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ON C*-ALGEBRA EXTENSIONS RELATIVE TO A FACTOR OF TYPE II_∞

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Abstract.

The commutative semigroup of strong equivalence classes of unital extensions of the norm-closed two-sided ideal of an infinite semifinite countably decomposable factor by a separable unital C*-algebra which is the direct limit of a sequence of C*-algebras with continuous trace is a group. The identity of this group is the class of any trivial unital extension. In the case that the quotient C*-algebra is the direct limit of a sequence of finite-dimensional C*-algebras, and the factor is of type II, this group consists of a single element.

1. Introduction.

Let M be an infinite semifinite countably decomposable factor and denote by I the norm closure of the ideal of elements of finite rank. Let A be a separable unital C*-algebra. In [5], Brown, Douglas, and Fillmore introduced the commutative semigroup $\mathrm{Ext}_s(A,I)$ of strong equivalence classes of unital (essential) extensions of I by A. This consists of the unital embeddings of A into M/I, up to unitary equivalence by the image in M/I of a unitary in M, with the sum of extensions τ_1 and τ_2 defined as the class of $u_1\tau_1u_1^*+u_2\tau_2u_2^*$ where u_1 and u_2 are the images in M/I of isometries in M whose range projections are orthogonal with sum 1. If a unital embedding of A into M/I is the image of a unital embedding of A into M, we shall say that it is trivial.

Brown, Douglas, and Fillmore showed that $\mathrm{Ext}_s(A,I)$ is a group, with identity the class of any trivial unital extension, if M is of type I and A is commutative. If M is of type II, this was shown by Fillmore in [7].

In the case that M is ot type I, it was shown by Voiculescu in [12] that for arbitrary (separable) A the class of a trivial unital extension is unique, and is an identity for the semigroup $\mathrm{Ext}_s(A,I)$. It was shown by Choi and Effros in [6], using an idea of Arveson in [1], that if A is nuclear then $\mathrm{Ext}_s(A,I)$ is a group. For a unified exposition of these results, see [2].

While we cannot extend the result of Voiculescu in full generality to the case

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that M is of type II, we can take a step beyond the result of Fillmore, by dealing with C*-algebras with continuous trace (not just commutative ones), and also with direct limits of sequences of such C*-algebras.

The result of Arveson, Choi, and Effros holds with the same proof in the case that M is of type II. In the absence of the analogue of Voiculescu's theorem, the result must be stated as follows: if A is nuclear then for any unital extension there is another unital extension such that the sum is trivial. This of course says that $\mathrm{Ext}_s(A,I)$ modulo the trivial classes is a group.

We can compute the group $\mathrm{Ext}_s(A,I)$ if A is the direct limit of a sequence of finite-dimensional C*-algebras. In the case that M is of type I, this was done by Pimsner and Popa in [10] and Pimsner in [11] (for a different generalization of this see [8]); the answer is $\mathrm{Ext}(K_0A/\mathbf{Z}, \mathbf{Z})$, where $\mathbf{Z}$ is embedded in K_0A as multiples of the class [1] of the unit of A. In the case that M is of type II, the group is zero.

2. THEOREM. *Let A be the C*-algebra direct limit of a unital sequence $A_1 \to A_2 \to \ldots$ of unital C*-algebras with continuous trace. Then there is a unique element of* $\mathrm{Ext}_s(A,I)$ *which is the class of a trivial extension, and this element is an identity for the semigroup* $\mathrm{Ext}_s(A,I)$.

PROOF. We must show that if $\sigma \in \mathrm{Ext}_s(A,I)$, and if $\tau \in \mathrm{Ext}_s(A,I)$ is the class of a trivial extension, then

$$\sigma+\tau = \sigma .$$

It is sufficient to prove that if σ and τ are as above, then there exists $\sigma' \in \mathrm{Ext}_s(A,I)$ such that

$$\sigma = \sigma'+\tau .$$

Indeed, then, as in [12], we may take in place of τ a trivial extension τ' for which

$$\tau'+\tau = \tau' ,$$

and then

$$\sigma+\tau = (\sigma'+\tau')+\tau = \sigma'+(\tau'+\tau) = \sigma'+\tau' = \sigma .$$

(τ' may be taken, as in [12], to be the class of the extension defined by the orthogonal sum of infinitely many copies of a unital splitting of τ, i.e., the sum $u_1\pi u_1^* + u_2\pi u_2^* + \ldots$ where π is a unital splitting of τ and $u_1, u_2, \ldots$ are isometries in M with $u_1u_1^* + u_2u_2^* + \ldots = 1$.)

