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A NATURAL MAP FROM THE EXT KÜNNETH SEQUENCE TO THE TOR ONE

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Dedicated to the memory of my brother Göran.

Under certain conditions, there are two Künneth sequences for the (co)homology of the tensor product of two chain complexes. We show that they are related as indicated in the title.

This paper is a slight expansion of a result in my thesis [2] written under the direction of Professor Brayton Gray. I would like to thank him for his help and guidance and the Department of Mathematics at the University of Illinois at Chicago Circle for the pleasant time I spent there.

ASSUMPTIONS. G is a module and C and D are chain complexes over a principal ideal domain R. C and D are bounded below and C is free and of finite type.

Then the cochain complexes $\mathrm{Hom}(C, \mathrm{Hom}(D, G)) \cong \mathrm{Hom}(C \otimes D, G)$ and $\mathrm{Hom}(C, R) \otimes \mathrm{Hom}(D, G)$ are isomorphic by [1, 5.5.6], since the boundedness condition ensures that the direct products of the former coincide with the direct sums of the latter.

The assumptions may be relaxed in various ways at the expense of complicating the statement; let us just notice that by (a minor extension of) [1, 5.5.9] it is enough that $H(C)$ rather than C be bounded below and of finite type.

We use the notations $H_q(C; G) = H_q(C \otimes G)$, $H^q(C; G) = H^q(\mathrm{Hom}(C, G))$, and $H^q(C) = H^q(C; R)$.

THEOREM. *The two Künneth sequences for $H^*(C \otimes D; G)$ are connected by a natural map*

$$\begin{array}{ccccccc} 0 \to & \mathrm{Ext}(H_*(C), H^*(D; G))^{q-1} & \to & H^q(C \otimes D; G) & \to & \mathrm{Hom}(H_*(C), H^*(D; G))^q & \to 0 \\ & \downarrow & & \downarrow = & & \downarrow & \\ 0 \longrightarrow & (H^*(C) \otimes H^*(D; G))^q & \longrightarrow & H^q(C \otimes D; G) & \longrightarrow & (H^*(C) * H^*(D; G))^{q+1} & \longrightarrow 0 \end{array}$$

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COROLLARY. *There is a natural exact sequence*

$$0 \to \operatorname{Ext}(H_*(C), H^*(D;G))^{q-1} \to (H^*(C) \otimes H^*(D;G))^q \to$$
$$\to \operatorname{Hom}(H_*(C), H^*(D;G))^q \to (H^*(C) * H^*(D;G))^{q+1} \to 0$$

and this sequence is non-naturally split.

Here by splitting of an exact sequence of arbitrary length, we mean that if regarded as a chain complex, it is chain contractible.

The Corollary is immediate from the Theorem by tracing the path $\sqcup\!\!\sqcap\!\downarrow$ in the diagram; a splitting is obtained from splittings of the Künneth sequences. Note that the middle map is induced by the Kronecker pairing $H^*(C) \otimes H_*(C) \to R$.

To prove the Theorem, let $Z_n \subseteqq C_n$ be the n-cycles, $B_{n+1} \subseteqq Z_n$ the n-boundaries of C; let $Z^n \subseteqq C^n$ be the n-cocycles, $B^{n-1} \subseteqq Z^n$ the n-coboundaries of $C^* = \operatorname{Hom}(C, R)$. Regard $Z = \{Z_n\}$, $B = \{B_n\}$, $Z^* = \{Z^n\}$, and $B^* = \{B^n\}$ as (co)chain complexes with trivial (co)boundary operator. These complexes are all free and of finite type. Let $D^* = \operatorname{Hom}(D, G)$.

LEMMA. *There is a natural map of short exact sequences of cochain complexes*

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Hom}(B_*, D^*) & \to & \operatorname{Hom}(C_*, D^*) & \to & \operatorname{Hom}(Z_*, D^*) & \to & 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & Z^* \otimes D^* & \longrightarrow & C^* \otimes D^* & \longrightarrow & B^* \otimes D^* & \longrightarrow & 0 \end{array}$$

PROOF OF THEOREM FROM LEMMA. There is induced a ladder of long exact cohomology sequences

$$\begin{array}{ccccccccc} \to & \operatorname{Hom}(Z_*, H^*)^{q-1} & \xrightarrow{\delta_1} & \operatorname{Hom}(B_*, H^*)^q & \to & H^q(C \otimes D; G) & \to & \operatorname{Hom}(Z_*, H^*)^q & \to \\ & \downarrow & & \downarrow & & \downarrow = & & \downarrow & \\ \longrightarrow & (B^* \otimes H^*)^{q-1} & \xrightarrow{\delta_2} & (Z^* \otimes H^*)^q & \longrightarrow & H^q(C \otimes D; G) & \longrightarrow & (B^* \otimes H^*)^q & \longrightarrow \end{array}$$

where $H^* = H^*(D; G)$. The Theorem follows from this by identifying the (co)kernels of the maps δ_1 and δ_2.

