_h$.

LEMMA 11. *Let $C \in \hat{\mathscr{R}}_h$ and let $D=[0,d]\circ C$. Further let*

$$D^* = D \cup \{n_D+1\} \cup \{n_D+(h-1)d+2+\delta \mid \delta \in \bar{D}\} .$$

Then

$$\# D^* = \# \hat{D}+1 ,$$
$$n(h, D^*) = n(h, \hat{D})+(h-1)d+1 .$$

PROOF. First we show that $[0, 2n_D+(h-1)d+1] \subset hD^*$. We have $[0, n_D] = hD \subset hD^*$,

$$[n_D+1, n_D+1+(h-1)d] \subset \{n_D+1\} \cup (h-1)[0,d] \subset hD^* ,$$

and

$$[n_D+(h-1)d+2, 2n_D+(h-1)d+1] \subset$$
$$\subset \{n_D+(h-1)d+2+\delta \mid \delta \in \bar{D}\}+(h-1)D \subset hD^* ,$$

since

$$[n_D+1, 2n_D] \subset \{n_D+1+\delta \mid \delta \in \bar{D}\}$$

by Corollary 4. Next, suppose $2n_D+(h-1)d+2 \in hD^*$. Since

$$(n_D+(h-1)d+2+\delta)+(n_D+1) > 2n_D+(h-1)d+2$$

for $\delta \geqq 0$, the possible representations are

$$\text{(i)} \quad 2n_D+(h-1)d+2 = (n_D+(h-1)d+2+\delta)+n'$$

where $\delta \in \bar{D}$ and $n' \in (h-1)D$, or

$$\text{(ii)} \quad 2n_D+(h-1)d+2 = l(n_D+1)+n''$$

where $l \in [0,h]$ and $n'' \in (h-l)D$.

However, (i) implies that $2n_D+1=(n_D+1+\delta)+n' \in h\hat{D}$ which contradicts $n(h,\hat{D})=2n_D$. Next, consider (ii). We get $l(n_D+1) \leqq 2n_D+hd-d+2<3(n_D+1)$. Hence $l \leqq 2$.

On the other hand, $n'' \leqq n_D$ and so

$$2(n_D+1)+(h-1)d+1 \leqq (l+1)(n_D+1) .$$

Hence $l+1>2$, and so $l \geqq 2$, that is $l=2$. However, this implies that $(h-1)d = n'' \in (h-2)D$. Since $D=[0,d]\circ C$, $[0,hd]\cap D=[0,d]$. Hence $n'' \leqq (h-2)d$ which contradicts $n''=(h-1)d$.

4. Bases with range close to $2k$.

In this section we show that $\hat{\mathscr{R}}_h$ contains bases with range close to $2k$.

NOTATION. For $B \in \mathscr{R}_h$, let $B_m = B\circ B\circ \ldots \circ B$ (m factors).

PROPOSITION 12. *Let* $B \in \hat{\mathscr{R}}_h$. *Then*

$$n(h,\hat{B}_m) = 2(n(h,B)+1)^m - 2\,,$$

$$\#\hat{B}_m = (\#B)^m + (n(h,B)+1)^m - (n(h,B)+1-\#\bar{B})^m\,,$$

and

$$\frac{n(h,\hat{B}_m)}{\#\hat{B}_m - 1} \to 2 \quad \text{when } m \to \infty\,.$$

PROOF. By Corollary 4 and Lemma 6 (iii),

$$n(h,\hat{B}_m) = 2n(h,B_m) = 2(n(h,B)+1)^m - 2\,.$$

By Lemma 6 (ii), $\#B_m = (\#B)^m$, and by Lemma 9 (iii)

$$\#\bar{B}_m = n(h,B_m)+1-(n(h,B)+1-\#\bar{B})^m\,.$$

Hence, by Lemma 5 (i),

$$\#\hat{B}_m = (\#B)^m + (n(h,B)+1)^m - (n(h,B)+1-\#\bar{B})^m\,.$$

Since $\#B < n(h,B)+1$ and $n(h,B)+1-\#\bar{B} < n(h,B)+1$ we get

$$\frac{n(h,\hat{B}_m)}{\#\hat{B}_m - 1} \to 2 \quad \text{when } m \to \infty\,.$$

As a corollary of Proposition 12 we get the next theorem.

