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SHARPNESS OF YOUNG'S INEQUALITY FOR CONVOLUTION

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1. Introduction.

One of the most basic results in harmonic analysis is the following convolution theorem, which is usually referred to as *Young's inequality*.

THEOREM. *Let G be a unimodular locally compact group. Let p, q be real numbers such that* $1<p<\infty$, $1<q<\infty$ *and* $1/p+1/q>1$, *and let r be defined by* $1/r=1/p+1/q-1$. *Then*

(i) $L_p(G)*L_q(G)\subseteqq L_r(G)$,
(ii) *for* $f\in L_p(G)$ *and* $g\in L_q(G)$, *we have*

$$\|f*g\|_r \leqq 1\cdot\|f\|_p\|g\|_q .$$

This result suggests several natural questions. For example, one may ask

(a) when do we have equality in (i)?
(b) for given p and q, is the index r in (i) optimal?
(c) is the constant 1 in (ii) the best possible?

The answer to question (a) is "never" (except for the trivial case when G is finite). In fact, Yap [14, Theorem 1.1] has proved that the subspace spanned by $L_p(G)*L_q(G)$ is a dense subspace of the first category in $L_r(G)$ for all infinite unimodular locally compact groups G. The answer to question (c) is "no". In fact, Beckner [1, Theorem 3] has shown that for the n-dimensional Euclidean space $\mathbf{R}^n$, $n\geqq 1$,

$$\text{(iii)}\qquad \|f*g\|_r \leqq (A_pA_qA_{r'})^n\|f\|_p\|g\|_q$$

for all $f\in L_p(\mathbf{R}^n)$, $g\in L_q(\mathbf{R}^n)$, where

$$A_m = [(m^{1/m})/(m')^{1/m'}]^{\frac{1}{2}}, \qquad 1/m+1/m' = 1 .$$

Moreover, the constant $(A_pA_qA_{r'})^n$ in (iii) is the best possible. (See Fournier [4] and Brascamp and Lieb [2] for related results.)

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In [8, Theorem 9], Kunze and Stein have shown that if G is the unimodular group of 2×2 real matrices and $1\leqq p<2$, then

$$L_2(G)*L_p(G) \subseteqq L_2(G).$$

This result, now called the Kunze–Stein Phenomenon, shows that the answer to (b) is "no" in general. In fact, Cowling [3] has shown that the Kunze–Stein Phenomenon holds if G is any connected semi-simple Lie group with finite center. However, the answer to question (b) is "yes" for all locally compact *abelian* groups G. More precisely, we have the following result.

THEOREM 1.1. *Let G be an infinite locally compact abelian group. Let p, q be real numbers such that $1<p<\infty$, $1<q<\infty$ and $1/p+1/q>1$, and let r be defined by $1/r=1/p+1/q-1$. Then we have*

(i) *if G is compact, then*

$$L_p(G)*L_q(G) \nsubseteqq \bigcup \{L_s(G) : r<s\};$$

(ii) *if G is discrete, then*

$$l_p(G)*l_q(G) \nsubseteqq \bigcup \{l_s(G) : s<r\};$$

(iii) *if G is neither compact nor discrete, then*

$$L_p(G)*L_q(G) \nsubseteqq \bigcup \{L_s(G) : s\neq r\}.$$

The method we use in the proof of Theorem 1.1 can also be used to prove Theorem 1.2 below. Before we state Theorem 1.2 we recall that if G is any unimodular locally compact group and $1<p<\infty$, then $L_p(G)*L_{p'}(G)\subseteqq C_0(G)$, where $1/p+1/p'=1$ and $C_0(G)$ denotes the space of all continuous functions on G which vanish at infinity. In particular, if G is a compact group and $1<p<\infty$, then

$$L_p(G)*L_{p'}(G) \subseteqq \bigcap \{L_s(G) : 1\leqq s<\infty\}.$$

THEOREM 1.2. *Let G be a non-compact locally compact abelian group, and let $1<p<\infty$. Then*

$$L_p(G)*L_{p'}(G) \nsubseteqq \bigcup \{L_s(G) : 1\leqq s<\infty\}.$$

The following result of N. Rickert, which complements Young's inequality and Theorems 1.1 and 1.2 above, will be useful to us later.

THEOREM 1.3 (Rickert [12]). *Let $1<p<\infty$, $1<q<\infty$, and $1/p+1/q<1$. Let G be a non-compact locally compact group and U a neighborhood of the zero* *element of G with compact closure. Then there exist functions* $f \in L_p(G)$ *and* $g \in L_q(G)$ *such that* $f * g(y)$ *is undefined for all* y *in* U.

As an easy consequence of Theorems 1.1, 1.2 and 1.3, we have the following corollary which shows that the generalized L_p-conjecture (see Rajagopalan [11]) is true for locally compact abelian groups.

COROLLARY 1.4. *Let G be a locally compact abelian group. Let* $1 < p < \infty$ *and* $1 < q < \infty$. *Then* $L_p(G) * L_q(G) \subseteqq L_p(G)$ *if and only if G is compact.*

Before we state our next corollary, we give a simple definition.

DEFINITION 1.5. For real numbers p, q and s such that $1 \leqq p, q, s < \infty$, we shall say that $(p, q;\ s)$ is *admissible* for the locally compact group G if there exists a constant C_{pq} such that

$$\|f * g\|_s \leqq C_{pq}\|f\|_p\|g\|_q$$

for all $f \in L_p(G)$ and all $g \in L_q(G)$.

We note that $(p, q;\ 2)$ is admissible for G if and only if $(1/p, 1/q)$ belongs to the indicator diagram $\Delta(G)$, where $\Delta(G)$ is as defined in Lipsman [9, Section 2]. It is easy to see that Theorem 3 of Lipsman [9] follows from Corollary 1.6 below.

