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AN INTEGRAL INEQUALITY FOR CAPACITIES

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1. Introduction.

Let C_s, $0<s<n$, and C_0 be the potential-theoretic capacities in the euclidean n-space R^n corresponding to the kernels $|x-y|^{-s}$ and

$$\log_+|x-y|^{-1} = \max\{\log|x-y|^{-1},0\}\,,$$

respectively. We shall prove that if $f\colon \mathsf{R}^n \to \mathsf{R}^m$ is a Lipschitzian mapping and $m<s<n$, then for any compact subset F of R^n,

$$\int C_{s-m}(F\cap f^{-1}\{y\})\,d\mathscr{L}^m y \leqq c(\operatorname{Lip} f)^m C_s(F)\,,$$

where $\mathscr{L}^m$ is the Lebesgue measure on R^m, Lip f the Lipschitz constant of f and c a constant depending only on m, n and s. This inequality holds for arbitrary subsets of R^n provided the capacities are replaced by the corresponding outer capacities and the integral by the upper integral. If $s=m$ we also give an inequality in 3.2, which however is much more complicated.

In the special case where f is the projection $\mathsf{R}^m\times\mathsf{R}^{n-m}\to\mathsf{R}^m$ and F is a product set, $F=F_1\times F_2$, $F_1\subset\mathsf{R}^m$, $F_2\subset\mathsf{R}^{n-m}$, similar inequalities were proved by Ohtsuka in [4, § 2]. An inequality of the above type for Hausdorff measures can be found in [1, 2.10.27].

Let V be an $n-m$ dimensional linear subspace of R^n, and denote by V_y the $n-m$ plane through y parallel to V, where $y\in V^\perp$, the orthogonal complement of V. Taking f as the orthogonal projection of R^n onto $V^\perp$ the above inequality becomes

$$\int_{V^\perp} C_{s-m}(F\cap V_y)\,d\mathscr{H}^m y \leqq cC_s(F)\,,$$

with $\mathscr{H}^m$ the m dimensional Hausdorff measure (whose restriction to $V^\perp$ is the Lebesgue measure of $V^\perp$). Here the left hand side may be zero and the right hand side positive, as examples where F is a suitable Cantor set show. However, if one integrates also over all V's with respect to the orthogonally invariant probability measure on the space of $n-m$ dimensional linear

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subspaces of $\mathbf{R}^n$, one has also a reversed inequality, see [3, 4.6]. Combining these two results we obtain

$$c^{-1}C_s(F) \leqq \int C_{s-m}(F \cap A)\, d\lambda_{n,n-m}A \leqq cC_s(F),$$

where $\lambda_{n,n-m}$ is an isometry invariant measure on the space of all $n-m$ dimensional affine subspaces of $\mathbf{R}^n$ and c depends only on n, m and s.

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2. Preliminaries.

Throughout the whole paper m and n will be positive integers and s a real number with $0 \leqq s < n$.

2.1. *Radon measures.* Let $\mathscr{M}_n$ be the space of non-negative Radon measures on $\mathbf{R}^n$ with compact support. We equip $\mathscr{M}_n$ with the vague topology. Then a sequence (μ_i) in $\mathscr{M}_n$ converges to μ if and only if $\int g\, d\mu_i \to \int g\, d\mu$ for every real-valued continuous function g on $\mathbf{R}^n$ with compact support. We denote the support of a measure μ by $\operatorname{spt}\mu$.

2.2. *Capacities.* For any $\mu \in \mathscr{M}_n$ the s potential of μ is defined for $x \in \mathbf{R}^n$ by

$$U_s^\mu(x) = \int |x-y|^{-s}\, d\mu y, \quad \text{if } s>0,.$$

$$U_0^\mu(x) = \int \log_+ |x-y|^{-1}\, d\mu y.$$

The (inner) s capacity of a compact set $F \subset \mathbf{R}^n$ is

$$C_s(F) = \sup\{\mu(\mathbf{R}^n) : \mu \in \mathscr{M}_n,\ \operatorname{spt}\mu \subset F,\ U_s^\mu \leqq 1 \text{ on } \operatorname{spt}\mu\},$$

and for an arbitrary subset E of $\mathbf{R}^n$

$$C_s(E) = \sup\{C_s(F) : F \text{ compact}, F \subset E\}.$$

The outer s capacity of $E \subset \mathbf{R}^n$ is defined by

$$C_s^*(E) = \inf\{C_s(G) : G \text{ open}, E \subset G\}.$$

It is well-known that C_s and C_s^* agree for Suslin (i.e. analytic) sets, and hence for Borel sets, [2, Theorem 4.5].

