MATH. SCAND. 54 (1984), 5-16

# THE BOUNDED MODEL FOR HYPERBOLIC 3-SPACE AND A QUATERNIONIC UNIFORMIZATION THEOREM

WILLIAM ABIKOFF*

In 1971, Lars Ahlfors lectured at the Mittag-Leffler Institute and introduced me to a simple and elegant method using quaternions for computing the action of Möbius transformations on the upper half-space model of $\mathscr{H}^3$, the 3-dimensional hyperbolic space. After substantially completing this work I traced the origins of Ahlfors' exposition, but not his methods to Fueter [5]. There is some overlap between Fueter's results, but again not his methods, and the results given here. On seeing a preprint of this paper, Ahlfors informed me that Dennis Hejhal had discovered an earlier and more general form of the quaternionic representation in Vahlen [12], Vahlen uses Clifford algebras to represent the orientation-preserving hyperbolic motions as Möbius transformations. Many of the results given here may generalize to $n$-dimensional hyperbolic space using the Vahlen representation. Further investigations along the lines initiated by Vahlen are being conducted by Ahlfors.

As best I can discern, Fueter was unaware of Vahlen's work; but Vahlen also seems to have been unaware of Clifford's earlier work which, via the Klein model, gives a representation of the motions of hyperbolic space.

As in the case of Fuchsian groups, it is often preferable to utilize a bounded model for $\mathscr{H}^3$. Bounded models allow us to obtain elementary estimates for the convergence of automorphic forms and give qualitative and quantitative information on group actions.

Here we show that quaternionic actions may also be utilized to describe the action of orientation-preserving motions on the unit ball model of $\mathscr{H}^3$. Of lesser importance currently is that the method has application to $\mathscr{H}^4$.

We also extend several standard results from the theory of Fuchsian or Kleinian groups to general discrete groups of motions of $\mathscr{H}^3$. These results are originally due to Poincaré in the Fuchsian case. In addition we utilize the bounded model group action to define Poincaré $\Theta$ series and interpret their

Received March 28, 1983.

* Research partially supported by the National Science Foundation.

quotients as automorphic quaternion-valued functions on the orbit space of the group. This immediately leads to a quaternionic uniformization theorem.

The fundamental result in $n$-dimensional conformal geometry is the theorem of Liouville as generalized by Gehring and Rešetnjak (see [7]) which states that when $n \geqq 3$ every conformal transformation is a Möbius transformation; i.e. is a composition of inversions in spheres. The generalization removes all differentiability assumptions. We shall make decisive, hidden and terminological use of this theorem.

I would like to thank Troels Jørgensen for several helpful conversations and Werner Fenchel who informed me of several results and methods mentioned in Section 4.

## 1. The group action in the bounded model.

We denote by $\mathsf{H}$ the skew field of quaternions and use the usual identification with $\mathsf{R}^4$ and basis elements $\mathbf{1} = (1,0,0,0)$, $\boldsymbol{i} = (0,1,0,0)$, $\boldsymbol{j} = (0,0,1,0)$ and $\boldsymbol{k} = (0,0,0,1)$. We consider the upper half of 3-space $\mathsf{R}^{3+}$ as $w = z + y\boldsymbol{j}$ with $z \in \mathsf{C}$, $y \in \mathsf{R}^+$. Ahlfors' arguments, as given in [1] (or using the results contained herein), show that the Möbius transformation $z \mapsto (az+)(cz+d)^{-1}$ may have its action extended to $\mathsf{R}^{3+}$ by simply writing $w \mapsto (aw+b)(cw+d)^{-1}$. This extension is conformal and is the orientation-preserving isometry group of $\mathscr{H}^3$ modelled on $\mathsf{R}^{3+}$.

We shall need to know that

$$\boldsymbol{j}c = \bar{c}\boldsymbol{j} \quad \text{for } c \in \mathsf{C}\,, \tag{1}$$

the conjugate of $w = a + \boldsymbol{i}b + \boldsymbol{j}c + \boldsymbol{k}d$ for $a, b, c, d \in \mathsf{R}$ is

$$\bar{w} = a - \boldsymbol{i}b - \boldsymbol{j}c - \boldsymbol{k}d \tag{2}$$

and

$$w^{-1} = \bar{w}/\|w\|^2\,. \tag{3}$$

The absolute value or modulus of a quaternion is multiplicative. Also we define

$$\tilde{w} = a - \boldsymbol{i}b - \boldsymbol{j}c + \boldsymbol{k}d\,.$$

For $A, B, C, D \in \mathsf{H}$ with $(AC^{-1}D - B)C \neq 0$, set

$$\gamma : \hat{\mathsf{H}} \to \hat{\mathsf{H}}$$
$$w \mapsto (Aw+B)(Cw+D)^{-1}$$

where $\hat{\mathsf{H}} = \mathsf{H} \cup \{\infty\}$.