PROOF OF LEMMA. $\partial_n \colon C_n \to C_{n-1}$ factors

$$C_n \xrightarrow{\partial_n^B} B_n \rightarrowtail Z_{n-1} \rightarrowtail C_{n-1}$$

and dually $\delta^{n-1} \colon \mathrm{C}^{n-1} \to \mathrm{C}^n$ factors

$$\operatorname{Hom}(C_{n-1}, R) \twoheadrightarrow \operatorname{Hom}(Z_{n-1}, R) \to \operatorname{Hom}(B_n, R) \xrightarrow{\delta_B^{n-1}} \operatorname{Hom}(C_n, R)$$

where the first map is epi since it is the dual of a split monomorphism. Now,

$$\delta^n \circ \delta_B^{n-1} = \mathrm{Hom}(\partial_{n+1}, R) \circ \mathrm{Hom}(\partial_n^B, R) = \mathrm{Hom}(\partial_n^B \circ \partial_{n+1}, R) = 0$$

since $\partial_n \circ \partial_{n+1} = 0$ and ∂_n is ∂_n^B followed by a monomorphism. Thus the image of δ_B^{n-1} is contained in $\ker \delta^n = Z^n$, so δ_B^{n-1} factors through Z^n; the factorization of δ^{n-1} shows further that the image of $\mathrm{Hom}(Z_{n-1}, R)$ in $\mathrm{Hom}(C_n, R)$ is precisely $\mathrm{im}\,\delta^{n-1} = B^{n-1}$. We get a map of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}(B_n, R) & \to & \mathrm{Hom}(C_n, R) & \to & \mathrm{Hom}(Z_n, R) & \to & 0 \\ & & \downarrow & & \downarrow = & & \downarrow & & \\ 0 & \longrightarrow & Z^n & \longrightarrow & C^n & \longrightarrow & B^n & \longrightarrow & 0 \end{array}$$

Since B_n, C_n, Z_n, and B^n are finitely generated free, tensoring this with D^m gives

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}(B_n, D^m) & \to & \mathrm{Hom}(C_n, D^m) & \to & \mathrm{Hom}(Z_n, D^m) & \to & 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & Z^n \otimes D^m & \longrightarrow & C^n \otimes D^m & \longrightarrow & B^n \otimes D^m & \longrightarrow & 0 \end{array}$$

Taking the direct sum for $n+m=q$ now establishes the Lemma: it is easily checked that all maps commute with the coboundary operators.

Our assumptions are also appropriate for the chain complexes $C \otimes D \otimes G$, $\mathrm{Hom}(\mathrm{Hom}(C, R), R) \otimes D \otimes G$, and $\mathrm{Hom}(\mathrm{Hom}(C, R), D \otimes G)$ to be isomorphic, and in an entirely analogous fashion we may prove the Theorem with homology and cohomology interchanged:

THEOREM. *The two Künneth sequences for $H_*(C \otimes D;\, G)$ are connected by a natural map*

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ext}(H^*(C), H_*(D;\, G))_{q+1} & \to & H_q(C \otimes D;\, G) & \to & \mathrm{Hom}(H^*(C), H_*(D;\, G))_q & \to & 0 \\ & & \downarrow & & \downarrow = & & \downarrow & & \\ 0 & \longrightarrow & (H_*(C) \otimes H_*(D;\, G))_q & \longrightarrow & H_q(C \otimes D;\, G) & \longrightarrow & (H_*(C) * H_*(D;\, G))_{q-1} & \longrightarrow & 0 \end{array}$$

COROLLARY. *There is a natural exact sequence*

$$0 \to \mathrm{Ext}(H^*(C), H_*(D;\, G))_{q+1} \to (H_*(C) \otimes H_*(D;\, G))_q \to$$
$$\to \mathrm{Hom}(H^*(C), H_* D;\, G))_q \to (H_*(C) * H_*(D;\, G))_{q-1} \to 0$$

and this sequence is non-naturally split.

Finally we make the usual observation that by taking $D_0 = R$, $D_q = 0$, $q \neq 0$, we get the corresponding results for (co)homology universal coefficient sequences. In this case, of course, C need not be bounded.

REFERENCES

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2. T. Weibull, *Steenrod operations in spectral cohomology theories*, Thesis, University of Illinois at Chicago Circle, 1980.

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