THEOREM 13. *For each* $h \geqq 2$ *and each* $\varepsilon > 0$, *there exists a* k *such that*

$$\nu(h,k) < (2+\varepsilon)k\,.$$

EXAMPLE. If we choose $B=[0,1]$, then $n(h,B)+1=h+1$ and $\#\bar{B}=1$. Hence

$$n(h,\hat{B}_m) = 2(h+1)^m - 2 \quad \text{and} \quad \#\hat{B}_m = 2^m + (h+1)^m - h^m\,.$$

In fact

$$\hat{B}_m = \left\{ \sum_{i=0}^{m-1} \varepsilon_i (h+1)^i \;\middle|\; \varepsilon_i \in \{0,1\} \right\}$$

$$\cup \left\{ (h+1)^m + \sum_{i=0}^{m-1} d_i (h+1)^i \;\middle|\; d_i \in [0,h] \text{ and } (\exists j)(d_j=0) \right\}.$$

Let

$$\varrho_m = \frac{n(h, \hat{B}_m)}{\# \hat{B}_m - 1}.$$

Then $\varrho_1 = h$, and $\varrho_2 = h$. On the other hand

$$\varrho_3 = \frac{2h^3+6h^2+6h}{3h^2+3h+8} < h \quad \text{for } h>2.$$

E.g. for $h=3$, $\hat{B}_3$ is a base with 44 positive elements and range 126. Also, the construction of Lemma 11 gives B_3^* with 45 positive elements and range 129.

Given Theorem 13 it is natural to ask how frequent are the k's such that $v(h,k) < (2+\varepsilon)k$.

NOTATION. For $h \geqq 2$ and $y>2$, let

$$L(h,y) = \liminf_{x \to \infty} \frac{1}{x} \cdot \#\{k \leqq x \mid v(h,k) < yk\}.$$

We are going to show that $L(h,y)>0$ for all h and $y>2$. First we prove another lemma.

LEMMA 14. *Let $d_1, d_2, \ldots, d_r$ be positive integers, and let*

$$D(d_1, d_2, \ldots, d_r) = [0, d_1] \circ [0, d_2] \circ \ldots \circ [0, d_r]$$

and

$$\varrho(d_1, d_2, \ldots, d_r) = \frac{n(h, \hat{D}(d_1, d_2, \ldots, d_r))}{\# \hat{D}(d_1, d_2, \ldots, d_r) - 1}.$$

Then $\varrho(d_1, d_2, \ldots, d_r)$ is a non-increasing function in d_i for each $i \in [1, r]$.

PROOF. By Prop. 8, Lemma 6, and Lemma 9 we get, by induction, that

$$n(h, \hat{D}) = 2\left\{ \prod_{i=1}^{r} (hd_i + 1) - 1 \right\},$$

$$\sharp\hat{D} = \prod_{i=1}^{r} (hd_i+1) - \prod_{i=1}^{r} (hd_i+1-d_i) + \prod_{i=1}^{r} (d_i+1) .$$

Both are symmetric functions in $d_1, d_2, \ldots, d_r$. Therefore, it is enough to prove that $\varrho(d_1, d_2, \ldots, d_{r-1}, d)$ is non-increasing in d. By definition,

$$\varrho(d) = \varrho(d_1, d_2, \ldots, d_{r-1}, d) = \frac{sd+t}{ud+v},$$

where $s = 2h\prod_{i=1}^{r-1} (hd_i+1)$, $t = 2\prod_{i=1}^{r-1} (hd_i+1) - 2$,

$$u = h\prod_{i=1}^{r-1} (hd_i+1) - (h-1)\prod_{i=1}^{r-1} (hd_i+1-d_i) + \prod_{i=1}^{r-1} (d_i+1),$$

$$v = \prod_{i=1}^{r-1} (hd_i+1) - \prod_{i=1}^{r-1} (hd_i+1-d_i) + \prod_{i=1}^{r-1} (d_i+1) - 1 .$$