COROLLARY 1.6. *Let G be an infinite locally compact abelian group and let p, q and s be real numbers such that* $1 \leqq p, q, s < \infty$. *Then we have*:

(i) *If G is compact, then* $(p, q;\ s)$ *is admissible if and only if* $1/p + 1/q - 1 \leqq 1/s$.
(ii) *If G is discrete, then* $(p, q;\ s)$ *is admissible if and only if* $1/p + 1/q - 1 \geqq 1/s$.
(iii) *If G is neither compact nor discrete, then* $(p, q;\ s)$ *is admissible if and only if* $1/p + 1/q - 1 = 1/s$.

2. Preliminary Results.

DEFINITION 2.1. Let G denote a locally compact abelian group with Haar measure λ. Let f be a measurable function defined on (G, λ). For $y \geqq 0$, we define

$$m(f, y) = \lambda\{x \in G : |f(x)| > y\}.$$

For $x \geqq 0$, we define

$$\begin{aligned} f^*(x) &= \inf\{y : y > 0 \text{ and } m(f, y) \leqq x\} \\ &= \sup\{y : y > 0 \text{ and } m(f, y) > x\}, \end{aligned}$$

with the conventions $\inf \varnothing = \infty$ and $\sup \varnothing = 0$. For $x > 0$, we define

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)\,dt\,.$$

We also define

$$\|f\|^*_{(p,q)} = \left[\int_0^\infty (x^{1/p}f^*(x))^q \frac{dx}{x}\right]^{1/q}, \qquad (0<p<\infty, 0<q<\infty)$$

$$\|f\|^*_{(p,\infty)} = \sup_{x>0} x^{1/p}f^*(x), \qquad (0<p<\infty)$$

$$L(p,q)(G) = \{f:\ \|f\|^*_{(p,q)}<\infty\}\,.$$

It is quite easy to see that we have

$$\int_0^\infty f^*(x)^p\,dx = \int_G |f(x)|^p\,d\lambda(x)\,,$$

and hence $L_p(G)=L(p,p)(G)$. We shall write $L(p,q)$ instead of $L(p,q)(G)$ when the underlying group G is understood.

If we replace $f^*(x)$ by $f^{**}(x)$ in the definition of $\|f\|^*_{(p,q)}$, the resulting number will be denoted by $\|f\|_{(p,q)}$. For $1<p<\infty$, $0<q\leqq\infty$, it is known (see Yap [15, Lemma 3.2] or O'Neil [10, (6.8)]) that

$$\|f\|^*_{(p,q)} \leqq \|f\|_{(p,q)} \leqq C\|f\|^*_{(p,q)}\,,$$

where C is a constant (depending only on p and q).

In the sequel the symbol C will be used to denote a generic constant, which need not be the same at different occurrences.

The following proposition is taken from Lemma 4.4 and its proof in Hunt [7].

PROPOSITION 2.2. *Let $1<r<\infty$ and $1\leqq q\leqq\infty$. Suppose $f(t)$ is non-negative, locally integrable and an even function of t, $-\infty<t<\infty$. Further, suppose $f(t)$ is non-increasing on $(0,\infty)$ and $f(t)\to 0$ as $t\to\infty$. Then the function $f^\#$, defined by*

$$f^\#(x) = \int_0^\infty f(t)\cos xt\,dt\,,$$

is in $L(r,q)$ if and only if f is in $L(r',q)$. Moreover, there exists a constant C such that

$$|f^\#(x)| \leqq C(1/|x|)f^{**}(1/|x|), \qquad x\neq 0;$$

and

$$\|f^\#\|_{(r,q)} \leqq C\|f\|_{(r',q)}$$

for all $f \in L(r', q)$.

PROPOSITION 2.3. *Let* p, q *be real numbers such that* $1 < p < \infty$, $1 < q < \infty$ *and* $1/p + 1/q > 1$. *Let* H *be a compact subgroup of a locally compact abelian group* G, *and let* η *be the natural homomorphism of* G *onto* G/H. *Suppose that* $f \in L_p(G/H)$ *and* $g \in L_q(G/H)$. *Then* $f \circ \eta \in L_p(G)$ *and* $g \circ \eta \in L_q(G)$. *Moreover, for* $s \geqq 1$, *we have* $f * g \in L_s(G/H)$ *if and only if* $(f \circ \eta) * (g \circ \eta) \in L_s(G)$.

PROOF. Let the Haar measure on H be normalized. Let λ and λ_1 be, respectively, the Haar measures on G and G/H such that Hewitt and Roos [6, (28.54 (iv)] can be applied. It is now easy to see that $f \circ \eta \in L_p(G)$ and $g \circ \eta \in L_q(G)$. By Young's inequality $f * g$ is well-defined on G/H and thus, by Hewitt and Ross [6, (28.55 (iii))], $f * g \in L_s(G/H)$ if and only if $(f * g) \circ \eta \in L_s(G)$.

We now show that $(f * g) \circ \eta = (f \circ \eta) * (g \circ \eta)$ λ-a.e. Since $f \circ \eta \in L_p(G)$ and $g \circ \eta \in L_q(G)$, it follows from the proof of Hewitt and Ross [5, (20.18)] that $(f \circ \eta)(g \circ \eta)_x^\star \in L_1(G)$ for λ-almost all x, where $(g \circ \eta)_x^\star(y) = g(x - y + H)$. It is easy to see that $(f \circ \eta)(g \circ \eta)_x^\star = (f g_{x+H}^\star) \circ \eta$. Thus for λ-almost all x, we have $(f g_{x+H}^\star) \circ \eta \in L_1(G)$. It follows from Hewitt and Ross [6, (28.55 (iii))] that $f g_{x+H}^\star \in L_1(G/H)$. Following Hewitt and Ross [6, p. 96], we have

$$
\begin{aligned}
(f \circ) * (g \circ \eta)(x) &= \int_G (f \circ \eta)(y)(g \circ \eta)(x - y)\, d\lambda(y) \\
&= \int_G (f g_{x+H}^\star) \circ \eta(y)\, d\lambda(y) \\
&= \int_{G/H} (f g_{x+H}^\star)(y + H)\, d\lambda_1(y + H) \\
&\qquad\qquad \text{(by Hewitt and Ross [6, (28.54 (iv))])} \\
&= \int_{G/H} f(y + H) g(x + H - (y + H))\, d\lambda_1(y + H) \\
&= (f * g)(x + H) \\
&= (f * g) \circ \eta(x)\,.
\end{aligned}
$$

Thus $f * g \in L_s(G/H)$ if and only if $(f * g) \circ \eta \in L_s(G)$.

