MATH. SCAND. 53 (1983), 185–192

SOME SPECIAL W-ALMOST PERIODIC SETS ON THE INTEGERS

ERLING FØLNER

We begin by specializing, starting from an arbitrary topological group and passing to the discrete abelian group $\mathbf{Z}$ of the integers, some definitions and theorems concerning W-almost periodic (W-ap) functions on the group. For this we refer mainly to [1] and [2, I].

DEFINITION. A bounded complex-valued function $f(n)$ on $\mathbf{Z}$ is called a W-almost periodic function on $\mathbf{Z}$ if to every $\varepsilon > 0$ there exists a trigonometric polynomial on $\mathbf{Z}$

$$s(n) = \sum_{p=1}^{P} c_p e^{i\gamma_p n}$$

where the c_p are complex numbers and the γ_p are real numbers, determined modulo 2π, such that

$$\|f(n)-s(n)\|_W = \bar{M}|f(n)-s(n)| < \varepsilon .$$

Here for a bounded real function $g(n)$ on $\mathbf{Z}$

$$\bar{M}g(n) = \inf_A \sup_n \sum_{q=1}^{Q} \alpha_q g(n+n_q) , \tag{1}$$

where

$$A = \{\alpha_1,\ldots,\alpha_Q;\, n_1,\ldots,n_Q\}, \qquad \alpha_q > 0,\ \sum_{q=1}^{Q} = 1,\ n_q \in \mathbf{Z} .$$

A set B on $\mathbf{Z}$ is called W-almost periodic on $\mathbf{Z}$ if its characteristic function $B(n)$ is W-ap.

Analogously to (1) we put

$$\underline{M}g(n) = \sup_A \inf_n \sum_{q=1}^{Q} \alpha_q g(n+n_q) . \tag{2}$$

Received September 3, 1982; in revised form November 8, 1982.

If $\bar{M}g(n) = \underline{M}g(n)$ we denote this number by $Mg(n)$. For a complex-valued bounded function $f(n) = g(n) + ih(n)$ we put

$$Mf(n) = Mg(n) + iMh(n)$$

if $Mg(n)$ and $Mh(n)$ exist.

We omit the proof of the simple fact that if $Mf(n)$ exists, then

$$(3) \qquad \frac{1}{N} \sum_{n=1}^{N} f(n) \to Mf(n) \quad \text{for } N \to \infty .$$

The sum and the product of two W-ap functions are W-ap, and the numerical value of a W-ap function is W-ap. For a W-ap function $f(n)$ the mean value $Mf(n)$ exists.

To a real W-ap function $f(n)$ we ascribe a Fourier series,

$$(4) \qquad f(n) \sim \tfrac{1}{2}a_0 + \sum_{r=1}^{\infty} \{a_r \cos\gamma_r n + b_r \sin\gamma_r n\}$$

where

$$a_r = 2M\{f(n)\cos\gamma_r n\}, \qquad b_r = 2M\{f(n)\sin\gamma_r n\}$$

and we consider at least the at most denumerably many γ_r, determined modulo 2π, for which a_r or b_r are different from zero.

The Fourier series W-converges to $f(n)$, that is

$$M\left| f(n) - \tfrac{1}{2}a_0 - \sum_{r=1}^{R} \{a_r \cos\gamma_r n + b_r \sin\gamma_r n\} \right| \to 0 \quad \text{for } R \to \infty .$$

Main Theorem. *A bounded function $f(n)$ on* **Z** *is W-ap on* **Z** *if and only if to every $\varepsilon > 0$ there exists an*

$$A = \{\alpha_1, \ldots, \alpha_Q;\, n_1, \ldots, n_Q\}, \qquad \alpha_q > 0,\ \sum_{q=1}^{Q} \alpha_q = 1,\ n_q \in \mathbf{Z}$$

and finitely many integers m_s such that to every integer m there exists an m_s with

$$\|f(n+m) - f(n+m_s)\|_{S_A} = \sup_n \sum_{q=1}^{Q} \alpha_q |f(n+n_q+m) - f(n+n_q+m_s)| < \varepsilon .$$

If $X_1, X_2, \ldots$ are at most denumerably many characters on **Z**, then by $S(X_1, X_2, \ldots)$ we understand the at most denumerably many characters which are finite products of some, arbitrarily chosen, of the characters $X_1, \bar{X}_1, X_2, \bar{X}_2, \ldots$. Then we have

THEOREM 2. *Let $f(n)$ be a real W-ap function on* **Z**. *For real β we consider on* **Z** *the set*

$$A_\beta = [f(n) \leqq \beta] .$$

With exception of the at most denumerably many β where

$$\psi(\beta) = \bar{M} A_\beta$$

makes a jump, A_β is a W-ap set and

$$M\{[f(n) = \beta]\} = 0 .$$

If $X_1, X_2, \ldots$ are the Fourier characters of $f(n)$, the Fourier characters of A_β are in $S(X_1, X_2, \ldots)$.