LEMMA 1. *$\gamma$ is a conformal diffeomorphism of $S^4$.*

PROOF (The elegance of this argument clearly shows the hand of Lars Ahlfors.) If $C=0$, $\gamma$ is trivially conformal, so we may assume that $C \neq 0$.

$$\gamma(w)(Cw+D) = Aw+B$$

and

$$\gamma(w')(Cw'+D) = Aw'+B .$$

Therefore

$$\gamma(w)C(w-w')+[\gamma(w)-\gamma(w')][Cw'+D] = A(w-w')$$

or

$$[\gamma(w)-\gamma(w')][Cw'+D] = [A-\gamma(w)C][w-w'] .$$

Thus

$$d\gamma \cdot (Cw+D)(dw)^{-1} = A-\gamma(w)C \tag{4}$$

or

$$\|d\gamma\| \cdot \|Cw+D\| \, \|dw\|^{-1} = \|A-\gamma(w)C\| . \tag{5}$$

We compute the right hand side of (4),

$$A-\gamma(w)C = [AC^{-1}D-B][Cw+D]^{-1}C . \tag{6}$$

Hence

$$\|A-\gamma(w)C\| = \|AC^{-1}D-B\| \, \|Cw+D\|^{-1}\|C\| . \tag{7}$$

Thus using (5) and (7) we obtain

$$\|d\gamma\| = \frac{\|(AC^{-1}D-B)C\| \, \|dw\|}{\|Cw+D\|^2} = \text{Const.}\frac{\|dw\|}{\|Cw+D\|^2} . \tag{8}$$

It is then immediate that $\gamma$ is conformal at each point. By Liouville's theorem, $\gamma$ is locally, hence globally the restriction of a Möbius transformation, hence is a diffeomorphism of $S^4$.

Set

$$\hat{\gamma} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{for } A, B, C, D \in \mathsf{H} .$$

We denote by Gl $(2, \mathsf{H})$ the set of invertible 2 by 2 matrices over $\mathsf{H}$. The simple determinant condition for invertibility when the ground ring is commutative is not valid here. Invertibility is determined by the solvability of the four linear equations given by $\hat{\gamma}(\hat{\gamma})^{-1} = \text{Id}$. If

$$(\hat{\gamma})^{-1} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

then

$$\begin{aligned} A' &= (A-BD^{-1}C)^{-1} \\ B' &= (C-DB^{-1}A)^{-1} \\ C' &= -D^{-1}C(A-BD^{-1}C)^{-1} \\ D' &= -B^{-1}A(C-DB^{-1}A)^{-1} \end{aligned} \tag{9}$$

provided that these terms make sense. If the product $ABCD=0$, i.e. at least one entry is zero, either $AD\neq 0$ or $BC\neq 0$ suffices to insure invertibility. Thus in examining (9) we may assume that $ABCD\neq 0$. Note that

$$(A-BD^{-1}C)C^{-1}D = AC^{-1}D-B$$

or $(A-BD^{-1}C)^{-1}$ exists if and only if $AC^{-1}D-B\neq 0$. Similarly $(C-DB^{-1}A)^{-1}$ exists under precisely the same conditions. This gives us the generalization of the determinant condition, namely either $AC^{-1}D-B\neq 0$ or $AD\neq 0$ and $C=0$. (See also Section 4.)

LEMMA 2.

$$\begin{aligned} \Phi &: \mathrm{Gl}\,(2,\mathsf{H}) \to \mathrm{Conf}_+\,\hat{\mathsf{H}} \\ &: \hat{\gamma} \mapsto \gamma \end{aligned}$$

*is a group epimorphism. Here* $\mathrm{Conf}_+\,\hat{\mathsf{H}}$ *is the group of directly conformal self-maps of* $\hat{\mathsf{H}}=S^4$.

