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ROTATION-AUTOMORPHIC FUNCTIONS NEAR THE BOUNDARY

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In the paper [4] we defined rotation-automorphic functions with respect to some Fuchsian group. The function f, meromorphic in the unit disk D, was said to be rotation-automorphic with respect to a Fuchsian group Γ acting on D if it satisfies the equation $f(T(z))=S_T(f(z))$ where $T\in\Gamma$ and S_T is a rotation of the Riemann sphere. The fundamental domain F of Γ was said to be thick if it satisfies the following condition: There are positive constants r, r' such that for any sequence of points $(z_n)\subset F$ there is a sequence of points (z_n') for which the hyperbolic distance $d(z_n, z_n')\leqq r$ and the hyperbolic disk $U(z_n', r')\subset F$ for each $n=1,2,\ldots$. If we suppose that F is thick, we proved the following theorem (cf. [4], Theorem 5).

THEOREM 1. *Let F be a fundamental domain of Γ and f a rotation-automorphic function with respect to Γ. If F is thick and*

$$\iint_F \left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma_z < \infty , \tag{1}$$

where $d\sigma_z$ is the euclidean area element, then f is a normal function in D.

Further, we showed there the existence of a rotation-automorphic function with at least two rotation axes and obtained a general principle to construct rotation-automorphic functions. Note that character-automorphic functions are special cases of rotation-automorphic functions with only one rotation axis (0∞-axis).

In [2] we constructed a non-normal function which satisfies the condition (1).

In the paper [5] we restricted Γ to be a finitely generated Fuchsian group and obtained the following theorem:

THEOREM 2. *Let Γ be a finitely generated Fuchsian group and f a rotation-automorphic function with respect to Γ. If*

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$$\iint_F \left(\frac{|f'(z)|}{1+|f(z)|^2} \right)^2 d\sigma_z < \infty ,$$

then f is a normal function in D.

Here we continue to study rotation-automorphic functions and prove a theorem which is in a close relation to Theorem 1.

1.

Let D and ∂D be the unit disk and the unit circle respectively. We shall denote the hyperbolic distance by $d(z_1, z_2)$ $(z_1, z_2 \in D)$ and the hyperbolic disk $\{z \mid d(z, z_0) < r\}$ by $U(z_0, r)$. The spherical distance is denoted by $d^*(w_1, w_2)$ $(w_1, w_2 \in \hat{\mathbf{C}})$. Let Γ be a Fuchsian group acting on D and

$$n(K, z) = \operatorname{card}\{\Gamma z \cap K\}, \quad K \subset D .$$

Let f be a meromorphic function in D. Then f is called rotation-automorphic with respect to Γ, if

$$f(T(z)) = S_T(f(z)), \quad z \in D,\ T \in \Gamma , \tag{1.1}$$

where S_T is a rotation of the Riemann sphere. The meromorphic function f is said to be normal in D [8], if

$$\sup_{z \in D} (1-|z|^2) \frac{|f'(z)|}{1+|f(z)|^2} < \infty . \tag{1.2}$$

We note that, for a rotation-automorphic function f with respect to Γ, the expression $(1-|z|^2)|f'(z)|/(1+|f(z)|^2)$ is invariant under the transformations of Γ.

Fix the fundamental domain F of Γ to be some normal polygon in D.

Just as we defined [1] additive automorphic functions (that is, integral functions of automorphic forms) of the second kind, we can define rotation-automorphic functions of the second kind.

1.1. Definition. A rotation-automorphic function f is said to be of the second kind if there exists a sequence of points (z_n) in the closure $\bar{F}$ such that the sequence of functions

$$g_n(\zeta) = f\left(\frac{\zeta + z_n}{1 + \bar{z}_n \zeta} \right) \tag{1.3}$$

tends uniformly to a constant limit in some neighbourhood of $\zeta = 0$.

For proving the main result we need the following lemma (cf. [4, Lemma]).

1.2. LEMMA. *Let* $(z_n) \subset F$ *be a sequence of points such that* $\lim_{n\to\infty} |z_n| = 1$, $r > 0$ *and* $R > 0$. *Then* $T(U(z_n, r)) \cap U(0,R) \neq \emptyset$ *for finitely many* $T \in \Gamma$ *and* $n \in \mathrm{N}$ *only.*

1.3. THEOREM. *Let* Γ *be a Fuchsian group satisfying the following conditions:*

(1.4) Γ *consists of* id *and hyperbolic transformations. There exists a constant* $t_0 > 2$ *such that*

(1.5) $|\mathrm{trace}\, T| \geqq t_0$ *for all* $T \in \Gamma$, $T \neq \mathrm{id}$.

Let f *be a rotation-automorphic function with respect to* Γ *and* F *a fundamental domain of* Γ. *If*

$$\iint_F \left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma_z < \infty , \tag{1.6}$$

then f *is normal in* D.