In other words, it is sufficient to prove that if σ and τ are unital embeddings of A in M/I, with τ trivial, then there exists a proper isometry $v \in M/I$ such that vv^* commutes with $\sigma(A)$, and

$$v^*\sigma v = \tau .$$

We shall begin by establishing two basic facts about extensions of I by A_1 (instead of A).

First, in the language of [3] (Definition 6.6), every point of $\hat{A}_1$ is pull-out-able. In the case that M is of type II, we define this as follows. If $\lambda \in \hat{A}_1$, that is, λ is the unitary equivalence class of a morphism of A_1 onto $\mathrm{M}_n\mathrm{C}$ for some $n = 1,2,\ldots$, embed $\mathrm{M}_n\mathrm{C}$ unitally in M/I and consider the composed morphism $A_1 \to M/I$. Note that, as M is of type II, any two unital embeddings of $\mathrm{M}_n\mathrm{C}$ in M/I are strongly equivalent. The strong equivalence class of the morphism $A_1 \to M/I$ therefore depends only on λ; denote this class by λ_M. We shall say that λ is pull-out-able (with respect to M) if for any $\sigma \in \mathrm{Ext}_s(A_1, I)$, $\sigma + \lambda_M = \sigma$.

The proof that any $\lambda \in \hat{A}_1$ is pull-out-able in the case that M is of type II is similar to the proof in the case that M is of type I, given in the first half of the paragraph following 6.7 of [3]. This proof is designed for weak equivalence rather than strong, but is easily modified for strong (and, furthermore, by the remark preceding 5.3 of [3], and in view of 6.8d of [3], does not even need to be so modified). We note that the case A_1 is commutative, to which the general case is reduced, is now to be deduced from 2.9 of [7], rather than [5].

The second basic fact about A_1 is as follows. For any unital embedding $\pi: A_1 \to M$, any finite projection f in M, any finite subset S of A_1, and any $\varepsilon > 0$, there exist a finite projection g in M with $g \geqq f$, a finite-dimensional sub-C*-algebra C of gMg containing g, and a unital morphism $\varrho: \pi(A_1) \to C$ such that, for each $a \in \pi(S)$,

$$\|ga - ag\| < \varepsilon ,$$

$$\|gag - \varrho(a)\| < \varepsilon .$$

To prove this, first note that it suffices to prove the same statement with $\pi(A_1)$ replaced by a larger sub-C*-algebra of M. We shall prove the statement with $\pi(A_1)$ replaced by a larger separable C*-algebra with continuous trace, D, with totally disconnected spectrum, chosen as follows. Since $\pi(A_1)$ is unital with continuous trace, its weak closure $\pi(A_1)''$ is a finite direct sum of finite homogeneous von Neumann algebras of type I, and is therefore the algebra generated by its centre and a finite-dimensional sub-C*-algebra D_1. Choose a separable approximately finite-dimensional sub-C*-algebra D_2 of the centre of $\pi(A_1)''$ containing the centres of $\pi(A_1)$ and of D_1, and denote by D the algebra generated by D_1 and D_2. Then D is a C*-algebra, $\pi(A_1) \subseteq D$, D has continuous trace, and the centre of D, D_2, is separable and approximately finite-dimensional. We shall use that D_2 is the C*-algebra generated by a single selfadjoint element h.