To state an alternative definition for $C_s(F)$, we denote by $I_s(\mu)$ the s energy of $\mu \in \mathscr{M}_n$,

$$I_s(\mu) = \int U_s^\mu \, d\mu \, .$$

Then, see [2, § 2.5],

$$C_s(F) = \sup \{ I_s(\mu)^{-1} : \mu \in \mathcal{M}_n, \operatorname{spt} \mu \subset F, \mu(\mathbf{R}^n) = 1 \} \, .$$

For any compact subset F of $\mathbf{R}^n$ there is a unique measure $\mu_s^F \in \mathcal{M}_n$ such that

$$U_s^{\mu_s^F}(x) \leqq 1 \quad \text{for } x \in \operatorname{spt} \mu_s^F \text{ and } \mu_s^F(\mathbf{R}^n) = C_s(F) \, ,$$

see [2, § 2.5]. By [2, Theorem 2.5], we have also

$$I_s(\mu_s^F) = C_s(F) \, .$$

2.3. Lemma. *Suppose that K is a non-negative lower semicontinuous function on $\mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^n$, F a compact subset of $\mathbf{R}^n$ and $f\colon F \to \mathbf{R}^m$ a continuous map. Let $\mu_y = \mu_s^{f^{-1}\{y\}}$ for $y \in \mathbf{R}^m$. Then the functions*

$$(y, z, v) \mapsto \int K(y, z, u, v) \, d\mu_y u \, ,$$

$$(y, z) \mapsto \iint K(y, z, u, v) \, d\mu_y u d\mu_z v \, ,$$

are Borel functions.

Proof. By the monotone convergence theorem we may assume that K is continuous with compact support. We first show that $y \mapsto \int g \, d\mu_y$ is a Borel function on $\mathbf{R}^m$ whenever g is a continuous function on $\mathbf{R}^m$ with compact support. For $y \in \mathbf{R}^m$ and $i = 1, 2, \ldots$ let $F(y, i)$ be the union of all closed dyadic cubes with side-length 2^{-i} which meet $f^{-1}\{y\}$. Denote $\mu_y^i = \mu_s^{F(y,i)}$. Then $\mu_y^i \to \mu_y$ vaguely, see [2, 4.2.1]; therefore it suffices to show that each function φ_i, $\varphi_i(y) = \int g \, d\mu_y^i$, is a Borel function. Clearly φ_i assumes only finitely many values and by approximating g we may assume that $\int g \, d\mu_y^i \neq \int g \, d\mu_z^i$ whenever $\mu_y^i \neq \mu_z^i$. Let A_j be the set of those $t \in \mathbf{R}^1$ for which there is $y \in \varphi_i^{-1}\{t\}$ such that $F(y, i)$ is a union of j dyadic cubes of side-length 2^{-i}. The continuity of f implies that $\varphi_i^{-1}\{t\}$ is open for all $t \in A_1$, $\varphi_i^{-1}\{t\}$ is relatively open in $\mathbf{R}^m \setminus \varphi_i^{-1}(A_1)$ for $t \in A_2$, and so on. Thus φ_i is a Borel function.

The Lemma follows now because by the Stone–Weierstrass theorem, K can be approximated uniformly by finite linear combinations of the functions $(y, z, u, v) \mapsto K_1(y) K_2(z) K_3(u) K_4(v)$ where $K_1, \ldots, K_4$ are continuous with compact support.