Hence, $\varrho(d) \geqq \varrho(d+1)$ if and only if $tu \geqq sv$. Let

$$\alpha = \prod_{i=1}^{r-1} (hd_i+1), \quad \beta = \prod_{i=1}^{r-1} (hd_i+1-d_i), \quad \text{and} \quad \gamma = \prod_{i=1}^{r-1} (\mathrm{d}_i+1) .$$

Then $s = h\alpha$, $t = 2\alpha - 2$, $u = h\alpha - (h-1)\beta + \gamma$, $v = \alpha - \beta + \gamma - 1$. Hence $tu - sv = 2\alpha(\beta + \gamma - h\gamma) + 2h\beta - 2\beta - 2\gamma$. If $r = 1$, $r = 2$, or $h = 2$, then $tu - sv = 0$. Next consider $h \geqq 3$ and $r \geqq 3$. Then

$$(hd_1+1-d_1)(hd_2+1-d_2) \geqq (d_1+1)(d_2+1)(h-1) .$$

Hence

$$\beta \geqq (hd_1+1-d_1)(hd_2+1-d_2)\prod_{i=3}^{r-1} (d_i+1) \geqq (h-1)\gamma$$

and so $tu - sv \geqq 2\alpha((h-1)\gamma - \gamma - h\gamma) + 2(h-1)^2\gamma - 2\gamma = 2h(h-2)\gamma > 0$. This proves that $\varrho(d+1) \geqq \varrho(d)$.

THEOREM 15. *For all* $h \geqq 2$ *and all* $y > 2$, $L(h, y) > 0$.

PROOF. Let $d_i = 1$ for all $i \geqq 1$. By Proposition 12, $\varrho(d_1, d_2, \ldots, d_r) \to 2$ when $r \to \infty$. Choose r such that $\varrho(d_1, d_2, \ldots, d_r) < y$. By Lemma 14, $\varrho(d_1, d_2, \ldots, d_{r-1}, d) < y$ for all $d \geqq 1$. In the proof of Lemma 14 we showed that $\sharp\hat{D}(d_1, d_2, \ldots, d_{r-1}, d) - 1 = ud + v$ where

$$u = h(h+1)^{r-1} - (h-1)h^{r-1} + 2^{r-1} \quad \text{and} \quad v = (h+1)^{r-1} - h^{r-1} + 2^{r-1} - 1 .$$

Hence $L(h, y) \geqq 1/u > 0$.

THEOREM 16. *For all* $h \geqq 3$, $L(h,h)=1$.

PROOF. We consider bases of the form $\hat{D}$ and D^* where $D=D(a,b,c)$. First we note that

$$\begin{aligned}\#\hat{D}(a,b,c)-1 &= (ha+1)(hb+1)(hc+1)-(ha+1-a)(hb+1-b)(hc+1-c)\\ &\quad +(a+1)(b+1)(c+1)-1\\ &= 2\{\{(3(h^2-h)/2+1)a+h\}b+(ha+1\}c+2\{(ha+1)b+a\}\ .\end{aligned}$$

Hence $\#\hat{D}(a,b,c)-1$ is always even. Choose a such that $ha+1$ is a prime. Then

$$\begin{aligned}2a\{(3(h^2-h)/2+1)a+h\} &= 3h^2a^2-3ha^2+2a^2+2ha\\ &\equiv 3+3a+2a^2-2 = (a+1)(2a+1)\\ &\not\equiv 0 \bmod (ha+1)\ .\end{aligned}$$

Hence $gcd((3(h^2-h)/2+1)a+h, ha+1)=1$ and so there exist infinitely many b's such that $p_b=\{(3(h^2-h)/2+1)a+h\}b+(ha+1)$ is a prime. Let B denote the set of such b's.