PROOF. Concerning Theorem 1 we note that if the fundamental domain F of Γ is thick, then Γ satisfies the condition (1.5). Fix $r > 0$ to be small enough (less than Marden's constant, cf. [9]). Then there exists a positive integer m such that

$$n(U(z,r),\zeta) \leqq m \tag{1.7}$$

for all $z \in D$, $\zeta \in D$. By 1.2. Lemma, (1.6), and (1.7) we can choose $R > 0$ so large that $z \in F \setminus U(0,R)$ implies

$$S(r) = \frac{1}{\pi} \iint_{U(z,r)} \left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma_z < \tfrac{1}{2} . \tag{1.8}$$

Define the transformations

$$T(w) = \frac{w+z}{1+\bar{z}w}$$

and the functions

$$f_T(w) = f(T(w)) .$$

By [6, Theorem 6.1.] we have

$$\left(\frac{|f'_T(0)|}{1+|f_T(0)|^2}\right)^2 \leqq \frac{1}{x^2}\frac{S(r)}{1-S(r)} \tag{1.9}$$

where $x = (e^{2r}-1)/(e^{2r}+1)$. Suppose now that $z \in F \setminus U(0,R)$. Then, by (1.7), (1.8), and (1.9), we obtain

$$
\begin{aligned}
(1.10) \qquad \left((1-|z|^2)\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 &\leqq \frac{2}{x^2}S(r) \\
&= \frac{2}{\pi x^2}\iint_{U(z,r)}\left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma_z \\
&\leqq \frac{2}{\pi x^2}(m+1)\iint_F\left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma_z .
\end{aligned}
$$

Thus

$$
(1.11) \qquad \sup_{z\in F\setminus U(0,R)} (1-|z|^2)\frac{|f'(z)|}{1+|f(z)|^2} = a_1 < \infty .
$$

Denote the closure of $U(0,R)$ by $\overline{U(0,R)}$. Then, in the compact set $\overline{U(0,R)}$, it holds

$$
(1.12) \qquad \sup_{z\in F\cap \overline{U(0,R)}} (1-|z|^2)\frac{|f'(z)|}{1+|f(z)|^2} = a_2 < \infty .
$$

Connecting (1.11) and (1.12) we have

$$
(1.13) \qquad \sup_{z\in F} (1-|z|^2)\frac{|f'(z)|}{1+|f(z)|^2} = \max\{a_1, a_2\} < \infty
$$

and thus f is normal in D.

1.4. COROLLARY. *By the assumptions of* 1.3. *Theorem we can prove the following: For each sequence of points* $(z_n)\subset F$ *converging to* ∂D *it holds*

$$
(1.14) \qquad \lim_{n\to\infty} (1-|z_n|^2)\frac{|f'(z_n)|}{1+|f(z_n)|^2} = 0 .
$$

PROOF. Let $(z_n)\subset F$ be any sequence of points converging to ∂D. Let $r>0$ be chosen as in 1.3. Theorem. By (1.10) we must only prove that

$$
\lim_{n\to\infty}\iint_{U(z_n,r)}\left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma_z = 0 .
$$

Suppose, on the contrary, that there is a subsequence (z_k) of (z_n) such that

$$
(1.15) \qquad \inf_k \iint_{U(z_k,r)}\left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma_z = a > 0 .
$$

Because of the finiteness of the integral $\iint_F (|f'(z)|/(1+|f(z)|^2))^2\, d\sigma_z$ we have an $R>0$ such that

$$\iint_{F \setminus U(0,R)} \left(\frac{|f'(z)|}{1+|f(z)|^2} \right)^2 d\sigma_z < \frac{a}{2m} . \tag{1.16}$$

By (1.15) and (1.16) there is a positive integer k_0 such that, for each $k \geqq k_0$, there exists $T_k \in \Gamma$ such that $T_k(U(z_k, r)) \cap U(0, R) \neq \emptyset$. This contradicts 1.2. Lemma. The corollary is proved.

2.

In this section we shall consider the existence and non-existence of angular limits of a rotation-automorphic function f at certain points of the unit circle ∂D. Let first Ω be the group of all Moebius transformations of D onto itself. The function f is called normal in D if the family of combined functions $\{f \circ T \mid T \in \Omega\}$ is normal in D (cf. [10]). For these functions a theorem of Pommerenke [11, Theorem 4] can be written in the following form:

2.1. THEOREM. *Let f be a normal rotation-automorphic function with respect to Γ. Then f has an angular limit at the parabolic vertices of the fundamental domains $T(F)$, $T \in \Gamma$.*

Related to 1.1 we prove the following theorem:

2.2. THEOREM. *Let f be a rotation-automorphic function with respect to Γ. If f has an angular limit at some point of ∂D, then f is of the second kind.*