In the following lemma we consider the truncated Riesz kernels K_d^s, $0 < d < \infty$:

$$K_d^s(x,y) = |x-y|^{-s} \quad \text{for } |x-y| \geqq d\,,$$
$$= d^{-s} \quad \text{for } |x-y| < d\,.$$

2.4. LEMMA. *There is a constant C depending only on n such that for any $\mu, \nu \in \mathscr{M}_n$, $0<d<\infty$,*

$$\iint K_d^s(x,y)\,d\mu x\,d\nu y \leqq C\left(\iint K_d^s(x,y)\,d\mu x\,d\mu y + \iint K_d^s(x,y)\,d\nu x\,d\nu y\right).$$

PROOF. Let $M = d^{-s}$. By a well-known formula we have

$$\iint K_d^s(x,y)\,d\mu x\,d\nu y = \int_0^\infty \mu\times\nu\{(x,y) : K_d^s(x,y) > t\}\,dt$$

$$= \int_0^M \mu\times\nu\{(x,y) : |x-y|^{-s} > t\}\,dt = \int_0^M \int \mu U(y, t^{-1/s})\,d\nu y\,dt\,,$$

where $U(y,r)$ stands for the open ball with centre y and radius r. We estimate the inner integral and set $r = t^{-1/s}$. We can cover $\operatorname{spt}\nu$ with balls $U(y_i, r)$, $i = 1, \ldots, k$, such that any point of R^n is contained in at most N of the balls $U(y_i, 2r)$, where N is an integer depending only on n. Observing that $y \in U(y_i, r)$ implies $U(y,r) \subset U(y_i, 2r)$ we estimate

$$\int \mu U(y,r)\,d\nu y \leqq \sum_{i=1}^k \int_{U(y_i,r)} \mu U(y,r)\,d\nu y$$

$$\leqq \sum_{i=1}^k \mu U(y_i, 2r)\nu U(y_i, r)$$

$$\leqq \sum_{i=1}^k ((\mu U(y_i, 2r))^2 + (\nu U(y_i, r))^2)$$

$$= \sum_{i=1}^k (\mu\times\mu(U(y_i, 2r)\times U(y_i, 2r)) + \nu\times\nu(U(y_i, r)\times U(y_i, r)))$$

$$\leqq N\left(\mu\times\mu\left(\bigcup_{i=1}^k U(y_i, 2r)\times U(y_i, 2r)\right) + \nu\times\nu\left(\bigcup_{i=1}^k U(y_i, r)\times U(y_i, r)\right)\right)$$

$$\leqq N(\mu\times\mu\{(x,y) : |x-y| < 4r\} + \nu\times\nu\{(x,y) : |x-y| < 4r\})\,.$$

Thus we have as before

$$\iint K_d^s(x,y)\,d\mu x\,d\nu y$$

$$\leqq N\left(\int_0^M \mu\times\mu\{(x,y)\,:\,|x-y|<4t^{-1/s}\}\,dt+\int_0^M \nu\times\nu\{(x,y)\,:\,|x-y|<4t^{-1/s}\}\,dt\right)$$

$$= N4^s\left(\int_0^{4^{-s}M} \mu\times\mu\{(x,y)\,:\,|x-y|^{-s}>t\}\,dt+\int_0^{4^{-s}M} \nu\times\nu\{(x,y)\,:\,|x-y|^{-s}>t\}\,dt\right)$$

$$\leqq N4^n\left(\iint K_d^s(x,y)\,d\mu x\,d\mu y+\iint K_d^s(x,y)\,d\nu x\,d\nu y\right).$$

3. Integral inequalities for capacities.

We let $\alpha(m)$ denote the volume of the unit ball in $\mathbf{R}^m$ and $\beta(m)=m\alpha(m)$ the $m-1$ dimensional area of the unit sphere.

3.1. THEOREM. *There is a constant c depending only m, n and s with the following property: If $m<s<n$ and $f\colon \mathbf{R}^n\to\mathbf{R}^m$ is Lipschitzian, then for any compact set $F\subset\mathbf{R}^n$*

$$\int C_{s-m}(F\cap f^{-1}\{y\})\,d\mathscr{L}^m y \leqq c(\operatorname{Lip} f)^m C_s(F)$$

and for any set $E\subset\mathbf{R}^n$

$$\int^* C^*_{s-m}(E\cap f^{-1}\{y\})\,d\mathscr{L}^m y \leqq c(\operatorname{Lip} f)^m C^*_s(E)\,.$$