Let f be a finite projection in M, S_2 a finite subset of D_2, and $\varepsilon_1 > 0$. We shall show that there exist a finite projection $g \geqq f$ in M commuting with D_1, a finite-dimensional commutative sub-C*-algebra C_2 of gMg commuting with gD_1, and a surjective morphism $\varrho_2\colon D_2 \to C_2$ such that, for each $d \in S_2$,

$$\|gd-dg\| < \varepsilon_1 ,$$
$$\|gdg-\varrho_2(d)\| < \varepsilon_1 .$$

Since h is approximately contained in a finite-dimensional sub-C*-algebra of D_2, it follows that h is approximately contained in a finite-dimensional sub-C*-algebra of D containing D_1. Denote by g the smallest projection in M containing f and commuting with this finite-dimensional algebra. Then g is finite, and g approximately commutes with h. In particular, since the elements of S_2 are approximately polynomials in h, g approximately commutes with S_2. Furthermore, since $ghg+(1-g)h(1-g)$ is close to h, each point of the spectrum of ghg is close to some point in the spectrum of h. Therefore, ghg is close to a selfadjoint element h' of $\{ghg\}''$ such that the spectrum of h' is finite and is contained in the spectrum of h. This defines a surjective morphism ϱ_2 from D_2 onto a finite-dimensional subalgebra C_2 of $\{ghg\}''$, a subalgebra commuting with gD_1 since ghg does, such that gdg is close to $\varrho_2(d)$ for all $d \in S_2$. Thus, if the commutative finite-dimensional subalgebra of D_2 chosen above approximates h sufficiently well, we have $\|gd-dg\|<\varepsilon_1$, $\|gdg-\varrho_2(d)\|<\varepsilon_1$ for all $d \in S_2$. In particular, if a central projection e of D_1 belongs to S_2, and $\varepsilon_1 \leqq 1$, then g and e both commute with $\varrho_2(e)$, and $\|ge-\varrho_2(e)\|<1$, so $\varrho_2(e)=ge$.

It follows immediately from what was shown in the preceding paragraph that for any finite projection f in M, any finite subset S of D, and any $\varepsilon>0$, there exist a finite projection g in M with $g \geqq f$, a finite-dimensional sub-C*-algebra C of gMg containing g, and a surjective morphism $\varrho\colon D \to C$ such that $\|ga-ag\|<\varepsilon$, $\|gag-\varrho(a)\|<\varepsilon$ for all $a \in S$. Just choose g as above, with ε_1 sufficiently small, and with S_2 such that S is contained in the algebra generated by S_2 and D_1, and such that each central projection of D_1 belongs to S_2, and take for C the subalgebra generated by C_2 defined above and by gD_1, and for ϱ the unique extension of ϱ_2 from D_2 to D which on D_1 coincides with multiplication by g. (Note that if $\varepsilon_1<1$ then $\varrho_2(e)=ge$ for all $e \in D_2 \cap D_1$.)

This establishes the two basic facts about A_1 (or A_2, or $A_3,\ldots$) that we shall need. Now let σ and τ be unital embeddings of A in M/I, with τ trivial, and let π be a unital splitting of τ. Denote the preimage of $\sigma(A)$ in M by B, choose a dense sequence (a_k) in A, and choose a sequence (b_k) in B with $b_k+I=\sigma(a_k)$.

By the second basic fact established above, applied to $A_{n_1}, A_{n_2},\ldots$ and $S_1 \subseteq A_{n_1}$, $S_2 \subseteq A_{n_2},\ldots$ where S_k is a finite set containing elements strictly within distance 2^{-k} of $a_1,\ldots,a_k$, $k=1,2,\ldots$, there exist an increasing sequence

$$g_1 \leqq g_2 \leqq \cdots$$

of finite projections in M, with supremum 1, finite-dimensional sub-C*-algebras $C_1 \subseteqq g_1 M g_1, C_2 \subseteqq g_2 M g_2, \ldots$ with $g_1 \in C_1, g_2 \in C_2, \ldots$, and surjective morphisms $\varrho_1 \colon A_{n_1} \to C_1$, $\varrho_2 \colon A_{n_2} \to C_2, \ldots$ such that, for all $a \in \pi(S_k)$,

$$\|g_k a - a g_k\| < 2^{-k}$$

$$\|g_k a g_k - \varrho_k \pi^{-1}(a)\| < k^{-1} .$$

We remark that we do not need that the sequence $A_1, A_2, \ldots$ is increasing; the only assumption we need on the separable unital C*-algebra A is that any finite subset can be approximated by elements of some unital sub-C*-algebra with continuous trace. Thus, the class of separable unital C*-algebras to which A may belong is closed under taking direct limits.