PROOF. Let f have an angular limit c at $\zeta_0 \in \partial D$. Suppose that the sequence of points $(z_n) \subset \alpha$ (α a Stoltz angle at ζ_0) converges to ζ_0. Choose the transformations $L_n \in \Omega$, $T_n \in \Gamma$ such that $L_n(0) = z_n$, $T_n(z_n) = z_n' \in \bar{F}$ and $(T_n \circ L_n)(\zeta) = (\zeta + z_n')/(1 + \bar{z}_n'\zeta)$ for each $\zeta \in D$ and $n = 1, 2, \ldots$. Define the functions

$$g_n(\zeta) = f(L_n(\zeta)) . \tag{2.1}$$

Since $\lim f(z) = c$ for $z \in \bigcup_{n=1}^{\infty} U(z_n, r)$, $z_n \to \zeta_0$, then $\lim_{n\to\infty} g_n(\zeta) = c$ uniformly in $U(0, r)$. Let

$$h_n(\zeta) = f(T_n(L_n(\zeta))) = S_{T_n}(f(L_n(\zeta))) = S_{T_n}(g_n(\zeta)) . \tag{2.2}$$

We now prove that there is a subsequence (h_k) of (h_n) converging uniformly to a constant in $U(0, r)$. Applying the spherical rotation S_{T_n} to the points $g_n(\zeta)$ and c, we have $d^*(g_n(\zeta), c) = d^*(S_{T_n}(g_n(\zeta)), S_{T_n}(c))$. Choose a subsequence $(S_{T_k}(c))$ of $(S_{T_n}(c))$ such that there exists $\lim_{k\to\infty} S_{T_k}(c) = c_0$. Then applying the triangle inequality to (2.2) yields

$$d^*(h_k(\zeta), c_0) \leqq d^*(S_{T_k}(g_k(\zeta)), S_{T_k}(c)) + d^*(S_{T_k}(c), c_0) \to 0 \tag{2.3}$$

uniformly in $U(0, r)$. Thus the assertion is proved.

Next we consider the behaviour of a rotation-automorphic function in the neighbourhood of a hyperbolic fixed point.

2.3. THEOREM. *Let f be a non-constant rotation-automorphic function with respect to Γ. Then f has no angular limit at a hyperbolic fixed point ξ of Γ.*

PROOF. Choose $z_1, z_2 \in F$ such that $f(z_1) \neq f(z_2)$. Let $T \in \Gamma$ be a hyperbolic transformation which has the point ξ as an attractive fixed point. Then the sequences of points $(z_1^n) = (T^n(z_1))$, $(z_2^n) = (T^n(z_2))$ converge in a Stolz angle α to ξ. Now

$$\begin{aligned} 0 < d^*(f(z_1), f(z_2)) &= d^*(S_{T^n}(f(z_1)), S_{T^n}(f(z_2))) \\ &= d^*(f(T^n(z_1)), f(T^n(z_2))) = d^*(f(z_1^n), f(z_2^n)) \end{aligned} \tag{2.4}$$

for each $n = 1, 2, \ldots$. Hence f has no angular limit at ξ.

The following theorem is applied for proving the non-existence of non-constant analytic rotation-automorphic functions with respect to the Fuchsian group whose fundamental domain is compact.

2.4. THEOREM. *Let Γ be a Fuchsian group having the fundamental domain F with a finite number of sides and f a non-constant rotation-automorphic function with respect to Γ. If the radius $L = 0\xi$, $|\xi| = 1$, intersects infinitely many fundamental domains $T(F)$, $T \in \Gamma$, then f has no angular limit at ξ.*

PROOF. By the assumptions, ξ is a limit point of Γ and not a parabolic vertex of any $T(F)$, $T \in \Gamma$. Suppose, on the contrary, that f has an angular limit c at ξ. By [7, V, 5G, Theorem] one can find a sequence of points $(z_n) \subset L$ tending to ξ such that Γ-images $z'_n = T_n(z_n)$ lie in $\{z \mid |z| \leqq R\}$ for some $R < 1$ and $T_n \in \Gamma$. We may suppose, without loss of generality, that there exists $z'_0 = \lim_{n\to\infty} z'_n \in \{z \mid |z| \leqq R\}$. It suffices to show that f is constant in any hyperbolic disk $U(z'_0, r)$. Let $r > 0$ and choose a $w \in U(z'_0, r)$. We may suppose that $z'_n \in U(z'_0, r)$, $n = 1, 2, \ldots$. Denote $w_n = T_n^{-1}(w)$. Consider the functions

$$g_n(\zeta) = f\left(\frac{\zeta + w_n}{1 + \bar{w}_n \zeta}\right). \tag{2.5}$$

