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FUNCTIONS WITH H^p HYPERBOLIC DERIVATIVE

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1. Introduction.

Let B be the family of functions f holomorphic and bounded, $|f|<1$, in the disk $D=\{|z|<1\}$. We shall consider properties of f of B under some conditions on the hyperbolic derivative $f^{\#}=|f'|/(1-|f|^2)$ in terms of harmonic majorants.

The disk D is endowed with the non-Euclidean hyperbolic distance

$$\sigma(z,w) = \tanh^{-1}(|z-w|/|1-\bar{z}w|), \qquad z,w \in D;$$

we denote $\sigma(z)=\sigma(z,0)$, the hyperbolic counterpart of $|z|$. For $f \in B$ and for $0<p<\infty$ the functions $f^{\#p}=\exp(p\log f^{\#})$ and $\sigma(f)^p=\exp(p\log\sigma(f))$ both are subharmonic in D because the same is true of $\log f^{\#}$ and $\log\sigma(f)$.

A subharmonic function u in D is said to have a harmonic majorant h in D if h is harmonic and $u \leqq h$ in D. This is the case if and only if

$$\sup_{0\leqq r<1} \int_T u(re^{it})\,dt < \infty, \qquad T = [0,2\pi],$$

see [6, p. 26]. The (parabolic) Hardy class H^p $(0<p<\infty)$ consists of holomorphic functions f in D such that $|f|^p$ have harmonic majorants; the class H^∞ consists of all bounded and holomorphic functions in D. Analogously, the hyperbolic Hardy class H^p_σ $(0<p<\infty)$ consists of $f \in B$ such that $\sigma(f)^p$ has a harmonic majorant in D, while H^∞_σ consists of $f \in H^\infty$ bounded by a constant strictly less than one, or, $\sup\{\sigma(f)(z);\ z \in D\}<\infty$.

We shall prove the hyperbolic versions of the following (A) and (B).

(A) *A function f holomorphic in D is continuous on $D \cup \Gamma$, where $\Gamma=\{|z|=1\}$, and absolutely continuous on Γ if and only if $f' \in H^1$* [1, Theorem 3.11, p. 42].

(B) *If $f' \in H^p$ for some $p<1$, then $f \in H^q$ with $q=p/(1-p)$* [1, Theorem 5.12, p. 88]; [3, Theorem 33 with $\alpha=1$, p. 415].

It is well known that the converse of (B) is false [1, p. 92].

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THEOREM 1. *A function $f \in B$ is continuous on $D \cup \Gamma$ and hyperbolically absolutely continuous on Γ if and only if $f^\#$ has a harmonic majorant in D.*

More precisely, the part of Theorem 1 before "if and only if" means that f is continuous on $D \cup \Gamma$, $f \in H_\sigma^\infty$, and further, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{j=1}^{n} \sigma(f(\zeta_{2j-1}), f(\zeta_{2j})) < \varepsilon ,$$

provided that $(\zeta_{2j-1}\zeta_{2j})^\frown$ $(1 \leqq j \leqq n)$ are non-overlapping open subarcs on Γ with

$$\sum_{j=1}^{n} |\arg(\zeta_{2j}\zeta_{2j-1}^{-1})| < \delta .$$

THEOREM 2. *If $f \in B$ and if $f^{\#p}$, for some p, $0 < p < 1$, has a harmonic majorant in D, then $f \in H_\sigma^q$ with $q = p/(1-p)$.*

Again the converse is not true. Although q in (B) is sharp (see [1, p. 90]), the sharpness of q in Theorem 2 remains open.

Let PL be the family of functions $u \geqq 0$ in D such that $\log u$ is subharmonic in D; the notation PL is due to E. F. Beckenbach and T. Radó; see [7, p. 9]. Each $u \in$ PL is subharmonic in D, and further $u^\alpha \in$ PL for all α, $0 < \alpha < \infty$. Let PL^p be the family of $u \in$ PL such that u^p has a harmonic majorant in D $(0 < p < \infty)$. If f is holomorphic in D, then $|f| \in$ PL, while if $f \in B$, then $\sigma(f)$ is in PL. Therefore, for f holomorphic in D to belong to H^p it is necessary and sufficient that $|f| \in \mathrm{PL}^p$, while for $f \in B$ to belong to H_σ^p it is necessary and sufficient that $\sigma(f) \in \mathrm{PL}^p$ $(0 < p < \infty)$. We shall make use of the following

LEMMA 1. *Suppose that $u \in \mathrm{PL}^p$ $(0 < p < \infty)$. Then there exists a zero-free $f \in H^p$ such that $u \leqq |f|$ in D, and further that $u^*(t) = |f^*(t)|$ for* a.e. $t \in T$.