We now gather some basic facts about infinite, compact, 0-dimensional, abelian groups. Let G be such a group for the remainder of this section. By Hewitt and Ross [5, (7.7)] there exists a neighborhood basis $\{G_\alpha\}_{\alpha \in I}$ of the zero element in G consisting of compact open subgroups of G such that $\lim \lambda(G_\alpha)$

$=0$. Let $G_0 = G$ and choose a sequence $\{G_n\}_{n=1}^{\infty}$ from $\{G_\alpha\}_{\alpha\in I}$ such that $\{G_n\}$ is strictly decreasing.

For $n \geqq 0$, let X_n be the annihilator of G_n. By Hewitt and Ross [5, (23.29)], X_n is a finite group. Let m_n be the number of elements in X_n. Since X_n is strictly increasing, we can write

$$X_n = \{\chi_0, \chi_1, \ldots, \chi_{m_n - 1}\}, \quad n = 0, 1, 2, \ldots ,$$

where χ_0 is the identity character of G.

By Hewitt and Ross [5, (23.19)], we have

$$\hat{\xi}_{G_n} = \lambda(G_n)\xi_{X_n}, \tag{1}$$

where ξ_E denotes the characteristic function of E.

By Plancherel's theorem we have

$$(m_n)^{\frac{1}{2}}\lambda(G_n) = \|\hat{\xi}_{G_n}\|_2 = \|\xi_{G_n}\|_2 = \lambda(G_n)^{\frac{1}{2}} ,$$

and so

$$\lambda(G_n) = 1/m_n . \tag{2}$$

Now define D_n on G by

$$D_n(t) = \sum_{i=0}^{n-1} \chi_i(t) .$$

It follows from Hewitt and Ross [5, (23.19)] and (1) that

$$\hat{D}_{m_n} = \xi_{X_n} = \frac{1}{\lambda(G_n)}\hat{\xi}_{G_n} .$$

Since D_{m_n} is a continuous function and G_n is open, we have

$$D_{m_n} = \frac{1}{\lambda(G_n)}\xi_{G_n} .$$

It follows from (2) that

$$D_{m_n}(t) = \begin{cases} m_n & \text{if } t \in G_n , \\ 0 & \text{if } t \notin G_n; \end{cases} \tag{3}$$

and

$$\|D_{m_n}\|_p = (m_n)^{1/p'}, \quad 1 \leqq p \leqq \infty . \tag{4}$$

The following two simple lemmas are stated here for easy reference. We omit the simple proofs.

LEMMA 2.4. *Let* $1<p<\infty$ *and let* m,n *be two positive integers such that* $1<2m \leqq n+1$. *Then*

$$A\cdot n^{p-1} < \sum_{k=m}^{n} k^{p-2} < B\cdot n^{p-1} ,$$

where A *and* B *are constants depending only on* p.

LEMMA 2.5. *Let* m,n *be two positive integers such that* $1\leqq m<n$. *Then* $1/m-1/n<1/m^2+\ldots+1/(n-1)^2$.

The next lemma is similar to Lemma 6.6 in Zygmund [16, Chapter XII]. All notation not explained in this lemma and its proof are as described above.

LEMMA 2.6. *Let* G *be an infinite, compact, 0-dimensional, abelian group. Let* $1<t<\infty$ *and let* $\{a_k\}_{k=1}^{\infty}$ *be a non-increasing sequence of positive numbers tending to zero such that*

(i) $a_{m_n} = a_{m_n+1} = \ldots = a_{m_{n+1}-1}, \quad n=0,1,2,\ldots ,$

and

(ii) $\sum_{k=1}^{\infty} (a_k)^t k^{t-2} < \infty .$

Then the function f, *defined on* G *by* $f=\sum_{k=1}^{\infty} a_k\chi_k$, *is in* $L_t(G)$.

PROOF. For each $k\geqq 1$, let $A_k=a_1+\ldots+a_k$. For $x\in G_n\setminus G_{n+1}$, $n\geqq 1$, we have

$$\begin{aligned} f(x) &= \sum_{k=1}^{m_n-1} a_k\chi_k(x)+\sum_{k=m_n}^{\infty} a_k\chi_k(x) \\ &= \sum_{k=1}^{m_n-1} a_k+\sum_{j=n}^{\infty} a_{m_j}(D_{m_{j+1}}(x)-D_{m_j}(x)) \\ &= \sum_{k=1}^{m_n-1} a_k-m_n a_{m_n} \qquad (\text{since } D_{m_j}(x)=0 \text{ for } j>n) \\ &\leqq A_{m_n} . \end{aligned}$$

Hence we have

$$\int_G |f(x)|^t \, d\lambda(x)$$

$$= \int_{G_0 \setminus G_1} |f(x)|^t \, d\lambda(x) + \sum_{n=1}^{\infty} \int_{G_n \setminus G_{n+1}} |f(x)|^t \, d\lambda(x)$$

$$\leqq A_1^t + \sum_{n=1}^{\infty} \int_{G_n \setminus G_{n+1}} (A_{m_n})^t \, d\lambda(x)$$

$$\leqq A_1^t + \sum_{n=1}^{\infty} (A_{m_n})^t (1/m_n - 1/m_{n+1})$$

$$\leqq A_1^t + \sum_{n=1}^{\infty} (A_{m_n})^t (1/(m_n)^2 + \ldots + 1/(m_{n+1}-1)^2) \qquad \text{(by Lemma 2.5)}$$

$$\leqq A_1^t + \sum_{k=1}^{\infty} (A_k)^t k^{-2}$$

$$< \infty ,$$

where the last inequality follows from an argument similar to that in Zygmund [16, p. 129].

