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TANGENCIES OF GENERIC REAL PROJECTIVE HYPERSURFACES

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Introduction.

Let P^n denote the real projective n space, $R(n,d)$ the vector space of homogeneous polynomials of degree d in $n+1$ variables and $D \subset R(n,d)$ the algebraic subset of forms defining singular hypersurfaces.

A hypersurface $\{P=0\} \subset P^n$ is said to have its tangencies in general position if, given a hyperplane H tangent at $\{p_1,\ldots,p_k\} \subset \{P=0\}$, the points $p_1,\ldots,p_k$ are in general position on H.

In a recent paper [1] Bruce showed that in the complex case the set of P whose tangencies are not in general position form a constructible set of codimension $\geqq 1$. In the same paper he raised the similar question in the real case and showed that a positive answer will give interesting informations about the duals of generic real hypersurfaces (Remark 2.9).

In the present note we prove the following:

THEOREM. *The set of polynomials $P \in R(n,d) \setminus D$ whose tangencies are not in general position form a semialgebraic set of codimension* $\geqq 1$.

1. A simple result on semialgebraic sets.

We shall use the definition and the properties of semialgebraic sets as presented in [2, Chap. I.].

We will need the following result.

LEMMA 1.1. *Let f: $\mathbf{R}^n \to \mathbf{R}^p$ be a polynomial mapping and let $X \subset \mathbf{R}^n$ be a semialgebraic nonsingular subset. Then*

$$\dim f(X) \leqq \dim X - \min\{\dim f^{-1}(y) \cap X;\ y \in f(X)\}\ .$$

PROOF. Note that $Y=f(X)$ is semialgebraic and if $\sum Y$ is the singular set of Y, then $Y_0 = Y \setminus \sum Y$ is an open dense subset in Y and $\dim Y = \dim Y_0$.

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Let Y_1 be a connected component of Y_0 such that $\dim Y_1 = \dim Y$. Then $X_1 = f^{-1}(Y_1)$ is an open subset in X and let $g\colon X_1 \to Y_1$ be the smooth map induced by f.

By Sard's theorem, there is a regular value $y \in Y_1$ for g and we get

$$\dim Y_1 = \dim (X_1) \text{ at } x - \dim (g^{-1}(y)) \text{ at } x$$

for any $x \in g^{-1}(y) = f^{-1}(y) \cap X$.

And this clearly ends the proof of our lemma.

REMARK 1.2. A similar result obviously holds in a global situation i.e. when $\mathbf{R}^n$ and $\mathbf{R}^p$ are replaced by real algebraic manifolds and f by a real algebraic map.

2. The proof of the Theorem.

For any $p = 1, 2, \ldots, n-1$ let us consider the flag manifold

$$F(p, n-1) = \{(E, H);\ E \subset H \subset P^n,\ \dim E = p,\ \dim H = n-1\}.$$

Recall that $F(p, n-1)$ is a real algebraic manifold of dimension $n(p+2) - (p+1)^2$.

Next let us define a semialgebraic subset

$$G_p \subset (P^n)^{p+2} \times F(p, n-1)$$

as follows

$$G_p = \{(a_0, \ldots, a_{p+1}, E, H);\ a_0, \ldots, a_p \text{ span } E,\ a_{p+1} \in E \text{ and } a_{p+1} \text{ is not a linear combination of less than } p+1 \text{ points from } a_i,\ i = 0, \ldots, p\}.$$

Using the second projection, it follows that G_p is a real algebraic manifold of dimension $n(p+2) - 1$.

Let $B = R(n, d) \setminus D$ with the notations from introduction and consider the following semialgebraic set $Z_p \subset B \times G_p$,

$$Z_p = \{(P, a_0, \ldots, a_{p+1}, E, H);\ P(a_i) = 0 \text{ and } T_{a_i}\{P = 0\} = H \text{ for } i = 0, \ldots, p+1\}.$$

Let $f\colon Z_p \to B$ denote the restriction of the first projection to Z_p.

The Theorem then follows from

LEMMA 2.1. *For any* $p = 1, \ldots, n-1$, $f(Z_p)$ *is a semialgebraic set in* B *of codimension* $\geqq 1$.

PROOF. Let $g\colon Z_p \to G_p$ be the restriction of the second projection to Z_p and note that g is a fiber bundle projection. Change of coordinates shows that we can take as typical fiber $F=g^{-1}(a^0, E^0, H^0)$ with H^0: $x_n=0$, E^0: $x_n=\ldots=x_{p+1}=0$,

$$a^0 = (a_0^0,\ldots,a_{p+1}^0), \quad \text{where } a_0^0=(1,0,\ldots,0),\ldots,$$
$$a_p^0 = (0,\ldots,0,\underset{p}{1},0,\ldots,0), \quad a_{p+1}^0 = (1,1,\ldots,\underset{p}{1},0,\ldots,0).$$

We shall write a polynomial $P \in B$ in the form

$$P(x) = x_n P_1(x) + Q(x_0,\ldots,x_{n-1})$$

and also

$$Q = A(x_0,\ldots,x_p) + \sum_{i=p+1}^{n-1} B_i(x_0,\ldots,x_p)x_i + \sum_{i,j=p+1}^{n-1} C_{i,j}(x_0,\ldots,x_{n-1})x_i x_j .$$

With these notations it is easy to check that $P \in F$ iff

$$S:\ \begin{cases} B_i(a_j^0)=0 & \text{for } i=p+1,\ldots,n-1;\ j=0,\ldots,p+1 \\ \dfrac{\partial A}{\partial x_k}(a_j^0)=0 & k=0,\ldots,p. \end{cases}$$

This is a system of linear equations in the coefficients of the polynomial P and hence F is the intersection of B with a linear subspace in $R(n,d)$. In particular F (and hence also Z_p) is nonsingular.