PROOF. That $\Phi$ is a homomorphism is a direct computation. To see that it is surjective, observe that it is a simple consequence of Liouville's theorem that $\mathrm{Conf}_+\,\hat{\mathsf{H}}$ is the semi-direct product of the group of similarities of $\mathsf{H}=E^4$ and the group of order two generated by $\gamma_1$: $w \mapsto w^{-1}$. The dilations of $E^4$ clearly lie in $\mathrm{Im}\,\Phi=\Phi(\mathrm{Gl}\,(2,\mathsf{H}))$. We must only show that each rotation $\gamma \in \mathrm{SO}\,(4)$, i.e. the rotations about zero, lies in $\mathrm{Im}\,\Phi$. For $p,q \in \mathsf{H}$ having norm one, $w \mapsto qwp^{-1}$ is obviously a rotation and lies in $\mathrm{Im}\,\Phi$. Conversely, if $\gamma \in \mathrm{SO}\,(4)$ and $\gamma(1)=q^{-1}$, then $q\gamma$ is a rotation of $E^4$ fixing 0 and 1, hence is a rotation of 3-space belonging to the 3-dimensional connected Lie group SO (3). The group

$$\{w \mapsto pq\gamma(w)p^{-1} \mid p \in \mathsf{H},\ \|p\|=1\}$$

is also a 3-dimensional connected Lie group of directly conformal maps of $\mathsf{H}$ which fix 0 and 1. It follows that each $\gamma \in \mathrm{SO}\,(4)$ is of the form $w \mapsto qwp^{-1}$, hence lies in $\mathrm{Im}\,\Phi$ which completes the proof.

The *Cayley transform* is the map

$$\chi : w \mapsto j(w-j)(w+j)^{-1}$$

with matrix

$$\begin{pmatrix} j & 1 \\ 1 & j \end{pmatrix}.$$

Clearly $\chi$ satisfies the conditions of Lemma 1. Its inverse $\chi^{-1}$ has matrix

$$\hat{\chi}^{-1} = \begin{pmatrix} -\frac{j}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{j}{2} \end{pmatrix}.$$

THEOREM 1.

$$\chi \circ \mathrm{Conf}_+ \mathbf{R}^{3+} \circ \chi^{-1} = \mathrm{Conf}_+ \Delta^3$$

where

$$\Delta^3 = \{x \in \mathbf{H} : \|x\| < 1, \; x = (x_1, x_2, x_3, 0)\}.$$

PROOF. Suppose $w = a + ib + jy$. Since the norm is multiplicative, it follows that

$$\|\chi(w)\| = \frac{a^2+b^2+(y-1)^2}{a^2+b^2+(y+1)^2} < 1$$

precisely when $y > 0$. Direct computation also shows that $\chi(w)$ has no $k$ component.

Pure matrix multiplication yields

COROLLARY 1. *The directly conformal group of $\Delta^3$ is the set of maps*

(10) $$\chi \circ \hat{\gamma} \circ \chi^{-1} : w \mapsto (aw+b)(\tilde{b}w+\tilde{a})^{-1}$$

*where*

$$\hat{\gamma} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}(2, C)$$

*and*

(11) $$a = \tfrac{1}{2}[(\bar{A}+D)+j(B-\bar{C})]$$

(12) $$b = \tfrac{1}{2}[(C+\bar{B})+j(A-\bar{D})].$$

An obvious consequence of the preceding corollary is

COROLLARY 2. *The group of orientation—preserving isometries of* $\mathscr{H}^3$, *denoted* $\mathrm{Isom}_+ \mathscr{H}^3$, *is parametrized by a sublocus of the quaternionic hyperbola* $\|a\|^2 - \|b\|^2 = 1$ *of the form given by equation* (10) *with a and b given by equations* (11) *and* (12).