Since $\varrho(1,1,1)<h$, $\varrho(a,b,c)<h$ for all $a,b,c \geqq 1$. Further, since

$$\#\hat{D}(a,b,c)-1 = 2p_bc+2\{(ha+1)b+a\}\ ,$$

if $k=2p_bc+2\{(ha+1)b+a\}$ for some $b \in B$ and some $c \geqq 1$, then $\nu(h,k)<hk$. Hence

$$\begin{aligned}\tfrac{1}{2} &\geqq \liminf_{x\to\infty}\frac{1}{x}\cdot\#\{k\leqq x \mid k \text{ even and } \nu(h,k)<hk\}\\ &\geqq \tfrac{1}{2}\Big\{1-\prod_{b\in B}(1-1/p_b)\Big\} = \tfrac{1}{2}\ .\end{aligned}$$

Therefore, $\frac{1}{x}\#\{k\leqq x \mid k \text{ even and } \nu(h,k)<hk\} \to \frac{1}{2}$ when $x\to\infty$.

Next we consider k odd. We have $\#D^*=\#\hat{D}+1$, and so $\#D^*-1$ is always odd. Moreover, $\varrho(a,b,c)\leqq\varrho(1,1,1)<h$ and so

$$\begin{aligned}n(h,D^*) &= n(h,\hat{D})+(h-1)a+1\\ &\leqq \varrho(1,1,1)(\#\hat{D}-1)+(h-1)a+1\\ &< h(\#D^*-1)\end{aligned}$$

for b fixed and c sufficiently large. Hence

$$\frac{1}{x}\cdot\#\{k\leqq x \mid k \text{ odd and } \nu(h,k)<hk\} \to \tfrac{1}{2} \quad \text{when } x\to\infty$$

also. Therefore $L(h,h)=\frac{1}{2}+\frac{1}{2}=1$.

Theorem 16 says that for fixed h, the minimum conjecture fails for almost all k. A number of questions are left open:

Is $v(h,k)<hk$ for all large k?
Is $L(h,y)=1$ for all $y>2$?
Is it perhaps even true that for all $y>2$, $v(h,k)<yk$ for all large k?

We note that $\sharp D(1,1,1)-1=(h+1)^3-h^3+2^3-1=3h^2+3h+8$ and this is our smallest counter example to the minimum conjecture.

Is there a $k<3h^2+3h+8$ such that $v(h,k)<hk$?

5. 2-bases with minimal range.

In this section we characterize the 2-bases with minimal range and we give estimates for the number of such bases.

NOTATIONS. For $k\geqq 1$ and $u\in[0,k]$, let

$$\begin{aligned}
\mathscr{M}(k) &= \{A\in\mathscr{A}(2,k)\mid n(2,A)=2k\}\,,\\
\mathscr{S}(k,u) &= \{B\in\mathscr{A}(2,u)\mid B\subset[0,k],\ n(2,B)\geqq k\}\,,\\
\mathscr{T}(k,u) &= \{B\in\mathscr{S}(k,u)\mid n(2,B)=k\}\,,\\
m(k) &= \sharp\mathscr{M}(k)\,,\\
s(k,u) &= \sharp\mathscr{S}(k,u),\ s(-1,-1)=1\,,\\
t(k,u) &= \sharp\mathscr{T}(k,u)\,.
\end{aligned}$$

For $B\in\mathscr{S}(k,u)$,

$$\begin{aligned}
\bar{B} &= \bar{B}_k = \{k-d\mid d\in[0,k] \text{ and } d\notin B\}\,,\\
\hat{B} &= \hat{B}_k = \bar{B}\cup\{k+1+\beta\mid \beta\in\bar{B}_k\}\,.
\end{aligned}$$

We note that $s(k,0)=0$.

THEOREM 17. *For all* $k\geqq 1$,

$$\mathscr{M}(k)=\bigcup_{u=1}^{k}\{\hat{B}\mid B\in\mathscr{S}(k,u)\}\,.$$

PROOF. Let $A\in\mathscr{M}(k)$. Let $B=A\cap[0,k]$. Then $n(2,B)\geqq k$ and so $B\in\mathscr{S}(k,u)$ for some $u\in[1,k]$. We show that $A=\hat{B}$. Let $d\in[0,2k+1]$. If $d\in[0,k]$, then $d\in A$ if and only if $d\in B$. If $d\in[k+1,2k+1]$, the $2k+1-d\in[0,k]$ and so

$$d \in A \Leftrightarrow 2k+1-d \notin A \Leftrightarrow 2k+1-d \notin B \Leftrightarrow k-(2k+1-d) \in \bar{B}$$
$$\Leftrightarrow d = k+1+k-(2k+1-d) \in \hat{B} .$$

Hence $A=\hat{B} \in \bigcup_{u=0}^{k} \{\hat{B} \mid B \in \mathcal{S}(k,u)\}$.