As the reader will see, this can easily be deduced by the method put forward in [4, p. 78], confer [2, I, p. 34]. For the more general result, see [2, I, p. 60].

Now let α be an arbitrary fixed irrational number. Then for every integer $h \neq 0$

$$e^{2\pi i h \alpha n}$$

is a character on **Z** which is not the main character 1. Hence from (3)

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h \alpha n} \to M e^{2\pi i h \alpha n} = 0 \quad \text{for } N \to \infty .$$

From this Weyl showed, see [3] for general results concerning the asymptotic distribution of real numbers and in particular equi-distribution, that if $g(x)$ is real and periodic on **R** with period 1 and in a period interval is continuous, except for finitely many points of jump, if any, then

$$\frac{1}{N} \sum_{n=1}^{N} g(\alpha n) \to \int_0^1 g(x)\, dx \quad \text{for } N \to \infty . \tag{5}$$

Let $0 < \lambda < 1$ and let first

$$g(x) = \begin{cases} 1 & \text{for } 0 \leqq x \leqq \lambda \\ 0 & \text{elsewhere in } 0 \leqq x < 1 \end{cases}$$

and periodic with period 1.

Our aim is to prove that the set C_λ on **Z** *written*

$$C_\lambda = [g(\alpha n) = 1]$$

i.e. *the set of $n \in$* **Z** *for which*

$$0 \leqq \alpha n \leqq \lambda \pmod 1$$

is W-ap and to find its Fourier series. Note that

$$C_\lambda(n) = g(\alpha n)\,.$$

We have

$$\begin{aligned} C_\lambda &= \left[-\frac{\lambda}{2} \leqq \alpha n - \frac{\lambda}{2} \leqq \frac{\lambda}{2} \pmod 1\right] \\ &= \left[\cos 2\pi\left(\alpha n - \frac{\lambda}{2}\right) \geqq \cos 2\pi\frac{\lambda}{2}\right] \\ &= [\cos(2\pi\alpha n - \pi\lambda) \geqq \cos\pi\lambda]\,. \end{aligned}$$

Let $0 < \lambda_1 < \lambda$ and $\lambda < \lambda_2 < 1$ and put

$$\begin{aligned} C_{\lambda,\lambda_1} &= [\cos(2\pi\alpha n - \pi\lambda) \geqq \cos\pi\lambda_1] \\ &= \left[-\frac{\lambda_1}{2} + \frac{\lambda}{2} \leqq \alpha n \leqq \frac{\lambda_1}{2} + \frac{\lambda}{2} \pmod 1\right], \\ C_{\lambda,\lambda_2} &= [\cos(2\pi\alpha n - \pi\lambda) \geqq \cos\pi\lambda_2] \\ &= \left[-\frac{\lambda_2}{2} + \frac{\lambda}{2} \leqq \alpha n \leqq \frac{\lambda_2}{2} + \frac{\lambda}{2} \pmod 1\right]. \end{aligned}$$

In the last expression for C_λ above, $\cos(2\pi\alpha n - \pi\lambda)$ is a trigonometric polynomial on Z containing only the two characters $e^{i2\pi\alpha n}$ and $e^{-i2\pi\alpha n}$. Using Theorem 2 to

$$f(n) = -\cos(2\pi\alpha n - \pi\lambda)$$

we see that C_{λ,λ_1} and C_{λ,λ_2} are W-ap with Fourier characters amongst

$$e^{i2\pi r\alpha n}, \qquad r = 0, \pm 1, \pm 2, \ldots$$

when we assume that λ_1 and λ_2 avoid certain at most denumerably many values. Thus we may also let $\lambda_2 - \lambda_1 \to 0$.