By the first basic fact established above, for each $k = 1, 2, \ldots$ there exist an infinite projection p_k in M commuting modulo I with the preimage B_{n_k} in M of $\sigma(A_{n_k}) \subseteqq M/I$, a finite-dimensional sub-C*-algebra $\bar{C}_k$ of $p_k M p_k$ containing p_k, with $\bar{C}_k \cap I = 0$, and an isomorphism $\theta_k \colon C_k \to \bar{C}_k$ such that for all $b \in B_{n_k}$,

$$p_k b p_k - \theta_k \varrho_k \sigma^{-1}(b+I) \in I .$$

It follows that there exists an orthogonal sequence $(f_1, f_2, \ldots)$ of finite projections in M (with $f_k \leqq p_k$) such that $1 - \sum_1^\infty f_k$ is infinite in M, f_k commutes with $\bar{C}_k$, the map $f_k \theta_k \colon C_k \to f_k \bar{C}_k$ is determined by a partial isometry u_k in M, i.e.,

$$u_k^* u_k = g_k, \quad u_k u_k^* = f_k ,$$

$$u_k^* \theta_k(c) u_k = c, \quad c \in C_k ,$$

and for each b belonging to a fixed finite subset of B which maps onto $\sigma(S_k) \subseteqq \sigma(A_{n_k})$ and contains elements strictly within distance 2^{-k} of $b_1, \ldots, b_k$ (recall that b_i maps onto $\sigma(a_i)$),

$$f_k b f_j = f_j b f_k = 0, \quad 1 \leqq j < k ,$$

$$\| f_k b f_k - f_k \theta_k \varrho_k \sigma^{-1}(b+I) f_k\| < k^{-1} .$$

We now have the following chain of inequalities, whenever $1 \leqq i \leqq k$, where $b \in B_{n_k}$ is chosen as above close to b_i:

$$\|u_k^* b_i u_k - u_k^* b u_k\| < 2^{-k} ,$$

$$\|u_k^* b u_k - u_k^* \theta_k \varrho_k \sigma^{-1}(b+I) u_k\| < k^{-1} ,$$

$$u_k^* \theta_k \varrho_k \sigma^{-1}(b+I) u_k = \varrho_k \sigma^{-1}(b+I) ,$$

$$\|\varrho_k \sigma^{-1}(b+I) - g_k \pi(\sigma^{-1}(b+I)) g_k\| < k^{-1} ,$$

$$\|g_k\pi(\sigma^{-1}(b+I))g_k - g_k\pi(\sigma^{-1}(b_i+I))g_k\| < 2^{-k},$$

$$\sigma^{-1}(b_i+I) = a_i .$$

Hence by the triangle inequality, for $1 \leqq i \leqq k$,

$$\|u_k^* b_i u_k - g_k \pi(a_i) g_k\| < 4k^{-1} .$$

We also have, for $1 \leqq i \leqq k$,

$$\|g_k\pi(a_i) - \pi(a_i)g_k\| < 3 \cdot 2^{-k} < 2^{-k+2},$$

$$\|f_k b_i f_j\| < 2^{-k}, \quad \|f_j b_i f_k\| < 2^{-k}, \quad 1 \leqq j < k .$$

Finally, set $g_k - g_{k-1} = e_k$, $k = 1, 2, \ldots$, where $g_0 = 0$. Then $\sum_{k=1}^{\infty} e_k = 1$ in M, and if $1 \leqq i < k$, setting $u_k e_k = v_k$ we have

$$\|v_k^* b_i v_k - e_k \pi(a_i) e_k\| < 4k^{-1},$$

$$\|e_k\pi(a_i) - \pi(a_i)e_k\| < 2^{-k+3} + 2^{-k+2} < 2^{-k+4},$$

$$\|v_k^* b_i v_j\| < 2^{-k}, \quad \|v_j^* b_i v_k\| < 2^{-k}, \quad 1 \leqq j < k .$$