PROOF. For each $y\in\mathbf{R}^m$ we denote $\mu_y=\mu_{s-m}^{F\cap f^{-1}\{y\}}$, and define $\mu\in\mathscr{M}_n$ by

$$\int g\,d\mu = \iint g\,d\mu_y\,d\mathscr{L}^m y$$

whenever g is a real-valued continuous function on $\mathbf{R}^n$; this is possible by Lemma 2.3. Then the formula remains valid for every non-negative lower semicontinuous function g on $\mathbf{R}^n$. Obviously $\operatorname{spt}\mu\subset F$. We shall now estimate $I_s(\mu)$. The several applications of Fubini's theorem can all be justified with the help of Lemma 2.3. Denoting $L=\operatorname{Lip} f$, we have for $y,z\in\mathbf{R}^m$ (with $L\operatorname{dist}(\varnothing,A)=\infty$)

$$|y-z| \leqq L\operatorname{dist}(f^{-1}\{y\},f^{-1}\{z\}) \leqq L\operatorname{dist}(\operatorname{spt}\mu_y,\operatorname{spt}\mu_z)\,,$$

and using Fubini's theorem

$$I_s(\mu) = \iint |u-v|^{-s}\,d\mu u\,d\mu v$$

$$= \iiiint |u-v|^{-s}\, d\mu_y u\, d\mathscr{L}^m y\, d\mu_z v\, d\mathscr{L}^m z$$

$$= \iiiint |u-v|^{-s}\, d\mu_y u\, d\mu_z v\, d\mathscr{L}^m y\, d\mathscr{L}^m z$$

$$= \iiiint_{\{(u,v)\,:\,|y-z|\leqq L|u-v|\}} |u-v|^{-s}\, d\mu_y u\, d\mu_z v\, d\mathscr{L}^m y\, d\mathscr{L}^m z\,.$$

Applying Lemma 2.4 with $d=|y-z|/L$ we have

$$\iint_{\{(u,v)\,:\,|y-z|\leqq L|u-v|\}} |u-v|^{-s}\, d\mu_y u\, d\mu_z v$$

$$\leqq C\left(\iint K^s_{|y-z|/L}(u,v)\, d\mu_y u\, d\mu_y v + \iint K^s_{|y-z|/L}(u,v)\, d\mu_z u\, d\mu_z v\right).$$

Therefore by Fubini's theorem

$$I_s(\mu) \leqq 2C \iiiint K^s_{|y-z|/L}(u,v)\, d\mu_y u\, d\mu_y v\, d\mathscr{L}^m y\, d\mathscr{L}^m z$$

$$= 2C\left(\iiiint_{\{(u,v)\,:\,|y-z|\leqq L|u-v|\}} |u-v|^{-s}\, d\mu_y u\, d\mu_y v\, d\mathscr{L}^m y\, d\mathscr{L}^m z\right.$$

$$\left. + \iiiint_{\{(u,v)\,:\,|y-z|>L|u-v|\}} L^s|y-z|^{-s}\, d\mu_y u\, d\mu_y v\, d\mathscr{L}^m y\, d\mathscr{L}^m z\right).$$

We see by Fubini's theorem that the first integral in the above sum equals

$$\iiint \mathscr{L}^m\{z : |y-z|\leqq L|u-v|\}|u-v|^{-s}\, d\mu_y u\, d\mu_y v\, d\mathscr{L}^m y$$

$$= \alpha(m)L^m \iiint |u-v|^{m-s}\, d\mu_y u\, d\mu_y v\, d\mathscr{L}^m y$$

$$= \alpha(m)L^m \int I_{s-m}(\mu_y)\, d\mathscr{L}^m y\,.$$

Applying Fubini's theorem and the formula (which follows by integration in polar coordinates)

$$\text{(1)} \qquad \int_{\mathbf{R}^m\setminus B(y,r)} |y-z|^{-s}\, d\mathscr{L}^m z = \beta(m)(s-m)^{-1} r^{m-s}\,,$$

we obtain for the second integral

$$L^s \iiiint_{\{z\,:\,|y-z|>L|u-v|\}} |y-z|^{-s}\, d\mathscr{L}^m z\, d\mu_y u\, d\mu_y v\, d\mathscr{L}^m y$$

$$= \beta(m)(s-m)^{-1}L^s \iiint L^{m-s}|u-v|^{m-s}\, d\mu_y u\, d\mu_y v\, d\mathscr{L}^m y$$

$$= m\alpha(m)(s-m)^{-1}L^m \int I_{s-m}(\mu_y)\, d\mathscr{L}^m y\,.$$

Hence (recall 2.2)