To show the converse, let $B \in \mathcal{S}(k,u)$ for some $u \in [0,k]$. First we show that $\hat{B}$ satisfies condition (*). If $d \in [0,k]$, then

$$d \in \hat{B} \Leftrightarrow d \in B \Leftrightarrow k-d \notin \bar{B} \Leftrightarrow 2k+1-d \notin \hat{B} .$$

If $d \in [k+1, 2k+1]$, then

$$d \in \hat{B} \Leftrightarrow d \in \{k+1+\beta \mid \beta \in \bar{B}\} \Leftrightarrow d-(k+1) \in \bar{B} \Leftrightarrow 2k+1-d \notin B$$
$$\Leftrightarrow 2k+1-d \notin \hat{B} .$$

Hence $\hat{B}$ satisfies (*) with $\varkappa=k$ and, by Lemma 3, $n(2,\hat{B})=2k$. Since $\hat{B}$ satisfies (*), $\#\hat{B}-1=k$ and so $\hat{B} \in \mathcal{M}(k)$.

LEMMA 18.

(i) $m(k)=\sum_{u=1}^{k} s(k,u)$,
(ii) $s(k+1,u)=s(k,u)+s(k,u-1)-t(k,u)$ *for* $u \geqq 1$.

PROOF. (i) follows directly from Theorem 17. To prove (ii) we prove that

$$\mathcal{S}(k+1,u) = (\mathcal{S}(k,u) \setminus \mathcal{T}(k,u)) \cup \{B \cup \{k+1\} \mid B \in \mathcal{S}(k,u)\} .$$

Let $C \in \mathcal{S}(k+1,u)$. If $k+1 \notin C$, then $C \in \mathcal{S}(k,u) \setminus \mathcal{T}(k,u)$. If $k+1 \in C$, then $C=B \cup \{k+1\}$ where $B \in \mathcal{S}(k,u)$. This shows that $\mathcal{S}(k+1,u)$ is included in the right set. The inclusion of the right set in $\mathcal{S}(k+1,u)$ is similar. Therefore $s(k+1,u)=s(k,u)-t(k,u)+s(k,u-1)$.

LEMMA 19. *Let* $k \geqq 1$ *and* $\frac{1}{2}k \leqq u \leqq k$. *Then*

$$s(k,u) = \sum_{r=0}^{[\frac{1}{2}k]} \left\{\binom{k-2r}{u-r} - \binom{k-2r}{u-r+1}\right\} s(2r-1,r-1) .$$

PROOF. We prove this by induction on k. First we note that by Corollary 2, $t(k,u)=0$ for $\frac{1}{2}k < u \leqq k$. First consider $k=1$. Then $u=1$ and the sum has one term,

$$\left\{\binom{1-0}{1-0} - \binom{1-0}{1-0+1}\right\} s(-1,-1) = 1 = s(1,1) .$$

Next, let $k>1$. We consider k odd and k even separately. First consider k odd. If $u \geqq \frac{1}{2}k$, then $u \geqq \frac{1}{2}(k+1)$ and so $u-1 \geqq \frac{1}{2}(k-1)$. Hence $t(k-1,u)=0$ and by the induction hypothesis

$$\begin{aligned} s(k,u) &= s(k-1,u)+s(k-1,u-1) \\ &= \sum_{r=0}^{[\frac{1}{2}(k-1)]} \left\{ \binom{k-1-2r}{u-r} - \binom{k-1-2r}{u-r+1} + \binom{k-1-2r}{u-1-r} - \binom{k-1-2r}{u-r} \right\} \\ &\quad s(2r-1,r-1) \\ &= \sum_{r=0}^{[\frac{1}{2}k]} \left\{ \binom{k-2r}{u-r} - \binom{k-2r}{u-r+1} \right\} s(2r-1,r-1) \end{aligned}$$

since $[\frac{1}{2}k]=[\frac{1}{2}(k-1)]$.