Moreover, $\sum_{k=1}^{\infty} v_k$ is an isometry in M, with cokernel $1 - \sum_{k=1}^{\infty} v_k v_k^*$ containing $1 - \sum_{k=1}^{\infty} f_k$ which is infinite. Denote the image of $\sum_{k=1}^{\infty} v_k$ in M/I by v. Then v is a proper isometry in M/I, and for each $i = 1, 2, \ldots$,

$$\begin{aligned} v^*\sigma(a_i)v &= v^*(b_i+I)v \\ &= \left(\sum_{k=1}^{\infty} v_k^* b_i v_k\right) + I \\ &= \left(\sum_{k=1}^{\infty} e_k \pi(a_i) e_k\right) + I \\ &= \left(\sum_{k=1}^{\infty} e_k \pi(a_i)\right) + I \\ &= \pi(a_i) + I \\ &= \tau(a_i) . \end{aligned}$$

By continuity,

$$v^*\sigma(a)v = \tau(a), \quad a \in A .$$

This implies in particular that vv^* commutes with $\sigma(A)$ (see the proof of Corollary 1, page 338 of [2]), and so the proof of the Theorem is complete.

3. THEOREM. *Let A be a separable unital nuclear* C*-*algebra. Then for any $\tau \in \mathrm{Ext}_s(A, I)$ there exists $\tau' \in \mathrm{Ext}_s(A, I)$ such that $\tau + \tau'$ is trivial.*

PROOF. This follows from Corollary 3.11 of [6] (see also the Corollary of Theorem 7 of [2]) applied with $B=M$ and $J=I$, by using the idea of [1], as in the case that M is of type I.

4. COROLLARY. *Let A be as in 2. Then* $\mathrm{Ext}_s(A,I)$ *is a group.*

PROOF. This follows from 2 and 3.

5. REMARK. The methods used above, slightly modified, yield analogues of 2 and 4 for extensions of I by a nonunital separable C*-algebra A which is the direct limit of C*-algebras with continuous trace.

In this case, by 3.4 of [5], strong equivalence coincides with weak equivalence, and hence by 3.15 of [5], we may suppose that A is stable. Then A is the direct limit of stable C*-algebras with continuous trace. Since a stable C*-algebra with continuous trace and with totally disconnected spectrum is trivial, and is hence a direct limit of unital C*-algebras with continuous trace, the proof of 2 is applicable with only minor modifications.

6. THEOREM. *Let A be the C*-algebra direct limit of a unital sequence $A_1 \to A_2 \to \ldots$ of finite-dimensional C*-algebras, and suppose that M is of type* II. *Then every unital extension of I by A is trivial.*

PROOF. It would be enough to show that if $1 \in A_1 \subseteqq A_2 \subseteqq M/I$, and $1 \in B_1 \subseteqq M$ maps isomorphically onto A_1 by the quotient map $M \to M/I$, then there exists $B_2 \subseteqq M$ with $B_1 \subseteqq B_2$, mapping isomorphically onto A_2. (With $1 \in A_1 \subseteqq A_2 \subseteqq \ldots \subseteqq A \subseteqq M/I$, one could just choose successively $B_1 \subseteqq B_2 \subseteqq \ldots \subseteqq M$ mapping isomorphically onto $A_1 \subseteqq A_2 \subseteqq \ldots$.) If the finite-dimensional C*-algebra A, is not simple, however, this is not true.

The proof seems to require a less direct approach, using K-theory. The argument consists of five steps. Let $I \to B \to A$ be an essential unital extension of I by A.

STEP 1. The induced sequence $K_0I \to K_0B \to K_0A$ defines an abelian group extension of the group K_0I $(=\mathbb{R})$ by the group K_0A.