Since $d(z_n, w_n) = d(z'_n, w) < 2r$, the sequence $(g_n(\zeta))$ converges to c uniformly on every compact subset of D. Hence

$$\frac{|g_n'(0)|}{1+|g_n(0)|^2} = (1-|w_n|^2)\frac{|f'(w_n)|}{1+|f(w_n)|^2} \to 0 \tag{2.6}$$

as $n \to \infty$. By the invariance under the transformations of Γ

$$(1-|w_n|^2)\frac{|f'(w_n)|}{1+|f(w_n)|^2} = (1-|w|^2)\frac{|f'(w)|}{1+|f(w)|^2} \tag{2.7}$$

and thus

$$(1-|w|^2)\frac{|f'(w)|}{1+|f(w)|^2} = 0\,. \tag{2.8}$$

Since $1-|w|^2 \neq 0$, we have

$$\frac{|f'(w)|}{1+|f(w)|^2} = 0\,. \tag{2.9}$$

Hence f is a constant in $U(z_0', r)$ and as a meromorphic function in D. The theorem follows.

2.5. THEOREM. *Let Γ be a Fuchsian group and f an analytic rotation-automorphic function with respect to Γ. If the fundamental domain F of Γ is compact, then f is a constant in D.*

PROOF. Suppose, on the contrary, that f is not a constant in D. Since the expression $(1-|z|^2)|f'(z)|/(1+|f(z)|^2)$ is invariant under the transformations of Γ, f is a normal function in D. Let $\xi \in \partial D$ be an arbitrary point. Then the radius $L = 0\xi$ intersects infinitely many fundamental domains $T(F)$, $T \in \Gamma$. By 2.4 the function f has no angular limit at ξ. On the other hand, an analytic normal function f has at least one angular limit on ∂D. This contradiction shows that f is a constant in D.

2.6. REMARK. If the fundamental domain F of Γ is compact, then there exist always non-constant meromorphic rotation-automorphic functions with respect to Γ (for example, automorphic functions).

Finally we prove the non-existence of angular limits at transitive boundary points. Therefore let λ be a hyperbolic ray. Then λ crosses a finite or infinite number of fundamental domains, the images of the fundamental domain F. Each point of λ has a Γ-equivalent point in $\bar{F}$. If the set of these points is everywhere dense in F, then λ is said to be transitive (under Γ). A point $\xi \in \partial D$ is called transitive if every hyperbolic ray through ξ is transitive.

2.7. THEOREM. *Let f be a non-constant rotation-automorphic function with respect to Γ. Then f has no angular limit at a transitive point $\xi \in \partial D$.*

PROOF. Suppose, on the contrary, that f has an angular limit c at ξ. The radius $L=0\xi$ intersects infinitely many fundamental domains $F_n = T_n(F)$, $T_n \in \Gamma$, and the arcs $T_n^{-1}(L \cap F_n)$ have, by the transitivity, an accumulation continuum C in F. Choose two different points x', y' of C. Let $x'_k, y'_k \in T_{n_k}^{-1}(L \cap F_{n_k})$, $x'_k \neq y'_k$, be the points such that $\lim_{k\to\infty} x'_k = x'$ and $\lim_{k\to\infty} y'_k = y'$. Denoting $x_k = T_{n_k}(x'_k)$ and $y_k = T_{n_k}(y'_k)$ we have $\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} f(y_k) = c$. Now

$$
\begin{aligned}
(2.10) \qquad d^*(f(x'), f(y')) &= \lim_{k\to\infty} d^*(f(x'_k), f(y'_k)) \\
&= \lim_{k\to\infty} d^*(S_{T_{n_k}}(f(x'_k)), S_{T_{n_k}}(f(y'_k))) \\
&= \lim_{k\to\infty} d^*(f(T_{n_k}(x'_k)), f(T_{n_k}(y'_k))) \\
&= \lim_{k\to\infty} d^*(f(x_k), f(y_k)) = 0 .
\end{aligned}
$$

Hence $f(x') = f(y')$. Thus f is a constant on the nondegenerate continuum C which contradicts the assumption that f is a non-constant meromorphic function. The theorem is proved.

2.8. REMARK. In fact, in 2.7, we have proved that f has no radial limit at ξ. Further, for proving the assertion of 2.7. Theorem we need only suppose that some hyperbolic ray is transitive.

2.9. REMARK. In [3, 2.3. Theorem] we obtained the following theorem: If $W(T(z)) = W(z) + A_T$, $A_T \neq 0$, for a hyperbolic transformation $T \in \Gamma$, then the additive automorphic function W has the angular limit ∞ at the fixed points of T. The complex number A_T is known as the period of W with respect to T in Γ. If $A_T = 0$, it is easy to see that W has no angular limit at the fixed points of T. In [3] we proved a slightly weaker version of 2.7. Theorem for additive automorphic functions; namely, an additive automorphic function can have no finite angular limit at a transitive point.

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