3. Proof of Theorem 1.1, Part (i).

Let G be an infinite compact abelian group. We consider the following two cases.

CASE I. Suppose that G is not 0-dimensional. By Rudin [13, Theorem 2.5.6 (a)], the character group X of G has an element of infinite order. Therefore X contains $\mathbf{Z}$ (the group of integers) as a closed subgroup. Let H be the annihilator of this subgroup. Since H is a closed subgroup of G and G is compact, H is sompact. Moreover, the character group of G/H is isomorphic to $\mathbf{Z}$ and hence G/H is isomorphic to the circle group. By Proposition 2.3, we may assume that G is the circle group.

Define two sequence $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ by

$$a_k = 2^{-(n+1)/p'}(n+1)^{-2/p}, \quad 2^n \leqq k < 2^{n+1}, \quad n=0,1,2,\ldots;$$

$$b_k = 2^{-(n+1)/q'}(n+1)^{-2/q}, \quad 2^n \leqq k < 2^{n+1}, \quad n=0,1,2,\ldots .$$

Define two functions f and g on G by

$$f(x) = \sum_{k=1}^{\infty} a_k e^{ikx}, \quad g(x) = \sum_{k=1}^{\infty} b_k e^{ikx} .$$

By Lemma 2.4, both $\sum_{k=1}^{\infty} (a_k)^p k^{p-2}$ and $\sum_{k=1}^{\infty} (b_k)^q k^{q-2}$ are finite.

Hence, by Zygmund [16, Chapter XII, Lemma 6.6], we have $f \in L_p(G)$ and $g \in L_q(G)$. It is easy to see that

$$(f*g)(x) = \sum_{k=1}^{\infty} a_k b_k e^{ikx}.$$

By Lemma 2.4, $\sum_{k=1}^{\infty} (a_k b_k)^s k^{s-2} = \infty$ for all $s > r$. Hence, by Zygmund [16, Chapter XII, Lemma 6.6], we have $f*g \notin L_s(G)$. Thus $f \in L_p(G)$ and $g \in L_q(G)$, but $f*g \notin L_s(G)$ for all $s > r$.

CASE II. Suppose that G is 0-dimensional. Let $\{\chi_k\}_{k=1}^{\infty}$ and $\{m_n\}_{n=0}^{\infty}$ be as in Section 2. Define two sequences $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ by

$$a_k = (m_{n+1})^{-1/p'}(n+1)^{-2/p}, \quad m_n \leqq k < m_{n+1}, \quad n = 0, 1, 2, \ldots;$$

$$b_k = (m_{n+1})^{-1/q'}(n+1)^{-2/q}, \quad m_n \leqq k < m_{n+1}, \quad n = 0, 1, 2, \ldots.$$

Define two functions f and g on G by

$$f = \sum_{k=1}^{\infty} a_k \chi_k, \quad g = \sum_{k=1}^{\infty} b_k \chi_k.$$

We have, by Lemma 2.4, $\sum_{k=1}^{\infty} (a_k)^p k^{p-2} < \infty$ and $\sum_{k=1}^{\infty} (b_k)^q k^{q-2} < \infty$. Hence by Lemma 2.6, we have $f \in L_p(G)$ and $g \in L_q(G)$. It is easy to see that

$$f*g = \sum_{k=1}^{\infty} a_k b_k \chi_k.$$

Let $\{G_n\}_{n=0}^{\infty}$ and $\{D_{m_n}\}_{n=0}^{\infty}$ be as in Section 2. For $x \in G_n \setminus G_{n+1}$, where $n \geqq 1$, we have

$$\begin{aligned}
(f*g)(x) &= \sum_{k=1}^{\infty} a_k b_k \chi_k(x) \\
&= \sum_{k=1}^{m_n - 1} a_k b_k + \sum_{j=n}^{\infty} a_{m_j} b_{m_j} (D_{m_{j+1}}(x) - D_{m_j}(x)) \\
&= \sum_{k=1}^{m_n - 1} a_k b_k - (a_{m_n} b_{m_n})(m_n) \qquad (\text{since } D_{m_j}(x) = 0 \text{ for } j > n) \\
&\geqq (m_n - 1)(m_n)^{-1/p' - 1/q'}(n)^{-2/p - 2/q} - m_n (m_{n+1})^{-1/p' - 1/q'}(n+1)^{-2/p - 2/q} \\
&= (m_n)^{1 - 1/p' - 1/q'}(n)^{-2/p - 2/q}\left[\frac{m_n - 1}{m_n} - \frac{(m_n)^{1/p' + 1/q'}(n)^{2/p + 2/q}}{(m_{n+1})^{1/p' + 1/q'}(n+1)^{2/p + 2/q}}\right] \\
&\geqq C(m_n)^{1 - 1/p' - 1/q'}(n)^{-2/p - 2/q} \qquad (\text{since } 2m_n \leqq m_{n+1}) \\
&= C(m_n)^{1/r}(n)^{-2/p - 2/q}.
\end{aligned}$$

Hence for all $s > r$, we have

$$\int_G |(f*g)(x)|^s \, d\lambda(x)$$

$$= \int_{G_0 \setminus G_1} |(f*g)(x)|^s \, d\lambda(x) + \sum_{n=1}^{\infty} \int_{G_n \setminus G_{n+1}} |(f*g)(x)|^s \, d\lambda(x)$$

$$\geqq C \sum_{n=1}^{\infty} \int_{G_n \setminus G_{n+1}} (m_n)^{s/r} (n)^{-2s/p - 2s/q} \, d\lambda(x)$$

$$= C \sum_{n=1}^{\infty} (m_n)^{s/r} (1/m_n - 1/m_{n+1}) (n)^{-2s/p - 2s/q}$$

$$\geqq C \sum_{n=1}^{\infty} (m_n)^{s/r - 1} (n)^{-2s/p - 2s/q} \qquad (\text{since } 2m_n \leqq m_{n+1})$$

$$= \infty \qquad (\text{since } m_n \geqq 2^n \text{ and } s > r) .$$

Thus $f \in L_p(G)$ and $g \in L_q(G)$, but $f*g \notin L_s(G)$ for all $s > r$.