Next consider k even. If $u>\frac{1}{2}k$, then $u \geqq \frac{1}{2}k+1$ and so $u-1 \geqq \frac{1}{2}k$. As for k odd we get, using the induction hypothesis,

$$\begin{aligned} s(k,u) &= s(k-1,u)+s(k-1,u-1) \\ &= \sum_{r=0}^{[\frac{1}{2}(k-1)]} \left\{ \binom{k-2r}{u-r} - \binom{k-2r}{u-r+1} \right\} s(2r-1,r-1) . \end{aligned}$$

In this case $[\frac{1}{2}k]=[\frac{1}{2}(k-1)]+1$. However the term for $r=\frac{1}{2}k$ is zero so we get the expression of Lemma 19 also in this case. Finally,

$$\begin{aligned} s(k,\tfrac{1}{2}k) &= s(k-1,\tfrac{1}{2}k-1)+s(k-1,\tfrac{1}{2}k) \\ &= s(k-1,\tfrac{1}{2}k-1)+\sum_{r=0}^{[\frac{1}{2}(k-1)]} \left\{ \binom{k-1-2r}{\frac{1}{2}k-r} - \binom{k-1-2r}{\frac{1}{2}k-r+1} \right\} s(2r-1,r-1) \\ &= s(k-1,\tfrac{1}{2}k-1)+\sum_{r=0}^{\frac{1}{2}k-1} \left\{ \binom{k-2r}{\frac{1}{2}k-r} - \binom{k-2r}{\frac{1}{2}k-r+1} \right\} s(2r-1,r-1) \\ &= \sum_{r=0}^{\frac{1}{2}k} \left\{ \binom{k-2r}{\frac{1}{2}k-r} - \binom{k-2r}{\frac{1}{2}k-r+1} \right\} s(2r-1,r-1) \end{aligned}$$

since for all $\varkappa \geqq 1$,

$$\binom{2\varkappa-1}{\varkappa} - \binom{2\varkappa-1}{\varkappa+1} = \binom{2\varkappa}{\varkappa} - \binom{2\varkappa}{\varkappa+1} .$$

THEOREM 20. *For* $k \geqq 1$,

$$m(k) = \sum_{u=1}^{\lceil\frac{1}{2}k\rceil-1} s(k,u) + \binom{k}{[\frac{1}{2}k]} + \sum_{r=1}^{[\frac{1}{2}k]} \binom{k-2r}{[\frac{1}{2}k]-r} s(2r-1,r-1)$$

PROOF. Let $\varkappa=\lceil\frac{1}{2}k\rceil$. By Lemmata 18 and 19,

$$m(k)-\sum_{u=1}^{\varkappa-1} s(k,u) = \sum_{u=\varkappa}^{k} s(k,u)$$

$$= \sum_{u=\varkappa}^{k} \sum_{r=0}^{[\frac{1}{2}k]} \left\{ \binom{k-2r}{u-r} - \binom{k-2r}{u-r+1} \right\} s(2r-1, r-1)$$

$$= \sum_{r=0}^{[\frac{1}{2}k]} s(2r-1, r-1) \sum_{u=\varkappa}^{k} \left\{ \binom{k-2r}{u-r} - \binom{k-2r}{u-r+1} \right\}$$

$$= \sum_{r=0}^{[\frac{1}{2}k]} s(2r-1, r-1) \binom{k-2r}{\varkappa - r} .$$

Theorem 20 proves in particular that

$$m(k) \geqq \binom{k}{[\frac{1}{2}k]} \sim 2^{k+\frac{1}{2}} / \sqrt{\pi k} .$$

Hence $m(k)$ is growing exponentially. Numerical data suggest that $m(k) \sim c \cdot 2^k$ for some $c > 0$. We describe these data next. Using a FORTRAN program on a NORD-10 at the University of Bergen we first generated all bases $A \in \mathscr{A}(2, u)$ where $u \leqq 14$ and $A \subset [0, 30]$. This gave $s(k, u)$ for $u \leqq 14$ and $k \leqq 30$. Using Lemma 19 we found $s(k, u)$ for all $u \leqq 30$ and $k \leqq 30$, and using Theorem 20 we found $m(k)$ for $k \leqq 30$. In Table 1 we give $m(k)$ and $\alpha(k) = m(k) 2^{-k}$ for $k \leqq 30$.