This can easily be shown directly, using that any projection in M/I is the image of a projection in M (Theorem 3.2 of [13]), so that any projection in A is the image of a projection in B. What is needed—namely, that the sequence of abelian groups

$$0 \to K_0I \to K_0B \to K_0A \to 0$$

is exact—can also be seen, as Brown has pointed out in the different situation in which I is an approximately finite-dimensional C*-algebra ([4]), by observing that in the six-term exact sequence

$$\begin{array}{ccccc} K_0I & \rightarrow & K_0B & \rightarrow & K_0A \\ \uparrow & & & & \downarrow \\ K_1A & \leftarrow & K_1B & \leftarrow & K_1I \end{array}$$

of Bott periodicity, the groups K_1A and K_1I are zero.

STEP 2. The abelian group extension $K_0I \rightarrow K_0B \rightarrow K_0A$ defined above splits.

To see this note that the subgroup K_0I $(=\mathbf{R})$ of K_0B is divisible. It is then by Theorem 2 of [9] a direct summand of K_0B.

STEP 3. There exists a splitting map $K_0A \rightarrow K_0B$ for the abelian group extension $K_0I \rightarrow K_0B \rightarrow K_0A$ which takes [1] into [1] (i.e., takes the class of the unit of A into the class of the unit of B).

To see this, first choose by Step 2 some splitting map $K_0A \rightarrow K_0B$. To get a different splitting we must add a map from K_0A to K_0I, and it is sufficient for us to show that there is such a map which is nonzero on $[1] \in K_0A$. (Since $K_0I = \mathbf{R}$ it follows that there is such a map which is arbitrary on this element.) The existence of such a map $K_0A \rightarrow K_0I$ follows from the facts that K_0I $(=\mathbf{R})$ is divisible and that K_0A is torsion free. (A maximal additive extension of a nonzero map from $\mathbf{Z} \subseteqq K_0A$ into K_0I must then be defined on all of K_0A.)

STEP 4. Any splitting map $K_0A \rightarrow K_0B$ for the abelian group extension $K_0I \rightarrow K_0B \rightarrow K_0A$ is positive with respect to the natural preorder in K_0A and K_0B.

It is sufficient to show that if h is an element of K_0B such that the image of h in K_0A is nonzero and positive, then h is positive in K_0B. After tensoring with a suitable full matrix algebra of finite order, what we must show is that if e and f are orthogonal projections in B with $f \in I$ and $e \notin I$, then there exists a projection $f' \in I$ equivalent to f with $f' \leqq e$. That this is true for B follows, as $B \subseteqq M$, from the fact that it is true for M.

STEP 5. Let φ: $K_0A \rightarrow K_0B$ be a splitting of $K_0I \rightarrow K_0B \rightarrow K_0A$ which is unital, i.e., takes [1] into [1]. Then there is a unital splitting $A \rightarrow B$ of the extension of C*-algebras $I \rightarrow B \rightarrow A$ which induces φ.

Denote by

$$\varphi_n : K_0A_n \to K_0B$$

the composition of φ with the induced map $K_0A_n \to K_0A$ (which may not be injective). It will suffice to show that if $n=1,2,\ldots$ and if

$$\psi_{n-1} : A_{n-1} \to B$$

is a unital morphism for which the given map $B \to A$ is a left inverse, where $A_0 = \mathbf{C} \subseteqq A$, and such that the induced map $K_0A_{n-1} \to K_0B$ is φ_{n-1}, where $\varphi_0[1] = [1]$, then there is an extension of ψ_{n-1} to a morphism

$$\psi_n : A_n \to B$$

for which the map $B \to A$ is a left inverse, and such that the induced map $K_0A_n \to K_0B$ is φ_n. The closure of the common extension of $\psi_1, \psi_2, \ldots$ is then a unital splitting $A \to B$, inducing φ: $K_0A \to K_0B$.