Here and elsewhere $g^*(t)$ means the radial limit at e^{it} of Γ of the function g considered. The function f is called a Hardy majorant of u. It is apparent that u^* is then of $L^p(T)$.

2. Proof of Lemma 1.

Suppose for the moment that Lemma 1 is valid for $p = 1$, and let $u \in \mathrm{PL}^p$ $(0 < p < \infty)$. Since $u^p \in \mathrm{PL}^1$, there exists a Hardy majorant $g \in H^1$ of u^p. Since g has no zero in D, we may consider a branch f of $g^{1/p}$ in D. Then $f \in H^p$ is a Hardy majorant of u.

To prove Lemma 1 for $p=1$ we set $v=\log u$ for $u \in \mathrm{PL}^1$, and we set $\varphi(x)=e^x$, $-\infty \leqq x < +\infty$. Then $\varphi(v)=u$ has a harmonic majorant in D. By a theorem of E. D. Solomentsev [8] (see [2] also), the least harmonic majorant $v\hat{}$ of v exists, and is expressed by the Poisson integral,

$$v\hat{}(z) = (1/2\pi)\int_T (1-|z|^2)|e^{it}-z|^{-2}\,d\mu(t) \qquad (z \in D)$$

of the signed measure

$$d\mu(t) = v^*(t)dt + d\mu_S(t) \qquad \text{on } T,$$

where $d\mu_S(t) \leqq 0$ on T (and $d\mu_S(t)$ is singular with respect to dt). Furthermore, $\varphi(v^*)=u^* \in L^1(T)$ and $v^* \in L^1(T)$.

Letting h be the Poisson integral of the function v^* on T, one observes the inequality $v \leqq h$ in D. Let $f=e^{h+ik}$, where k is a conjugate of h in D. The Jensen inequality then reads $|f|=e^h \leqq U$, where U is the Poisson integral of $\varphi(v^*)=u^*$. Therefore, $f \in H^1$ and $u=e^v \leqq e^h=|f|$ with $e^{h^*}=|f^*|=e^{v^*}=u^*$, or, f is a Hardy majorant of u in D.

3. Proof of Theorem 1.

We may suppose that f is nonconstant. To prove the "if" part we first notice that $f' \in H^1$ because $|f'| \leqq f^\#$ and $f^\# \in \mathrm{PL}^1$. By (A) f is then continuous on $D \cup \Gamma$ and absolutely continuous on Γ. It now suffices to show that

$$r = \max\{|f(e^{it})|;\ t \in T\} < 1 .$$

In effect, the hyperbolic absolute continuity of f on Γ then follows from the inequality

$$\sigma(w_1, w_2) \leqq K|w_1 - w_2|, \qquad |w_j| \leqq r,\ j=1,2 ,$$

where $K>0$ is a constant, say,

$$K = (1+r^2)(1-r^2)^{-1}(2r)^{-1}\log[(1+r)/(1-r)] .$$

We now set

$$A = \sup_{0 \leqq a < 1} \int_T f^\#(ae^{it})\,dt < \infty . \tag{3.1}$$

For each fixed $z \neq 0$ of D we consider the function $u(w)=f^\#(zw)$ of $w \in D$. Since $u \in \mathrm{PL}^1$, a Hardy majorant g of u exists, where

$$|g^*(t)| = u^*(t) = f^\#(ze^{it}) \qquad \text{for a.e. } t \in T.$$

By the theorem of L. Fejér and F. Riesz (see [1, p. 46]), together with (3.1), one observes that

$$\int_{-1}^{1} |g(x)|\,dx \leqq (1/2)\int_T |g^*(t)|\,dt = (1/2)\int_T f^\#(ze^{it})\,dt \leqq A/2\,.$$

Therefore, setting $\zeta = zx$ for $0 \leqq x \leqq 1$, one obtains the following chain of inequalities:

$$\sigma(f(z), f(0)) \leqq \int_0^z f^\#(\zeta)\,|d\zeta| = \int_0^1 f^\#(zx)|z|\,dx$$
$$\leqq \int_0^1 u(x)\,dx \leqq \int_{-1}^{1} |g(x)|\,dx \leqq A/2\,.$$

Since z is arbitrary, the proof of the "if" part is herewith complete.

The "only if" part is immediate. Since $|z-w| \leqq \sigma(z,w)$ for $z, w \in D$, it follows that f is absolutely continuous on Γ. It then follows from (A) that $f' \in H^1$. Since $f \in H_\sigma^\infty$, we have

$$r = \max\{|f(z)|\ ;\ z \in D \cup \Gamma\} < 1\,.$$

Therefore it follows from $f^\# \leqq |f'|/(1-r^2)$ that $f^\# \in \mathrm{PL}^1$.