Let

$$\sigma(k, v) = \{ s(k, \tfrac{1}{2}(k-v)) + s(k, \tfrac{1}{2}(k+v)) \} / \left\{ 2 \binom{k}{\frac{1}{2}(k-v)} \right\} ,$$

where $v \equiv k \pmod 2$. For reasons that will become apparent below, it is more convenient for work with $\sigma(k, v)$ rather than with $s(k, u)$. In Table 2 we give $\sigma(k, v)$ for $k \leqq 29$, $v = 1, 3, 5, 7$.

Table 1.

k	$m(k)$	$\alpha(k)$	k	$m(k)$	$\alpha(k)$
1	1	0.50000	16	16194	0.24710
2	2	0.50000	17	32058	0.24458
3	3	0.37500	18	63910	0.24380
4	6	0.37500	19	126932	0.24210
5	10	0.31250	20	253252	0.24152
6	20	0.31250	21	503933	0.24029
7	37	0.28906	22	1006056	0.23986
8	73	0.28516	23	2004838	0.23900
9	139	0.27148	24	4004124	0.23866
10	275	0.26855	25	7937149	0.23804
11	533	0.26025	26	15957964	0.23779
12	1059	0.25854	27	31854676	0.23734
13	2075	0.25330	28	63660327	0.23715
14	4126	0.25183	29	127141415	0.23682
15	8134	0.24823	30	254136782	0.23668

Table 2.

k	$\sigma(k,1)$	$\sigma(k,3)$	$\sigma(k,5)$	$\sigma(k,7)$
1	0.5000			
3	0.3333	0.5000		
5	0.2500	0.4000	0.5000	
7	0.2286	0.3333	0.4286	0.5000
9	0.2183	0.2857	0.3750	0.4444
11	0.2165	0.2545	0.3333	0.4000
13	0.2159	0.2397	0.3014	0.3636
15	0.2167	0.2313	0.2769	0.3341
17	0.2175	0.2272	0.2596	0.3099
19	0.2185	0.2256	0.2487	0.2901
21	0.2194	0.2250	0.2418	0.2746
23	0.2204	0.2249	0.2377	0.2631
25	0.2212	0.2251	0.2352	0.2550
27	0.2220	0.2254	0.2336	0.2493
29	0.2227	0.2257	0.2327	0.2454

LEMMA 21 (i) *For* $k \geqq 1$, $\alpha(k+1) < \alpha(k)$.
(ii) $\alpha(\infty) = \lim_{k\to\infty} \alpha(k)$ *exists and* $0 \leqq \alpha(\infty) < 0.2367$.

PROOF. (i) By Lemma 18

$$m(k+1) = \sum_{u=1}^{k+1} s(k+1,u) = \sum_{u=1}^{k+1} s(k,u) + \sum_{u=1}^{k+1} s(k,u-1) - \sum_{u=1}^{k+1} t(k,u)$$
$$< m(k)+m(k) = 2m(k)\,.$$

Hence $\alpha(k+1) < \alpha(k)$.

(ii) The existence of $\alpha(\infty)$ follows from (i). Further $\alpha(\infty) < \alpha(30) \approx 0.2367$.

Looking at Table 2 and similar tables for other values of v, it appears that $\sigma(k,v)$ satisfies the following conditions:

(σ1) $\sigma(k,v+2) > \sigma(k,v)$ for $0 \leqq v \leqq k-2$.

(σ2) For a fixed v, $\sigma(k,v)$ is first decreasing, then increasing for increasing k.
We have not been able to prove (σ1) or (σ2). However, we shall consider some of their consequences.