The system S consists of $n(p+2)$ equations which can be written explicitly due to the special form of the points a_j^0.

In the case $d=2$ the proof of (2.1) is trivial. Indeed, the hypersurface $\{P=0\}$ is then a smooth quadric and any hyperplane is tangent to it at no more than one point, and hence the sets Z_p are empty.

When $d \geqq 3$ the equations of the system S are linearly independent apart from the following "degenerate" cases:

$$(d,p) = (3,1), \quad (3,2) \quad \text{and} \quad (4,1).$$

To see this, note that the equations involving the polynomials B_i are always independent, while the equations involving the polynomial A are independent iff the following $(p+1)$ equations are independent.

$$E_k:\ \sum_\alpha \alpha_k a_\alpha = 0 \qquad k=0,1,\ldots,p$$

where $\alpha=(\alpha_0,\ldots,\alpha_p)$, $|\alpha|=d$, $\max \alpha_i < d-1$ and a_α indeterminates.

Suppose now $d=3$, $p \geqq 3$ and $\sum \lambda_k E_k=0$ for some $\lambda_i \in \mathbf{C}$. Using the multiindex

$$\alpha = (0 \dots \underset{i}{1} \dots 0 \dots \underset{j}{1} \dots 0 \dots \underset{k}{1} \dots 0)$$

we get $\lambda_i + \lambda_j + \lambda_k = 0$ for any $0 \leqq i < j < k \leqq p$.

Geometrically this means that any triangle (proper or degenerate) in the complex plane with vertexes in the set $\{\lambda_i\}$ has baricenter 0, which is possible only if $\lambda_i = 0$ for any i.

For $d \geqq 4$ we can use the multiindex

$$\alpha = (0 \dots \underset{i}{d-2} \dots 0 \dots \underset{j}{2} \dots 0)$$

and get $(d-2)\lambda_i + 2\lambda_j = 0$ for any $i \neq j$.

If $d > 4$ this already gives $\lambda_i = 0$ for any i by symmetry in i and j. For $d = 4$ we only get $\lambda_i + \lambda_j = 0$ for any $i \neq j$ and if $p > 1$ we find again as above $\lambda_i = 0$ for any i.

Therefore, apart from the special cases, codim F in $B = n(p+2)$ and hence $\dim Z_p \leqq \dim B - 1$ which gives the result. Finally we treat the degenerate cases one by one:

i) $(d, p) = (3, 1)$.

In this case the system S implies

$$A = B_2 = \dots = B_{n-1} = 0 .$$

Hence the line E^0 is contained in the hypersurface $\{P = 0\}$ and, if $P \notin D$, for any point $x \in E^0$ we have $T_x\{P = 0\} = H^0$. It follows that

$$\dim f^{-1}(P) \geqq 3 .$$

In this case rk $S = 3n - 2$ and hence $\dim Z_p = \dim B + 1$. Our result (1.1) then gives $\dim f(Z_p) < \dim B$.

ii) $(d, p) = (3, 2)$.

In this case the system S implies

$$A = 0 \quad \text{and} \quad B_i(x_0, x_1, x_2) = b_i x_0 x_1 + c_i x_1 x_2 + d_i x_0 x_2$$

with $b_i + c_i + d_i = 0$ for $i = 3, \dots, n-1$. In particular rk $S = 4n - 2$ and hence $\dim Z_p = \dim B + 1$.

Let us denote by s the composition

$$P^2 = E^0 \hookrightarrow \{P = 0\} \xrightarrow{\check{d}} \check{P}^n, \qquad \check{d} = (\partial P/\partial x_0, \dots, \partial P/\partial x_n) .$$

Note that $H \in S(P^2)$ implies $H \supset E^0$.

If $\dim s(P^2) < 2$ then it follows that there is a hyperplane $H \in \check{P}^n$ such that $\dim s^{-1}(H) > 0$ (use 3.3 in [2, p. 29]).

And a similar argument to that in i) ends the proof in this case.

If $\dim s(P^2)=2$, the it is easy to show that for any hyperplane H in a neighbourhood in image s of H^0, $s^{-1}(H)$ contains one point in each neighbourhood of the points a_j^0, $j=0,\ldots,3$ and hence $\dim f^{-1}(P)\geqq 2$.
As above, (1.1) gives the result.

iii) $(d,p)=(4,1)$.
In this case the system S gives:

$$A = 0 \quad \text{and} \quad B_i(x_0,x_1) = b_i x_0^2 x_1 + c_i x_0 x_1^2$$

with $b_i+c_i=0$ for $i=2,\ldots,n-1$. In particular $\operatorname{rk} S=3n-1$ and hence $\dim Z_p = \dim B$.

We consider the composition

$$t : P^1 = E^0 \hookrightarrow \{P=0\} \xrightarrow{\check{d}} \check{P}^n$$

and a completely similar argument with case ii) shows that $\dim f^{-1}(P)\geqq 1$ which gives the result.

REMARK 2.2. The proof of the Theorem given here works equally well over any algebraically closed field of zero characteristic since then (1.1) is a standard result.

REFERENCES

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2. C. G. Gibson, K. Wirthmüller, A. A. du Plessis, and E. J. N. Looijenga, *Topological stability of smooth mappings* (Lecture Notes in Math. 552), Springer-Verlag, Berlin - Heidelberg - New York, 1976.

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