4. Proof of Theorem 1.1, Part (ii).

Let G be an infinite discrete abelian group. Applying Rudin [13, Theorem 2.5.6 (a)] to the character group of G, we know that either G has an element of infinite order or its character group is a 0-dimensional compact abelian group. Thus it is sufficient to consider the following two cases.

CASE I. Suppose that G has an element a of infinite order. Define f and g on G by

$$f(x) = \begin{cases} (n+1)^{-1/p} (\log (n+2))^{-2/p} & \text{if } x = na, \ n = 0, 1, 2, \ldots , \\ 0 & \text{otherwise;} \end{cases}$$

$$g(x) = \begin{cases} (n+1)^{-1/q} (\log (n+2))^{-2/q} & \text{if } x = na, \ n = 0, 1, 2, \ldots , \\ 0 & \text{otherwise .} \end{cases}$$

Clearly $f \in l_p(G)$ and $g \in l_q(G)$. For each $k \geqq 0$, we have

$$\begin{aligned} (f*g)(ka) &= \sum_{n=0}^{k} f(na) g((k-n)a) \\ &= \sum_{n=0}^{k} \frac{1}{(n+1)^{1/p} (k-n+1)^{1/q} (\log (n+2))^{2/p} (\log (k-n+2))^{2/q}} \\ &\geqq \sum_{n=0}^{k} \frac{1}{(k+1)^{1/p} (k+1)^{1/q} (\log (k+2))^{2/p} (\log (k+2))^{2/q}} \\ &= (k+1)^{1 - 1/p - 1/q} (\log (k+2))^{-2/p - 2/q} \\ &= (k+1)^{-1/r} (\log (k+2))^{-2/p - 2/q} . \end{aligned}$$

It is now easy to see that $f*g \notin l_s(G)$ for all $s<r$. Thus $f \in l_p(G)$ and $g \in l_q(G)$, but $f*g \notin l_s(G)$ for all $s<r$.

CASE II. Suppose that the character group of G is a 0-dimensional group. It follows from Section 2 that there exist a strictly increasing sequence $\{m_n\}_{n=0}^{\infty}$ of positive integers with $m_0=1$ and a sequence $\{x_n\}_{n=0}^{\infty}$ in G such that x_0 is the zero element in G and $G_n \equiv \{x_0,\dots,x_{m_n-1}\}$ is a subgroup of G.

Define f and g on G by

$$f(x) = \begin{cases} (m_{n+1})^{-1/p}(n+1)^{-2/p} & \text{if } x \in G_{n+1} \setminus G_n,\ n=0,1,2,\dots, \\ 0 & \text{otherwise;} \end{cases}$$

$$g(x) = \begin{cases} (m_{n+1})^{-1/q}(n+1)^{-2/q} & \text{if } x \in G_{n+1} \setminus G_n,\ n=0,1,2,\dots, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f \in l_p(G)$ and $g \in l_q(G)$. For positive integers j such that $m_n \leqq j < m_{n+1}$, we have

$$\begin{aligned}
(f*g)(x_j) &= \sum_{k=0}^{\infty} f(x_k)g(x_j-x_k) \\
&\geqq \sum_{k=m_n}^{m_{n+1}-1} f(x_k)g(x_j-x_k) \\
&= \sum_{k=m_n}^{m_{n+1}-1} (m_{n+1})^{-1/p}(n+1)^{-2/p}(m_{n+1})^{-1/q}(n+1)^{-2/q} \\
&\qquad (\text{since } x_j \in G_{n+1},\ x_j-x_k \in G_{n+1} \text{ if } x_k \in G_{n+1}\setminus G_n) \\
&= (m_{n+1}-m_n)(m_{n+1})^{-1/p-1/q}(n+1)^{-2/p-2/q} \\
&\geqq (1/2)(m_{n+1})^{1-1/p-1/q}(n+1)^{-2/p-2/q} \qquad (\text{since } 2m_n \leqq m_{n+1}) \\
&= (1/2)(m_{n+1})^{-1/r}(n+1)^{-2/p-2/q}.
\end{aligned}$$

Now for $s<r$, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} |(f*g)(x_k)|^s &= \sum_{n=0}^{\infty} \sum_{j=m_n}^{m_{n+1}-1} |(f*g)(x_j)|^s \\
&\geqq C \sum_{n=0}^{\infty} (m_{n+1}-m_n)(m_{n+1})^{-s/r}(n+1)^{-2s/p-2s/q} \\
&\geqq C \sum_{n=0}^{\infty} (m_{n+1})^{1-s/r}(n+1)^{-2s/p-2s/q}
\end{aligned}$$

$$\geqq C \sum_{n=0}^{\infty} 2^{(n+1)(1-s/r)}(n+1)^{-2s/p-2s/q} \qquad (\text{since } 2^n \leqq m_n)$$

$$= \infty \qquad (\text{since } s<r)\,.$$

Thus $f \in l_p(G)$ and $g \in l_q(G)$, but $f*g \notin l_s(G)$ for all $s<r$.

5. Proof of Theorem 1.1, Part (iii).

Theorem 1.1 (iii) will be deduced from a series of lemmas and the structure theorem for locally compact abelian groups. We begin with two important lemmas concerning the real line R.