PROOF. $\gamma\colon w \mapsto (aw+b)(\tilde{b}w+\tilde{a})^{-1}$ is conformal by Lemma 1 since the condition for conformality is the same as the condition for invertibility of $\hat{\gamma}$. Direct computation shows that $\|a\|^2 - \|b\|^2 = 1$. Further $\|d\gamma\| = K\|dw\|/\|\tilde{b}w+\tilde{a}\|^2$ where $K = \|(a\tilde{b}^{-1}\tilde{a}-b)\tilde{b}\|$. If $K=1$, then $\gamma$ is a Poincaré isometry. To verify that indeed $K=1$, we note that $\gamma\,|\,\partial\Delta^3$ is a diffeomorphism, hence

$$\int_{\partial\Delta^3} dA = \int_{\partial\Delta^3} \|\gamma'\|^2\, dA = 4\pi .$$

Let $R = \|\tilde{b}^{-1}\tilde{a}\|$, then

$$(1)3 \qquad 4\pi = \int_{\partial\Delta^3} \|\gamma'\|^2\, dA = K^2\|\tilde{b}\|^{-4} \int_{\partial\Delta^3} \|w-\tilde{b}^{-1}\tilde{a}\|^{-4}\, dA .$$

In spherical coordinates, assume, by rotation of $\Delta^3$ if necessary, that $\tilde{b}^{-1}\tilde{a}$ lies on the vertical axis. Then

$$(14) \qquad \int_{\partial\Delta^3} \|w-\tilde{b}^{-1}\tilde{a}\|^{-4}\, dA = \int_0^{2\pi}\int_0^{\pi} \frac{\sin\varphi\, d\varphi d\theta}{(R^2+1-2R\cos\varphi)^2} = \frac{4\pi}{(R^2-1)^2} .$$

Using (13) and (14) we obtain

$$K^2 = \|\tilde{b}\|^4 (R^2-1)^2 .$$

But $R = \|\tilde{a}\|/\|\tilde{b}\| = \|a\|/\|b\|$, hence

$$K^2 = \|b\|^4 \left(\frac{\|a\|^2 - \|b\|^2}{\|b\|^2}\right)^2 = 1 .$$

Since $K \geqq 0$ we are done.

An alternate proof, using the fact that isometric spheres meet $\partial\Delta^3$ orthogonally, has been given by Andy Haas.

COROLLARY 3. *If* $w_0 \in \Delta^3$, *then the map*

$$\gamma : w \mapsto \frac{a}{\|a\|}(w+a^{-1}b)\frac{\bar{\tilde{a}}}{\|a\|}(1+\tilde{a}^{-1}\tilde{b}w)^{-1}$$

*with* $w_0 = -a^{-1}b$ *maps* $w_0$ *to* 0.

The proof of Corollary 3 is trivial but some care must be taken in the choice of $a$ and $b$ to assure that $\gamma$ is an automorphism of $\Delta^3$.

COROLLARY 4. *If* $\gamma\colon w \mapsto (aw+b)(\tilde{b}w+\tilde{a})^{-1}$ *with* $a$ *and* $b$ *as above, then*

$$d(0,\gamma(0)) = C\log(\|a\|+\|b\|)$$

*where* $d$ *is the hyperbolic distance and* $C$ *depends only on the curvature of the hyperbolic metric.* $C=1$ *for curvature* $-1$.

Again the proof is a trivial computation. The result together with the invariance of the hyperbolic metric under $\mathrm{Isom}_+\,\mathscr{H}^3$ has the immediate consequence that a subgroup $G$ of $\mathrm{Isom}_+\,\mathscr{H}^3$ is discrete if and only if it acts properly discontinuously on $\mathscr{H}^3$. One should also compare this result with some of the characterizations of the exponent of convergence of Poincaré series in Sullivan [11].

## 2. The Ford region and the Poincaré estimate.

When we study the bounded model for $\mathscr{H}^3$ we may conjugate a discrete group $G$ of motions so that 0 is not a fixed point. As in the case of $\mathscr{H}^2$, we may define the Ford fundamental region in the following manner. Each $\gamma \in G$ has an isometric sphere

$$I(\gamma) = \{w \in \Delta^3 \mid \|\gamma'(w)\|=1\} \quad \text{and} \quad \gamma(I(\gamma)) = I(\gamma^{-1}).$$

Clearly the condition that $\|\gamma'(w)\|=1$ is equivalent to $\|d\gamma(w)\|=\|dw\|$ or $\|\tilde{b}w+\tilde{a}\|=1$. The interior of

$$F = \bigcap_{\gamma\in G\setminus\{I\}} \mathrm{Ext}\, I(\gamma)$$

is the *Ford region* for the action of $G$ on $\mathscr{H}^3$. The standard proof (see Ford [4] or Lehner [9] for details) shows that it is a fundamental region for the action of $G$ on $\mathscr{H}^3$.

