MATH. SCAND. 54 (1984), 27-39

A FIXED POINT THEOREM FOR C*-CROSSED PRODUCTS WITH AN ABELIAN GROUP

GUY HENRARD

Abstract.

Let G be an abelian group and $[A, G, \alpha]$ a C*-dynamical system. We define a C*-algebra $B_u(A, G)$ (by giving generators), independent of the action α, and an action θ of G on $B_u(A, G)$ so that the C*-crossed product $G \times_\alpha A$ is the fixed point algebra in $B_u(A, G)$ for the action θ.

Introduction.

Let $(\mathcal{M}, G, \alpha)$ be a W*-dynamical system. It was proved by M. Takesaki [11], generalized or proved in another way by T. Digernes [3, 4], U. Haagerup [5] and A. Van Daele [13] that the W*-crossed product $G \times_\alpha \mathcal{M}$ is the fixed point algebra in $\mathcal{M} \otimes B(L^2(G))$ for the action $\theta = \alpha \otimes \operatorname{ad} \tilde{\lambda}$, where $B(L^2(G))$ is the Von Neumann algebra of bounded linear operators on the Hilbert space $L^2(G)$ and $\tilde{\lambda}$ is the right regular representation of G on $L^2(G)$. In the case of abelian groups this fixed point theorem can be seen as a consequence of the following two theorems:

1) $\mathcal{M}$ is isomorphic to the fixed point algebra of the W*-crossed product $G \times_\alpha \mathcal{M}$ for the dual action $\hat{\alpha}$, [9], [11], [13].

2) The duality theorem: $\hat{G} \times_{\hat{\alpha}} G \times_\alpha \mathcal{M}$ is isomorphic to $\mathcal{M} \otimes B(L^2(G))$, where $\hat{G}$ is the dual group of G, [9], [11], [13].

If $[A, G, \alpha]$ is a C*-dynamical system, then the duality theorem of Takai is available [10]. However a fixed point theorem for C*-crossed products is not obtained in this way as, unless G is discrete, the C*-algebra A is not imbedded in the C*-crossed product $G \times_\alpha A$. When G is compact ($\hat{G}$ is discrete), $G \times_\alpha A$ is the fixed point algebra in $A \otimes \mathcal{K}(L^2(G))$ for the action $\theta = \alpha \otimes \operatorname{ad} \tilde{\lambda}$, where $\mathcal{K}(L^2(G))$ is the C*-algebra of compact operators on $L^2(G)$. This is a consequnce of the duality theorem of Takai and a theorem of Landstad [8]. So, for compact groups the analogy to the W*-case is complete. For non-abelian compact groups this fixed point theorem is proved by A. J. Wassermann [14].

Received August 30, 1982.

If $[A, G, \alpha]$ is a C*-dynamical system with an abelian group G we define a C*-algebra $B(A, G, \alpha)$ so that $G \times_\alpha A \subseteq B(A, G, \alpha) \subseteq M(\hat{G} \times_{\hat{\alpha}} G \times_\alpha A)$, where $M(\hat{G} \times_{\hat{\alpha}} G \times_\alpha A)$ is the multiplier algebra of the double crossed product $\hat{G} \times_{\hat{\alpha}} G \times_\alpha A$ [8], [9]. We prove that $B(A, G, \alpha)$ admits the bidual action $\hat{\hat{\alpha}}$ and that $G \times_\alpha A$ is the fixed point algebra in $B(A, G, \alpha)$ for the action $\hat{\hat{\alpha}}$. By using the same isomorphisms as in the duality theory of Takai, $B(A, G, \alpha)$ is transformed to an algebra, $B(A, G)$, independent of α. In this way we get a fixed point theorem for the C*-crossed product $G \times_\alpha A$. The action θ on $B(A, G)$ is the transformed action of $\hat{\hat{\alpha}}$ by the isomorphisms mentioned above. This action is in general not continuous on $B(A, G)$ but for uniformly continuous C*-dynamical systems [9, 8.5] we can define a C*-subalgebra $B_u(A, G)$ of $B(A, G)$ on which θ is continuous. If G is compact the theorem coincides with the known theorem of Landstad, Takai and Wasserman. Unless G is compact the C*-algebras $B(A, G)$ and $B_u(A, G)$ are not tensorproducts of A with an operator algebra on $L^2(G)$; so in general this fixed point theorem has very little analogy to the Von Neumann case. Therefore, those C*-dynamical systems for which analogy is almost complete will be described in a forthcoming paper.