LEMMA 5.1. *Let p, q, and r be as in Young's inequality. Then*

$$L_p(\mathsf{R})*L_q(\mathsf{R}) \nsubseteq \bigcup \{L_s(\mathsf{R}) : s<r\}\,.$$

PROOF. Let $\beta=1/p+1/q$ and let n_0 be a positive integer such that $n_0 > \max\{e^{p'\beta}, e^{q'\beta}\}$. For $k \geqq n_0$, we define

$$U_k = [-1/k, -1/(k+1)) \cup (1/(k+1), 1/k];$$
$$a_k = (k+1)^{1/p'}(\log(k+1))^{-\beta}\,,$$
$$b_k = (k+1)^{1/q'}(\log(k+1))^{-\beta}\,.$$

Define f and h on R by

$$f = \sum_{k=n_0}^{\infty} a_k \xi_{U_k}, \qquad h = \sum_{k=n_0}^{\infty} b_k \xi_{U_k}\,.$$

By a calculation similar to that in the proof of Yap [14, Theorem 2.7], we have $f \in L_1(\mathsf{R}) \cap L(p',p)(\mathsf{R})$ and $h \in L_1(\mathsf{R}) \cap L(q',q)(\mathsf{R})$.

For $n \geqq n_0$ we define

$$f_n = \sum_{k=n_0}^{n} a_k \xi_{U_k}, \qquad h_n = \sum_{k=n_0}^{n} b_k \xi_{U_k}\,.$$

Clearly $\|f-f_n\|_{(p',p)} \to 0$ and $\|h-h_n\|_{(q',q)} \to 0$.

Let f^* be defined as in Proposition 2.2. Then, since $f, f_n \in L_1(\mathsf{R}) \cap L(p',p)(\mathsf{R})$ and $h, h_n \in L_1(\mathsf{R}) \cap L(q',q)(\mathsf{R})$, it follows from Proposition 2.2 that $f^*, f_n^* \in L_p(\mathsf{R}) \cap L_\infty(\mathsf{R})$, $h^*, h_n^* \in L_q(\mathsf{R}) \cap L_\infty(\mathsf{R})$, and there exists a constant C such that

$$\|f^*-f_n^*\|_p \leqq C\|f-f_n\|_{(p',p)} \to 0$$

and

$$\|h^{\#}-h_n^{\#}\|_q \leqq C\|h-h_n\|_{(q',q)} \to 0 .$$

Now, since $\{\|h_n^{\#}\|_q\}$ is bounded,

$$(5) \qquad \|f^{\#}*h^{\#}-f_n^{\#}*h_n^{\#}\|_r \leqq \|f^{\#}\|_p\|h^{\#}-h_n^{\#}\|_q+\|f^{\#}-f_n^{\#}\|_p\|h_n^{\#}\|_q \to 0 .$$

Hence, without loss of generality, we may assume that $f_n^{\#}*h_n^{\#} \to f^{\#}*h^{\#}$ a.e. Since $f_nh_n \to fh$ and $fh \in L_1(\mathrm{R})$, we have $(f_nh_n)^{\#} \to (fh)^{\#}$ a.e. Since $f_n, h_n \in L_2(\mathrm{R})$ and $2f_n^{\#}=\hat{f}_n$ and $2h_n^{\#}=\hat{h}_n$, we have $4(f^{\#}*h^{\#})=\hat{f}_n*\hat{h}_n=(f_nh_n)\hat{}$ $=2(f_nh_n)^{\#}$. Thus $2(f^{\#}*h^{\#})=(fh)^{\#}$ a.e. Now we have

$$\begin{aligned}(fh)(x) &= \sum_{k=n_0}^{\infty} (k+1)^{1/p'+1/q'}(\log(k+1))^{-2\beta}\xi_{U_k}(x)\\ &= \sum_{k=n_0}^{\infty} (k+1)^{1/r'}(\log(k+1))^{-2\beta}\xi_{U_k}(x) .\end{aligned}$$

By a calculation similar to that in the proof of Yap [14, Theorem 2.7], we have $fh \notin L(s',s)(\mathrm{R})$ for all $s<r$. Hence, by Proposition 2.2, we have $(fh)^{\#} \notin L_s(\mathrm{R})$. Thus $f^{\#}*h^{\#} \notin L_s(\mathrm{R})$ for all $s<r$.

LEMMA 5.2. *Let p, q, r be as in Young's inequality. Then*

$$L_p(\mathrm{R})*L_q(\mathrm{R}) \nsubseteq \bigcup\{L_s(\mathrm{R}) : r<s\} .$$

PROOF. Let $\beta=1/p+1/q$. For $k\geqq 0$, we define

$$\begin{aligned}V_k &= [-k-1,-k) \cup (k,k+1] ,\\ a_k &= (k+1)^{-1/p'}(\log(k+2))^{-\beta} ,\\ b_k &= (k+1)^{-1/q'}(\log(k+2))^{-\beta} .\end{aligned}$$

Define f and h on R by

$$f(x) = \sum_{k=0}^{\infty} a_k\xi_{V_k}(x), \qquad h(x) = \sum_{k=0}^{\infty} b_k\xi_{V_k}(x) .$$

By a calculation similar to that in the proof of Yap [14, Theorem 2.7], we have $f \in L(p',p)(\mathrm{R}) \cap L_\infty(\mathrm{R})$ and $h \in L(q',q)(\mathrm{R}) \cap L_\infty(\mathrm{R})$. By Proposition 2.2, we have $f^{\#} \in L_{p_2}(\mathrm{R})$ for all $p_2 \in (1,p]$, and $h^{\#} \in L_{q_2}(\mathrm{R})$ for all $q_2 \in (1,q]$.

For $n\geqq 0$, define

$$f_n(x) = \sum_{k=0}^{n} a_k\xi_{V_k}(x), \qquad h_n(x) = \sum_{k=0}^{n} b_k\xi_{V_k}(x) .$$

Since

$$f_n^\#(x) = \int_0^\infty f_n(t)\cos xt\,dt = \int_0^{n+1} f(t)\cos xt\,dt\,,$$

we have $f_n^\# \to f^\#$ a.e. Similarly, we have $h_n^\# \to h^\#$ a.e. and $(f_nh_n)^\# \to (fh)^\#$ a.e.