An immediate consequence is the Poincaré estimate.

THEOREM 2. *If* $G$ *is a group of orientation preserving isometries of* $\mathscr{H}^3$ *in the bounded model, then* $G$ *is non-discrete if and only if either*

(a) $\sum_* \|b\|^{-6}=\infty$ *where* $\sum_*$ *is the sum over* $\gamma \in G\setminus G_0$ *or*

(b) $G_0$ *is non-discrete.*

*Here $G_0$ is the subgroup of $G$ fixing* 0.

PROOF. Let us first assume $G$ is discrete and $G_0=\{1\}$. Then $0 \in F$ and for some $\varepsilon>0$, $B_\varepsilon(0)\subset F$. Now

$$\tfrac{4}{3}\pi > \operatorname{vol} GB_\varepsilon(0) = \sum_{\gamma\in G} \iint_{B_\varepsilon(0)} \|\gamma'(w)\|^3\, dv .$$

Since $\|\gamma'(w)\| = \|\tilde{b}\|^{-2}\|w-\tilde{b}^{-1}\tilde{a}\|^{-2}$ and $\|w-\tilde{b}^{-1}\tilde{a}\|>1-\varepsilon$ we have

$$\operatorname{vol} GB_\varepsilon(0) \geqq \sum_{G\setminus\{1\}} \|\tilde{b}\|^{-6}|1-\varepsilon|^{-6}\operatorname{vol} B_\varepsilon(0)$$

or

$$\infty > \sum_{G\setminus\{1\}} \|\tilde{b}\|^{-6} .$$

If $G_0 \neq \{1\}$ and $G$ is discrete we may find some $B_\varepsilon(w_0)$ lying in some fundamental set $F$ and repeat the above computation deleting all terms where $\gamma \in G_0$. Thus whenever $G$ is discrete

$$\sum_{*} \|b\|^{-6} < \infty .$$

If $G$ is not discrete then for any $\gamma \in G$ there is a sequence $\gamma_n \to \gamma$. Write

$$\gamma_n = \begin{pmatrix} * & * \\ c_n & * \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} * & * \\ c & * \end{pmatrix}.$$

Since $c_n \to c$, if $c\neq 0$ (a) is valid. If $c=0$ and some $c_n\neq 0$ repeat the above argument with $\gamma=\gamma_n$ and find a new sequence converging to $\gamma=\gamma_n$. Otherwise $c_n=0$ for all but finitely many $n$. But if $c_n=0$, then $\gamma_n \in G_0$ and $G_0$ is not discrete.

When $G$ is Kleinian of the second kind we may examine areas on $\partial\Delta^3$ instead of volumes and obtain

THEOREM 3. *If $G$ is a Kleinian group of the second kind,*

$$\sum_{*} \|b\|^{-4} < \infty .$$

For specific Kleinian groups of the second kind, Sullivan [11] has found sharp estimates of the exponent of convergence of these groups which are given in terms of the Hausdorff dimension of their limit sets. The above results may be rewritten as conditions on the matrices $\gamma$ in SL$(2,\mathbf{C})$ as follows:

THEOREM 4. *If* $G < \mathrm{SL}(2, \mathbf{C})$,

$$G = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\},$$

*then we have*

(i) $\sum_* (|C+\bar{B}|^2 + |A-\bar{D}|^2)^{-3} < \infty$ *if G is discrete.*

(ii) $\sum_* (|C+\bar{B}|^2 + |A-\bar{D}|^2)^{-2} < \infty$ *if G acts on* $\mathscr{H}^3$ *as a Kleinian group of the second kind.*

(iii) $\sum_* (|C+\bar{B}|^2 + |A-\bar{D}|^2)^{-3} = 0, \infty$ *if G is nondiscrete or* $G = G_0$ *where* $G_0$ *is the isotropy group of* 0.

There is a partial converse to Theorem 4 (ii) (see also Beardon and Nicholls [2] and Sullivan [11]).