$$I_s(\mu) \leqq cL^m \int I_{s-m}(\mu_y)\, d\mathscr{L}^m y$$

$$= cL^m \int C_{s-m}(F \cap f^{-1}\{y\})\, d\mathscr{L}^m y\,.$$

Since $\operatorname{spt}\mu \subset F$ and

$$\mu(\mathbf{R}^n) = \int \mu_y(\mathbf{R}^n)\, d\mathscr{L}^m y = \int C_{s-m}(F \cap f^{-1}\{y\})\, d\mathscr{L}^m y\,,$$

which we may assume to be positive, we have by 2.2

$$C_s(F) \geqq I_s(\mu(\mathbf{R}^n)^{-1}\mu)^{-1}$$

$$= \mu(\mathbf{R}^n)^2 I_s(\mu)^{-1} \geqq c^{-1}L^{-m} \int C_{s-m}(F \cap f^{-1}\{y\})\, d\mathscr{L}^m y\,.$$

This proves the first inequality.

It is sufficient to prove the second inequality for open sets. But every open set $G \subset \mathbf{R}^n$ is a union of an increasing sequence (F_i) of compact sets, and we have by [2, Theorem 4.2]

$$C_{s-m}(G \cap f^{-1}\{y\}) = \lim_{i\to\infty} C_{s-m}(F_i \cap f^{-1}\{y\}) \quad \text{for } y \in \mathbf{R}^m\,.$$

Hence the result follows from the monotone convergence theorem.

In the case $s=m$ we have the following inequality:

3.2. THEOREM. *There is a constant c depending only on m and n with the following property: If $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is Lipschitzian, then for any compact set $F \subset \mathbf{R}^n$*

$$\left(\int C_0(F \cap f^{-1}\{y\})\, d\mathscr{L}^m y\right)^2$$

$$\leqq c\big((1+\log_+ (\operatorname{Lip} f)^{-1}+\mathscr{L}^m(fF)\big)\int C_0(F\cap f^{-1}\{y\})^2\,d\mathscr{L}^m y+\int C_0(F\cap f^{-1}\{y\})\,d\mathscr{L}^m y)$$
$$\times (\operatorname{Lip} f)^m C_m(F)\,.$$

This can be proved by the same method as Theorem 3.1 when one observes that all the $\mathscr{L}^m$ integrations can be performed over fF. The formula (1) is replaced by the estimate

$$\int_{fF\setminus B(y,r)} |y-z|^{-m}\,d\mathscr{L}^m z$$
$$= \int_r^\infty t^{-m}\mathscr{H}^{m-1}(fF\cap\{z:|y-z|=t\})\,dt$$
$$\leqq \max\left\{\beta(m)\int_r^1 t^{-1}\,dt, 0\right\}+\int_1^\infty \mathscr{H}^{m-1}(fF\cap\{z:|y-z|=\})\,dt$$
$$\leqq m\alpha(m)\log_+ r^{-1}+\mathscr{L}^m(fF)\,.$$

Although clumsy, Theorem 3.2 is however sufficient for the following:

3.3. Corollary. *Let* $m\leqq s<n$ *and let* $f\colon \mathbf{R}^n\to\mathbf{R}^m$ *be Lipschitzian. If* $E\subset\mathbf{R}^n$ *and* $C_s^*(E)=0$, *then* $C_{s-m}^*(E\cap f^{-1}\{y\})=0$ *for* $\mathscr{L}^m$ *almost all* $y\in\mathbf{R}^m$.

Proof. If $s>m$ the result is immediate by Theorem 3.1. To settle the case $s=m$, we may assume that E is bounded. Then there is a decreasing sequence of bounded open sets G_i containing E such that $C_m(G_i)\to 0$. As in the proof of 3.1, the inequality of 3.2 extends to open sets. The integrals on the right hand side with F replaced by G_i form a bounded sequence. Hence

$$\int^* C_0^*(E\cap f^{-1}\{y\})\,d\mathscr{L}^m y \leqq \int C_0(G_i\cap f^{-1}\{y\})\,d\mathscr{L}^m y\to 0\,,$$

and the result follows.

Note. Recently A. Sadullaev has proved similar inequalities in the case of an orthogonal projection in the paper Rational approximation and pluripolar sets, Mat. Sb. (N.S.) 119 (161) (1982), 96–118 (Russian).

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