We now show that $f_n^\# \in L_p(\mathrm{R})$ and $\|f_n^\# - f^\#\|_p \to 0$. By Proposition 2.2, there exists a constant C such that

$$|f_n^\#(x)| \leqq C(1/|x|)f^{**}(1/|x|)$$

for $x \neq 0$. Now $f_n^\#$ is an even function and so

$$\begin{aligned}\int_{-\infty}^{\infty} |f_n^\#(x)|^p\,dx &= 2\int_0^\infty |f_n^\#(x)|^p\,dx \\ &\leqq C\int_0^\infty \left(\frac{1}{x}f^{**}\left(\frac{1}{x}\right)\right)^p dx \\ &= C\int_0^\infty (t^{1/p'}f^{**}(t))^p\frac{dt}{t} \\ &< \infty \qquad (\text{since } f \in L(p',p)(\mathrm{R}))\,.\end{aligned}$$

Thus $f_n^\# \in L_p(\mathrm{R})$ and $C(1/|x|)\,f^{**}(1/|x|) \in L_p(\mathrm{R})$. By Lebesgue's dominated convergence theorem, we have $\|f_n^\#\|_p \to \|f^\#\|_p$. But this and the fact that $f_n^\# \to f^\#$ a.e. imply that $\|f_n^\# - f^\#\|_p \to 0$. Similarly, $\|h_n^\# - h^\#\|_q \to 0$.

Next we show that $f^\# * h^\# \notin L_s(\mathrm{R})$ for all $s > r$. Arguing as in (5) of Lemma 5.1, we have $\|f_n^\# * h_n^\# - f^\# * h^\#\|_r \to 0$. Thus, without loss of generality, we may assume that $f_n^\# * h_n^\# \to f^\# * h^\#$ a.e. Since $f_n, h_n \in L_2(\mathrm{R})$, $2f_n^\# = \hat{f}_n$, $2h_n^\# = \hat{h}_n$, and $2(f_nh_n)^\# = (f_nh_n)\hat{}$, we have

$$4(f_n^\# * h_n^\#) = \hat{f}_n * \hat{h}_n = (f_nh_n)\hat{} = 2(f_nh_n)^\#\,.$$

But $(f_nh_n)^\# \to (fh)^\#$ a.e., and so $2(f^\# * h^\#) = (fh)^\#$ a.e. Since

$$fh = \sum_{k=0}^\infty (k+1)^{-1/p'-1/q'}(\log(k+2))^{-2\beta}\xi_{V_k}\,,$$

it follows that $fh \notin L(s',s)(\mathrm{R})$ for all $s > r$ (the calculation is similar to that in the proof of Yap [14, Theorem 2.7]). By Proposition 2.2, $(fh)^\# \notin L_s(\mathrm{R})$ for all $s > r$. Hence $f^\# * h^\# \notin L_s(\mathrm{R})$ for all $s > r$.

REMARK 5.3. (i) The proof of Lemma 5.1 shows that there exist functions φ_1 and ψ_1 with $\varphi_1 \in L_{p_1}(\mathrm{R})$ for all $p_1 \in [p, \infty]$ and $\psi_1 \in L_{q_1}(\mathrm{R})$ for all $q_1 \in [q, \infty]$, and $\varphi_1 * \psi_1 \notin L_t(\mathrm{R})$ for all $t < r$.

(ii) The proof of Lemma 5.2 shows that there exist functions φ_2 and ψ_2 with

$\varphi_2 \in L_{p_2}(\mathrm{R})$ for all $p_2 \in (1,p]$ and $\psi_2 \in L_{q_2}(\mathrm{R})$ for all $q_2 \in (1,q]$, and $\varphi_2 * \psi_2 \notin L_t(\mathrm{R})$ for all $t>r$.

LEMMA 5.4. *Let p, q, and r be as in Young's inequality. Then*

$$L_p(\mathrm{R}) * L_q(\mathrm{R}) \nsubseteq \bigcup \{L_s(\mathrm{R}) : s \neq r\}.$$

PROOF. Let $\varphi_1, \psi_1; \varphi_2$ and ψ_2 have the properties stated in Remark 5.3. Then $\varphi_1 * \psi_1 \notin L_t(\mathrm{R})$ for all $t<r$ and $\varphi_2 * \psi_2 \notin L_t(\mathrm{R})$ for all $t>r$. By Young's inequality we can find a number ε such that $0<\varepsilon<r$, and $\varphi_1 * \psi_2$, $\varphi_2 * \psi_1 \in L_{r-\varepsilon}(\mathrm{R}) \cap L_{r+\varepsilon}(\mathrm{R})$. It is now easy to see that

$$(\varphi_1+\varphi_2) * (\psi_1+\psi_2) \notin L_t(\mathrm{R}) \tag{6}$$

for all $t \in [r-\varepsilon, r) \cup (r, r+\varepsilon]$. Next we suppose that there exists $s \in (-\infty, r-\varepsilon) \cup (r+\varepsilon, \infty)$ such that $(\varphi_1+\varphi_2) * (\psi_1+\psi_2) \in L_s(\mathrm{R})$. By Young's inequality we have $(\varphi_1+\varphi_2) * (\psi_1+\psi_2) \in L_r(\mathrm{R})$. Hence, by applying Hölder's inequality, we have $(\varphi_1+\varphi_2) * (\psi_1+\psi_2) \in L_t(\mathrm{R})$ for all $t \in [r-\varepsilon, r) \cup (r, r+\varepsilon]$. But this contradicts (6). Hence $(\varphi_1+\varphi_2) * (\psi_1+\psi_2) \notin L_s(\mathrm{R})$ for all $s \neq r$.

PROOF OF THEOREM 1.1 (iii). Suppose that G is neither compact nor discrete. By the structure theorem for locally compact abelian groups (see Rudin [13, Theorem 2.4.1]), G has an open subgroup $\mathrm{R}^n \times F$ where $n \geqq 0$ and F is a compact abelian group. We consider the following two cases.