4. A lemma.

In the proof of Theorem 2 in Section 5 we shall make use of the following

LEMMA 2. *Let $u \in \mathrm{PL}^p$ $(0<p<\infty)$, and set*

$$U(t) = \sup\{u(re^{it})\ ;\ 0 \leqq r < 1\}, \qquad t \in T\,.$$

Then

$$\int_T U(t)^p\,dt \leqq C \int_T u^*(t)^p\,dt\,,$$

where $C>0$ is a constant independent of u.

This maximal theorem for PL^p is a consequence of the celebrated G. H. Hardy and J. E. Littlewood maximal theorem (see [1, p. 12]) applied to a Hardy majorant $f \in H^p$ of u. Since

$$U(t) \leqq \sup\{|f(re^{it})|;\ 0 \leqq r < 1\}, \qquad t \in T,$$

the inequality in Lemma 2 follows.

An obvious application of Lemma 2 is the maximal theorem for $f \in H^p_\sigma$ $(0<p<\infty)$ on considering $u=\sigma(f)$. Namely,

$$\int_T U_\sigma(t)^p\, dt \leqq C \int_T \sigma(f(t))^p\, dt ,$$

where

$$U_\sigma(t) = \sup\{\sigma(f^*)(re^{it});\ 0 \leqq r<1\} .$$

A merit of Lemma 2 is the estimate of $U(t)$ in the case $p<1$.

5. Proof of Theorem 2.

We may assume that $f(0)=0$. Otherwise we consider

$$g = [f-f(0)]/[1-\overline{f(0)}f]$$

for which $g^{\#}=f^{\#}$ and

$$|\sigma(f)-\sigma(g)| \leqq \sigma(f(0)) ,$$

so that $f \in H^q_\sigma$ if and only if $g \in H^q_\sigma$.

Let $u=f^{\#p}$. Then, by Lemma 2, applied to $u \in \mathrm{PL}^1$, we know that

$$U(t) = \sup\{u(re^{it});\ 0 \leqq r<1\}, \qquad t \in T,$$

is in $L^1(T)$. On the other hand, the Schwarz and Pick lemma (see [4, p. 226]) teaches that $f^{\#}(se^{it}) \leqq (1-s^2)^{-1}$ for all $0 \leqq s<1$ and all $t \in T$, so that

$$\int_0^1 u(se^{it})\, ds \leqq k_1 < \infty \qquad \text{for all } t \in T;$$

hereafter k_j $(j=1,2)$ are constants. Therefore, for $t \in T$, and for $0 \leqq R<1$,

$$\sigma(f)(Re^{it}) \leqq \int_0^R u(se^{it})^{1/p}\, ds \leqq k_1 U(t)^{1/p-1} ,$$

or

$$\sigma(f)(Re^{it})^q \leqq k_2 U(t)$$

because $q(1/p-1)=1$. We now obtain that, for all $0 \leqq R<1$,

$$\int_T \sigma(f)(Re^{it})^q\, dt \leqq k_2 \int_T U(t)\, dt < \infty$$

because $U \in L^1(T)$. This shows that $\sigma(f) \in \mathrm{PL}^q$ or $f \in H^q_\sigma$.

REMARK. In the proof of (B) [1, p. 88ff.] a deep theorem [1, Theorem 5.11, p. 87], which we shall call Theorem D, is used. Thanks to the Schwarz and Pick lemma, the proof of Theorem 2 is easier than that of (B). Combining Lemma 1 with Theorem D, one can easily prove the PL version of Theorem D, namely, *if* $0<p<q\leqq\infty$, $u\in \mathrm{PL}^p$, $\lambda\geqq p$, *and* $\alpha=1/p-1/q$, *then*

$$\int_0^1 (1-r)^{\lambda\alpha-1}\mu_q(r,u)^\lambda\,dr < \infty ,$$

where

$$\begin{aligned}\mu_q(r,u) &= \left[\frac{1}{2\pi}\int_T u(re^{it})^q\,dt\right]^{1/q}, && \text{if } q<\infty ,\\ &= \sup_{t\in T} u(re^{it}), && \text{if } q=\infty .\end{aligned}$$

Finally we must prove, as was promised in Section 1, that the converse of Theorem 2 is false. A. J. Lohwater, G. Piranian and W. Rudin [5, Theorem] proved the existence of a continuous function f on $D\cup\Gamma$ which is holomorphic in D, yet $|f'|$ has no radial limit at a.e. point of Γ. In particular,

$$\limsup_{r\to 1}|f'(re^{it})| = \infty \qquad \text{a.e.}$$

On dividing f by a suitable constant we may consider that $f\in H^\infty_\sigma$. Then $f^\#$ does not belong to PL^p for any p, $0<p<\infty$. For, otherwise, $f^\#$ has a finite radial limit $f^{\#*}(t)$ for a.e. $t\in T$. This is not the case because $|f(e^{it})|<1$, and $|f'|^*(t)$ does not exist.

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