THEOREM 5. *If* $G < \mathrm{SL}(2, \mathbf{C})$ *is discrete and has a Dirichlet region of finite volume, then*

$$\sum_* (|C+B|^2 + |A-D|^2)^{-2} = \infty .$$

PROOF. Nicholls [10] has shown that the hypothesis of the Theorem implies the existence of a constant $B(G) > 0$ so that

$$n(0, r) \geqq \frac{B(G)}{(1-r)^2},$$

where $n(0, r)$ is the cardinality of points lying both in the orbit of 0 and in the ball $A_r$ of Euclidean radius $r$ about 0. If $\gamma \in G$ satisfies $\gamma(0) \in A_r$, then $1 - \|\gamma(0)\| \geqq 1 - r$. However

$$1 - \|\gamma(0)\|^2 = 1 - \|\tilde{b}/\tilde{a}\|^2 = \|a\|^{-2}$$

and

$$\|\gamma'(0)\| = \|\tilde{a}\|^{-2} \leqq 2(1 - \|\gamma(0)\|) \leqq 2(1-r)$$

when $\gamma(0) \in A_r$, $\gamma\colon w \mapsto (aw+b)(\tilde{b}w+\tilde{a})^{-1}$ and $\|a\|^2 - \|b\|^2 = 1$. It follows that for all but finitely many $\gamma \in G$, $\|\gamma'(0)\|^2 \leqq 4(1-r)^2$ whenever $\gamma(0) \in A_r$. Thus

$$\sum_{\substack{\gamma \in G \\ \gamma(0) \in A_r}} \|\gamma'(0)\|^{-2} \geqq \frac{4(1-r)B(G)}{(1-r)^2}.$$

Letting $r \to 1$, this sum goes to $\infty$. The desired conclusion follows immediately from the observation that

$$\sum_{*} \|\tilde{b}\|^{-4} = \sum_{*} (\|a\|^2 - 1)^{-2} = \infty .$$

**3. Poincaré series for Kleinian groups $G$.**

When $w \in \mathsf{H}$ we may define a tangent vector $dw$ to $\mathsf{H}$ by identifying $\mathsf{H}$ with $\mathsf{R}^4$. Similarly we may take the $q$-fold tensor product $\mathsf{H}^q = \otimes_{k=1}^{q} \mathsf{H}$ which is both right and left multilinear. Similarly we may form $\mathsf{H}^q$-bundles $E^q$ over $\Delta^i = B_1(0)$ in $\mathsf{R}^i$ for $i = 3, 4$. A section $F\colon \Delta^i \to E^q$ is called a *$q$-form* for the discrete group $G < \mathrm{Isom}_+ \mathscr{H}^3$ if

$$(15) \qquad (F \circ \gamma)(w)\gamma'(w)^q = F(w)$$

for all $\gamma \in G$. By combining equations (4) and (6), we obtain a formula for $d\gamma$ in terms of $dw$. It is worthwhile to note however that $\gamma$ acts as a transformation of $\mathsf{R}^4 = \mathsf{C}^2$. In that notation, the derivative of $\gamma$ is a matrix $D_\gamma(w)$ and $d\gamma = D_\gamma(w)dw$. The chain rule then gives

$$(16) \qquad D_{\gamma\eta}(w) = D_\gamma(\eta(w))D_\eta(w) .$$

Since $\gamma$ and $\eta$ are assumed conformal, each $D_\gamma$ factors as a positive real multiple of the identity matrix composed with a special orthogonal matrix. A special orthogonal matrix is, for our purposes, multiplication by a unit quaternion. Thus the matrix $D_\gamma$ in a left action on $T\Delta^i$ may be represented by a quaternionic action satisfying the chain rule (16). By $\gamma'(w)$ we shall denote the quaternion $D_\gamma(w)$. An automorphic $q$-form is thus a tensor valued function satisfying (15). $\|D_\gamma(w)\| = \|Cw + D\|^{-2}$ where $\gamma(w) = (Aw + B)(Cw + D)^{-1}$. We may form the Poincaré $q$-series

$$\Theta f(w) = \sum_{G} (f \circ \gamma)(w)\gamma'(w)^q \underbrace{(1 \otimes 1 \otimes \ldots \otimes 1)}_{q \text{ times}}$$

$$= \sum_{G} (f \circ \gamma)(w)(\gamma'(w) \otimes \ldots \otimes \gamma'(w)) .$$

By the chain rule and multilinearity

$$\Phi f(\eta(w))\eta'(w)^q = \Theta f(w) .$$

Poincaré's proof (see Lehner [9]) for the convergence of such series remains valid for $q \geqq 3$. $f$ covers a tensor valued form on $\Delta^i/G$.