CASE I. Suppose that $n>0$. Let $\varphi_1, \psi_1; \varphi_2$ and ψ_2 have the properties stated in Remark 5.3. Define ϱ on $\mathrm{R}^{n-1} \times F$ by

$$\varrho(y) = \xi_{[0,1]^{n-1} \times F}(y).$$

Let $\varphi = \varphi_1 + \varphi_2$ and $\psi = \psi_1 + \psi_2$. Define f and g on $\mathrm{R}^n \times F$ by

$$f(x,y) = \varphi(x)\varrho(y), \qquad (x \in \mathrm{R},\ y \in \mathrm{R}^{n-1} \times F)$$

$$g(x,y) = \psi(x)\varrho(y), \qquad (x \in \mathrm{R},\ y \in \mathrm{R}^{n-1} \times F).$$

Then $f \in L_p(\mathrm{R}^n \times F)$, $g \in L_q(\mathrm{R}^n \times F)$. By Fubini's theorem,

$$(f*g)(x,y) = (\varphi * t)(x)(\varrho * \varrho)(y).$$

By Lemma 5.4, $\varphi * t \notin L_s(\mathrm{R})$ for all $s \neq r$. Since $\varrho * \varrho$ has a compact support of positive measure, it follows that $f*g \notin L_s(\mathrm{R}^n \times F)$ for all $s \neq r$. Now define f_0 and g_0 on G by

$$f_0(z) = \begin{cases} f(z) & \text{if } z \in \mathrm{R}^n \times F, \\ 0 & \text{otherwise;} \end{cases}$$

$$g_0(z) = \begin{cases} g(z) & \text{if } z \in \mathbf{R}^n \times F, \\ 0 & \text{otherwise}. \end{cases}$$

Since $\mathbf{R}^n \times F$ is an open subgroup of G, we have $f_0 \in L_p(G)$ and $g_0 \in L_q(G)$, but $f_0 * g_0 \notin L_s(G)$ for all $s \neq r$.

CASE II. Suppose that $n=0$. Since F is compact, there exist functions φ and ψ on F with $\varphi \in L_p(F)$, $\psi \in L_q(F)$ and $\varphi * \psi \notin L_s(F)$ for all $s>r$. Define φ_0 and ψ_0 on G by

$$\varphi_0 = \varphi \cdot \xi_F, \quad \psi_0 = \psi \cdot \xi_F .$$

Since F is a compact open subgroup of G, it follows that $\varphi_0 \in L_{p_0}(G)$ for all $p_0 \in [1,p]$, $\psi_0 \in L_{q_0}(G)$ for all $q_0 \in [1,q]$ and $\varphi_0 * \psi_0 \notin L_s(G)$ for all $s>r$.

Since G is non-compact and F is a compact open subgroup of G, G/F is an infinite discrete group. Thus there exist functions φ_1 and ψ_1 on G/F with $\varphi_1 \in l_p(G/F)$ and $\psi_1 \in l_q(G/F)$, but $\varphi_1 * \psi_1 \notin l_s(G/F)$ for all $s<r$. Let η be the natural homomorphism of G onto G/F. Then, by Proposition 2.3, $\varphi_1 \circ \eta \in L_{p_1}(G)$ for all $p_1 \in [p,\infty)$, $\psi_1 \circ \eta \in L_{q_1}(G)$ for all $q_1 \in [q,\infty)$, but $(\varphi_1 \circ \eta) * (\psi_1 \circ \eta) \notin L_s(G)$ for all $s<r$.

Now let $f=\varphi_0+\varphi_1 \circ \eta$ and $g=\psi_0+\psi_1 \circ \eta$. By an argument similar to that in the proof of Lemma 5.4 we have $f*g \notin L_s(G)$ for all $s \neq r$. This completes the proof of Theorem 1.1 (iii).

6. Proofs of Theorem 1.2 and of Corollaries 1.4 and 1.6.

As noted in Section 1, the proof of Theorem 1.1 can be easily adjusted to give us a proof of Theorem 1.2.

PROOF OF COROLLARY 1.4. Let $1<p<\infty$ and $1<q<\infty$. If G is compact, then clearly $L_p(G)*L_q(G) \subseteq L_p(G)$. Conversely, let G be non-compact. We consider the three cases

$$(1)\ 1/p+1/q > 1, \quad (2)\ 1/p+1/q = 1, \quad (3)\ 1/p+1/q < 1$$

and observe that case (j), $j=1,2,3$, follows from Theorem 1.j immediately.

PROOF OF COROLLARY 1.6. If $p>1$, $q>1$ and $1/p+1/q>1$, then the assertions in (i)–(iii) follow immediately from Theorem 1.1 and Young's inequality.

If $p=1$ and $q \geqq 1$, then the assertions in (i)–(iii) follow from Young's inequality and the fact that $L_1(G)*L_q(G)=L_q(G)$. (Note that for G compact, $L_q(G) \mid\subseteq L_s(G)$ if and only if $s>q$; for G discrete, $l_q(G) \nsubseteq l_s(G)$ if and only if $s<q$; for G neither compact nor discrete, $L_q(G) \nsubseteq L_s(G)$ if $q \neq s$.)

Next we consider the remaining possibility (i.e., when $1/p+1/q\leqq 1$) for each of the cases (i)–(iii).

(i) Suppose that $1/p+1/q\leqq 1$ and G is compact. Then $(p,q;s)$ is admissible for all $s\geqq 1$, and the condition $1/s\geqq 1/p+1/q-1$ is satisfied for all $s\geqq 1$.

(ii) Suppose that $1/p+1/q\leqq 1$ and G is discrete. Then, by Theorems 1.2 and 1.3, $(p,q;\ s)$ is not admissible for any $s\geqq 1$, and the condition $1/s\leqq 1/p+1/q-1$ is not satisfied by any $s\geqq 1$.

(iii) Suppose that $1/p+1/q\leqq 1$ and G is neither compact nor discrete, then the assertion follows as in case (ii).

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