From (5), taking

$$g(x) = \begin{cases} 1 & \text{for } -\frac{\lambda_1}{2} + \frac{\lambda}{2} \leqq x \leqq \frac{\lambda_1}{2} + \frac{\lambda}{2} \\ 0 & \text{elsewhere in } 0 \leqq x < 1 \end{cases}$$

and periodic with period 1 we get, using also (3)

$$MC_{\lambda,\lambda_1} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(\alpha n) = \int_0^1 g(x)\,dx = \lambda_1\,.$$

Analogously, taking $g(x)=1$ for

$$-\frac{\lambda_2}{2}+\frac{\lambda}{2} \leqq x \leqq \frac{\lambda_2}{2}+\frac{\lambda}{2},$$

and repeated with period 1, and $g(x)=0$ elsewhere, we get

$$MC_{\lambda,\lambda_2} = \lambda_2 .$$

We have

$$C_{\lambda,\lambda_1} \subset C_\lambda \subset C_{\lambda,\lambda_2}$$

so that

$$\begin{aligned} 0 &\leqq \bar{M}(C_\lambda(n)-C_{\lambda,\lambda_1}(n)) \leqq \bar{M}(C_{\lambda,\lambda_2}(n)-C_{\lambda,\lambda_1}(n)) \\ &= M(C_{\lambda,\lambda_2}(n)-C_{\lambda,\lambda_1}(n)) = \lambda_2-\lambda_1 \to 0 . \end{aligned}$$

Thus $C_\lambda(n)$ can be W-approximated by the W-ap functions $C_{\lambda,\lambda_1}(n)$ and hence is W-ap itself. The Fourier series of $C_{\lambda,\lambda_1}(n)$ will converge formally towards the Fourier series of $C_\lambda(n)$ so that the Fourier characters of $C_\lambda(n)$ are amongst

$$e^{i2\pi r\alpha n}, \qquad r=0,\ \pm 1,\ \pm 2,\dots .$$

Thus from (4)

$$C_\lambda(n) \sim \tfrac{1}{2}a_0+\sum_{r=1}^{\infty} \{a_r \cos 2\pi r\alpha n + b_r \sin 2\pi r\alpha n\}$$

and using (5) with

$$g(x) = \begin{cases} \cos 2\pi r x & \text{for } 0 \leqq x \leqq \lambda \\ 0 & \text{elsewhere in } 0 \leqq x < 1 \end{cases}$$

and periodic with period 1 we get, using also (3), that

$$a_r = 2M\{C_\lambda(n) \cos 2\pi r\alpha n\} = 2\int_0^\lambda \cos 2\pi r x \, dx .$$

Hence

$$\tfrac{1}{2}a_0 = \lambda$$

and for $r>0$,

$$a_r = \frac{\sin 2\pi r\lambda}{\pi r} .$$

Analogously, for $r>0$,

$$b_r = 2M\{C_\lambda(n)\sin 2\pi r\alpha n\} = 2\int_0^\lambda \sin 2\pi rx\,dx = \frac{1-\cos 2\pi r\lambda}{\pi r}.$$

It is evident that

$$\tfrac{1}{2}a_0 + \sum_{r=1}^{\infty} \{a_r \cos 2\pi rx + b_r \sin 2\pi rx\}$$

is the Fourier series of our first $g(x)$. Hence it converges to 1 for $0<x<\lambda$ (mod 1), to $\frac{1}{2}$ for

$$x \equiv \begin{cases} 0 \\ \lambda \end{cases} \pmod 1$$

and to 0 elsewhere.

Therefore, the Fourier series of

$$C_\lambda(n) = g(\alpha n)$$

converges to $C_\lambda(n)$ for all n, except for

$$\alpha n \equiv \begin{cases} 0 \\ \lambda \end{cases} \pmod 1,$$

i.e. $n=0$ and at most one further value of n, because α is irrational. For $n=0$ and the eventual other n it converges to $\frac{1}{2}$, and not 1.

The preceding result can be generalized from $\mathbf{Z}$ to $\mathbf{Z}\times\ldots\times\mathbf{Z}$. We state it for $\mathbf{Z}\times\mathbf{Z}$. If $1,\alpha_1,\beta_1$ or $1,\alpha_2,\beta_2$ are linearly independent with respect to rational coefficients and $0<\lambda_1<1$, $0<\lambda_2<1$ and a_1 and a_2 are real numbers, then the set C on $\mathbf{Z}\times\mathbf{Z}$, consisting of the $(n_1,n_2)\in\mathbf{Z}\times\mathbf{Z}$ for which both

$$a_1 \leqq \alpha_1 n_1 + \alpha_2 n_2 \leqq \lambda_1 + a_1 \pmod 1 \quad \text{and}$$
$$a_2 \leqq \beta_1 n_1 + \beta_2 n_2 \leqq \lambda_2 + a_2 \pmod 1$$