It is enough to consider the case $n=2$. Choose a maximal orthogonal set S_1 of minimal projections in A_1, and a maximal orthogonal set S_2 of minimal projections in A_2, such that every element of S_2 lies inside some element of S_1. Fix $p \in S_1$. Choose a morphism ψ from pA_2p to $\psi_1(p)M\psi_1(p)$ for which the map $M \to M/I$ is a left inverse; necessarily, $\psi(pA_2p) \subseteqq B$. For each $q \in S_2$ with $q \leqq p$, the element $\varphi_2[q]-[\psi q]$ of K_0B belongs to the kernel of $K_0B \to K_0A$ and hence, by Step 1, to the image of $K_0I \to K_0B$. Since $K_0I = \mathbf{R}$, each such $\varphi_2[q]-[\psi q]$ is either positive or negative. If $\varphi_2[q]-[\psi q]$ is negative, then, since $\psi q \in M \setminus I$, ψq may be decreased by a projection in I so that $\varphi_2[q] - [\psi q] = 0$. Furthermore, this may be done simultaneously for all the projections in S_2 equivalent in A_2 to q, in such a way that ψ is still a morphism. We may therefore suppose that $\varphi_2[q]-[\psi q]$ is positive for each $q \in S_2$ with $q \leqq p$. Since

$$[\psi_1 p] = \varphi_1[p] = \varphi_2[p] = \sum_{q \in S_2, q \leqq p} \varphi_2[q],$$

we have the decomposition

$$\left[\psi_1 p - \sum_{q \in S_2, q \leqq p} \psi q\right] = \sum_{q \in S_2, q \leqq p} (\varphi_2[q]-[\psi q])$$

in the positive part of the image of $K_0I \to K_0B$, and since any positive element of this image is the image of the class in K_0I of a projection in I, it follows that ψq may be increased by a projection in I, for each $q \in S_2$ with $q \leqq p$, with class $\varphi_2[q]-[\psi q]$ in the image of $K_0I \to K_0B$, in such a way that

$$\sum_{q \in S_2, q \leqq p} \psi q = \psi_1 p .$$

Moreover, since the change is by projections of the same dimension in I for different $q \in S_2$ which are equivalent to q in A_2, also the partial isometries between different ψq's may be extended so that ψ is still a morphism. We now have

$$\varphi_2[q]-[\psi q] = 0, \quad q \in S_2, q \leqq p .$$

Carry out the construction of ψ as above for each $p \in S_1$. Then we have a morphism

$$\psi: \sum_{p \in S_1} pA_2p \to B$$

such that for each $p \in S_1$,

$$\sum_{q \in S_2, q \leqq p} \psi q = \psi_1 p ,$$

and for each $q \in S_2$,

$$\varphi_2[q]-[\psi q] = 0 .$$

It is now straightforward to construct a common extension of ψ and of ψ_1 to a morphism ψ_2 of all of A_2 into B, necessarily inducing φ_2: $K_0A_2 \to K_0B$. We make only the following remarks. If p_1 and p_2 are distinct elements of S_1 which are equivalent in A_1 then a common extension of ψ and of ψ_1 to $(p_1+p_2)A_2(p_1+p_2)$ is unique. If p_1 and p_2 are elements of S_1 which are not equivalent in A_1, and if q_1 and q_2 are elements of S_2 with $q_1 \leqq p_1$ and $q_2 \leqq p_2$, such that q_1 and q_2 are equivalent in A_2, then

$$[\psi q_1] = \varphi_2[q_1] = \varphi_2[q_2] = [\psi q_2] .$$

Hence if v is a partial isometry in A_2 with $v^*v=q_1$ and $vv^*=q_2$, and u is any partial isometry in the preimage of v in $\psi(q_1)M\psi(q_2)$, then

$$[\psi q_1-u^*u] = [\psi q_1]-[u^*u] = [\psi q_2]-[uu^*] = [\psi q_2-uu^*] ,$$

and so u may be extended by adding a partial isometry in I in such a way that $u^*u=\psi q_1$, $uu^*=\psi q_2$.

7. Corollary. *Let A be the* C*-*algebra direct limit of a unital sequence $A_1 \to A_2 \to \dots$ of finite-dimensional* C*-*algebras, and suppose that M is of type* II. *Then* $\mathrm{Ext}_s(A,I)$ *has only one element.*

Proof. This follows from 2 and 6.

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