If $G$ is a Kleinian group and $w_1 \notin Gw_2$ and $\|w_i\| \neq 1$, then for $f_i = (w - w_i)^{-1}$, $\hat{T}(w) = \Theta f_1/\Theta f_2$ is an automorphic, non-constant extended quaternion valued function on $\mathscr{H}^3$, hence projects to a non-constant function $T(w)$ on $\Delta^3/G$.

THEOREM 6 (Quaternionic uniformization). *If $T_1$ and $T_2$ are as above, then the map*

$$\begin{aligned} \Delta^3 &\to \hat{\mathsf{H}}^2 = (\mathsf{H} \cup \{\infty\})^2 \\ w &\mapsto (T_1(w), T_2(w)) \end{aligned}$$

*is a quaternionic uniformization of* $\Delta^3/G$.

Note that Fueter [6] obtained a similar result for the Picard group using Eisenstein series.

**4. Concluding comments.**

Stereographic projection of the complex plane onto the unit sphere in $\mathsf{R}^3$ may be realized by composing the Cayley transform with complex conjugation (see also Gormley [8]). Although at first glance this may seem odd, looking at the unit sphere from the inside or outside involves an anti-conformal transformation of the position of the observer.

In addition to informing me of the existence of Gormley's paper cited above, Werner Fenchel has found an elementary proof unifying the treatments of Lemmas 1 and 2. In outline the proof follows. There are normal forms for quaternionic Möbius transformations, namely

$$w \mapsto awd^{-1} + bd^{-1} \qquad \text{if } c = 0$$

and

$$w \mapsto ac^{-1} - (ac^{-1}d - b)(cw + d)^{-1} \qquad \text{if } c \neq 0 \,.$$

Consequently, $\operatorname{Im} \Phi$ is generated, as is $\operatorname{Conf}_+ \hat{\mathsf{H}}$, by the map

$$\gamma_1 : w \mapsto w^{-1}$$

and by Euclidean similarities. This proof does not display the norm of $d\gamma$ which we later need. However he also notes that Corollary 2 and the conditions for invertibility of elements of $Gl(2, \mathsf{H})$ may be derived directly using Dieudonné's theory of determinants in skew fields [3]. I feel that the indicated integration and computations are sufficiently elementary to warrant inclusion here.

The results presented in the second and third sections suggest that hyperbolic 3 and 4-manifolds and orbifolds may possess algebraic geometric structure over $\mathsf{H}$. The main difficulties in developing this structure are immediately apparent. Neither the algebraic nor the analytic machinery seems well in hand.

REFERENCES

1. L. Ahlfors, *Möbius Transformations in Several Dimensions*, Lectures at the University of Minnesota, 1981.

2. A. Beardon and P. Nicholls, *On classical series associated with Kleinian groups*, J. London Math. Soc. (2) (1972), 645–655.
3. J. Dieudonne, *Les determinantes sur un corps non commutatif*, Bull. Soc. Math. France 71 (1943), 27–45.
4. L. Ford, *Automorphic Functions*, 2nd Ed., Chelsea, New York (1951).
5. R. Fueter, *Über automorphe Funktionen in bezug auf Gruppen, die in den Ebene uneigenlich diskontinuierlich sind*, J. Reine Angew. Math. 157 (1927), 66–78.
6. R. Fueter, *Über automorphe Funktionen der Picard'schen Gruppe* I, Comm. Math. Helv. 3 (1931), 42–68.
7. F. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. 103 (1962), 353–393.
8. P. Gormley, *Stereographic projection and the linear fractional group od quaternions*, Proc. Roy. Irish Acad. Sect. A 6 (1947), 67–85.
9. J. Lehner, *Discontinuous Groups and Automorphic Functions* (Math. Surveys 8), American Mathematical Society, Providence, R.I., 1964.
10. P. Nicholls, *On the distribution of orbits in a Kleinian group*, Proc. London Math. Soc. (3) 29 (1974), 193–215.
11. D. Sullivan, *A finiteness theorem for cusps*, Acta Math. 147 (1981), 289–299.
12. K. Vahlen, *Ueber Bewegungen und complexe Zahlen*, Math. Ann. 55 (1902), 585–593.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CONNECTICUT
STORRS, CONNECTICUT 06268
U.S.A.