is W-ap on the discrete abelian group $\mathbf{Z}\times\mathbf{Z}$. On $\mathbf{R}\times\mathbf{R}$ we put

$$g(x_1,x_2)$$

equal to 1 when both

$$a_1 \leqq x_1 \leqq \lambda_1 + a_1 \pmod 1 \quad \text{and} \quad a_2 \leqq x_2 \leqq \lambda_2 + a_2 \pmod 1,$$

and equal to 0 elsewhere. Then the characteristic function of C,

$$C(n_1,n_2) = g(\alpha_1 n_1 + \alpha_2 n_2,\ \beta_1 n_1 + \beta_2 n_2)$$

has, being W-ap on $\mathbf{Z}\times\mathbf{Z}$, a Fourier series on $\mathbf{Z}\times\mathbf{Z}$, and this series is obtained from the Fourier series on $\mathbf{R}\times\mathbf{R}$ of

$$g(x_1, x_2)$$

by replacing (x_1, x_2) by $(\alpha_1 n_1 + \alpha_2 n_2, \beta_1 n_1 + \beta_2 n_2)$.

From the preceding we can obtain a sharpening of Kronecker's approximation theorem.

Let $0 < \lambda_1 < 1, \ldots, 0 < \lambda_p < 1$ and let $a_1, \ldots, a_p$ be real numbers. Let further $1, \alpha_1, \ldots, \alpha_p$ be linearly independent with respect to rational coefficients. Then Kronecker's approximation theorem may be stated as follows. The set C of integers n which satisfy all the inequalities

$$a_1 \leqq \alpha_1 n \leqq a_1 + \lambda_1 \pmod 1$$
$$\ldots\ldots$$
$$a_p \leqq \alpha_p n \leqq a_p + \lambda_p \pmod 1$$

is not empty. *We shall prove moreover that C is W-almost periodic and has a positive mean value*

$$MC = \lambda_1 \ldots \lambda_p .$$

Let $C_1, \ldots, C_p$ be the sets of integers which satisfy the first, ..., the pth of the above inequalities. From the preceding it follows that they are W-almost periodic and that

$$MC_1 = \lambda_1, \ldots, MC_p = \lambda_p .$$

It also follows that the Fourier characters of $C_1, \ldots, C_p$ are among respectively

$$e^{i2\pi r_1 \alpha_1 n}, \quad r_1 = 0, \pm 1, \pm 2, \ldots$$
$$\ldots\ldots$$
$$e^{i2\pi r_p \alpha_p n}, \quad r_p = 0, \pm 1, \pm 2, \ldots .$$

We have

$$C = C_1 \cap \ldots \cap C_p$$

so that the characteristic functions satisfy

$$C(n) = C_1(n) \ldots C_p(n) ,$$

and hence $C(n)$ is W-almost periodic, and its Fourier series taken on complex form is obtained by formal multiplication of the Fourier series of $C_1(n), \ldots, C_p(n)$. We seek the constant term of the Fourier series of $C(n)$, and hence seek the integers $r_1, \ldots, r_p$ for which

$$e^{i2\pi r_1 \alpha_1 n} \ldots e^{i2\pi r_p \alpha_p n} = 1 ,$$

i.e.

$$r_1\alpha_1 + \ldots + r_p\alpha_p = \text{integer}$$

and hence $r_1 = \ldots = r_p = 0$, since $1, \alpha_1, \ldots, \alpha_p$ are linearly independent. Thus the only contribution to the constant term of the Fourier series of $C(n)$ is

$$MC = MC_1 \ldots MC_p = \lambda_1 \ldots \lambda_p .$$

REFERENCES

1. E. Følner, *W-almost periodic functions in arbitrary groups*, Mat. Tidskr. B (1946), 153–162.
2. E. Følner, *Almost periodic sets in topological groups*, I pp. 1–65, II pp. 1–16, Appendix pp. 1–10. Copenhagen University, Preprint Series 1979 No. 29. In II p. 8, l. 3–5 a reference to the topology on I p. 24 and I Theorem 6 is missing.
3. L. K. Hua, *Die Abschätzung von Exponentialsummen und ihre Anwendung in der Zahlentheorie*, Enzyklopädie der Mathematischen Wissenschaften Bd. I, 2, Heft 13, Teil I, B. G. Teubner Verlagsgesellschaft, Leipzig, 1959.
4. B. Jessen and A. Wintner, *Distribution functions and the Riemann zeta function*, Trans. Amer. Math. Soc. 38 (1935), 48–88.

LIONS PARK
DK-3400 HILLERØD
DENMARK