PROPOSITION 22. *If* (σ1) *is true and* (σ2) *is true for* $v=1$, *then* $\alpha(\infty) > 0$.

PROOF. By Lemma 18, since $s(2k+1,0)=0$,

$$m(2k+1) = \sum_{v=0}^{k} \{s(2k+1,\tfrac{1}{2}(2k+1-(2v+1))) + s(2k+1,\tfrac{1}{2}(2k+1+(2v+1)))\}$$

$$= \sum_{v=0}^{k} 2\sigma(2k+1, 2v+1)\binom{2k+1}{k-v}$$

$$\geqq \sum_{v=0}^{k} 2\sigma(2k+1, 1)\binom{2k+1}{k-v} \quad \text{by } (\sigma 1)$$

$$= \sigma(2k+1, 1)2^k \geqq \sigma(13, 1)2^k \quad \text{by } (\sigma 2) \text{ and Table 2}.$$

Hence $\alpha(2k+1) \geqq \sigma(13, 1) > 0$.

PROPOSITION 23. *If* $(\sigma 1)$ *and* $(\sigma 2)$ *are true, then*

$$\sigma(\infty, v) = \lim_{k \to \infty} \sigma(k, v)$$

exists and

(i) $\sigma(\infty, v+2) \geqq \sigma(\infty, v)$ *for all* $v \geqq 0$,
(ii) $\alpha(\infty) \geqq \sigma(\infty, v)$ *for all* $v \geqq 0$.

PROOF. The existence of $\sigma(\infty, v)$ follows from $(\sigma 2)$ and (i) then follows from $(\sigma 1)$. To prove (ii), suppose there exists a u such that $\sigma(\infty, 2u+1) > \alpha(\infty)$. Let ε be defined by $2\varepsilon = \sigma(\infty, 2u+1) - \alpha(\infty)$. Then there exists a K such that $\sigma(2k+1, 2u+1) > \alpha(\infty) + \varepsilon$ for all $k \geqq K$. Let $k \geqq K$. For $v \geqq u$,

$$\sigma(2k+1, 2v+1) \geqq \sigma(2k+1, 2v+1) > \alpha(\infty) + \varepsilon .$$

Hence

$$m(2k+1) = \sum_{v=0}^{k} 2\sigma(2k+1, 2u+1)\binom{2k+1}{k-v}$$

$$\geqq \sum_{v=u}^{k} 2\{\alpha(\infty) + \varepsilon\}\binom{2k+1}{k-v}$$

$$= \{\alpha(\infty) + \varepsilon\} \cdot \left\{2^{2k+1} - \sum_{v=0}^{u-1} 2\binom{2k+1}{k-v}\right\}$$

$$\geqq \{\alpha(\infty) + \varepsilon\} \cdot 2^{2k+1} \cdot \left\{1 - u \cdot 2 \cdot \binom{2k+1}{k} 2^{-(2k+1)}\right\}.$$

Hence

$$\alpha(2k+1) \geqq \alpha(\infty) + \varepsilon - u\binom{2k+1}{k} 2^{-(2k+1)}\{\alpha(\infty) + \varepsilon\} .$$

Let $k \to \infty$. Then we get $\alpha(\infty) \geqq \alpha(\infty) + \varepsilon$ which is a contradiction. Similarly we prove $\alpha(\infty) \geqq \sigma(\infty, v)$ for even v.

COROLLARY 24. *If* $(\sigma 1)$ *and* $(\sigma 2)$ *are true, then*

$$\alpha(\infty) > (29,3) \approx 0.22572 .$$

PROOF. Since $\sigma(25,3) > \sigma(23,3)$, $\sigma(2k+1,3)$ is increasing for $2k+1 \geqq 23$ by $(\sigma 2)$. By Proposition 23,

$$\alpha(\infty) \geqq \sigma(\infty,3) > \sigma(29,3) .$$

Extrapolation from the values of $\alpha(k)$ given in Table 1 indicates that the true value of $\alpha(\infty)$ is approximately 0.2355.

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