For the theory of C*-algebras and their crossed products we refer to the book of G. K. Pedersen [9]; for the abstract harmonic analysis we refer to the books of E. Hewitt and K. A. Ross [7].

I wish to express my gratitude to my supervisor Alfons Van Daele for giving both the idea of this paper and many suggestions. I wish also to thank Dorte Olesen and Gert K. Pedersen for their warm hospitality during my visit, where I had a chance to have some discussions on the subject. Finally I thank my colleagues at the Mathematics Institute of Leuven for their interest.

1. Notation and preliminaries.

Throughout this paper G will denote a locally compact abelian group and $\hat{G}$ its dual group. If A is a C*-algebra we denote by $C_c(G, A)$, $C_0(G, A)$, $C_u^b(G, A)$, $C^b(G, A)$ the sets of continuous functions from G to A, with compact support, vanishing at infinity, bounded uniformly continuous, bounded. If $\mathscr{H}$ is a Hilbert space we denote by $L^2(G, \mathscr{H}) = \mathscr{H} \otimes L^2(G)$ the Hilbert space of square integrable functions from G to $\mathscr{H}$. If A is a C*-algebra of operators on the Hilbert space $\mathscr{H}$ and y is in $C^b(G, A)$ we denote by m_y the operator on $L^2(G, \mathscr{H})$ defined by $m_y\xi(t) = y(t)\xi(t)$, where $\xi \in L^2(G, \mathscr{H})$ and $t \in G$. Then, $\|m_y\| = \|y\|_\infty$ where $\|y\|_\infty$ is the sup-norm of y. We denote by λ the left regular representation of G on $L^2(G, \mathscr{H})$: $\lambda_s\xi(t) = \xi(s^{-1}t)$ where $\xi \in L^2(G, \mathscr{H})$ and $s, t \in G$. The representation λ induces a representation of $L^1(G)$ on $L^2(G, \mathscr{H})$ which is also denoted by

$$\lambda : \lambda_f \xi(t) = \int \xi(s^{-1}t) f(s)\, ds$$

where $f \in L^1(G)$. $\tilde{\lambda}$ is defined by $(\tilde{\lambda}_t \xi)(s) = \xi(st)$. We denote by V the unitary representation on $\hat{G}$ on $L^2(G, \mathscr{H})$ defined by $V_\sigma \xi(t) = \langle t, \sigma \rangle \xi(t)$ where $\sigma \in \hat{G}$, $\xi \in L^2(G, \mathscr{H})$, $t \in G$ and $\langle t, \sigma \rangle$ is the image of t by the character σ. If $f \in L^1(G)$ and $\sigma \in \hat{G}$, we denote by f_σ the function in $L^1(G)$, so that $f_\sigma(s) = \langle s, \sigma \rangle f(s)$, where $s \in G$. If μ is a bounded measure on G, we denote by $\hat{\mu}$ the (inverse) Fourier transform of

$$\mu : \hat{\mu}(\sigma) = \int \langle t, \sigma \rangle \, d\mu(t)$$

where $\sigma \in \hat{G}$.

An action α of G on a C*-algebra A is a homomorphism of G into the group of automorphisms of A, Aut A. The action α is continuous if for each $a \in A$ the map $s \mapsto \alpha_s(a)$ is continuous; α is uniformly continuous if the continuity is uniformly on the unit ball of A. A triple $[A, G, \alpha]$ of a C*-algebra A, a locally compact group G and a continuous action α of G on A is called a C*-dynamical system [9].

Let $[A, G, \alpha]$ be a C*-dynamical system and $\mathscr{H}$ a Hilbert space so that A acts non-degenerately on $\mathscr{H}$. Denote by Π_α the representation on $L^2(G, \mathscr{H})$ of $C_c(G, A)$ with the $L^1(G, A)$-norm defined by

$$\Pi_\alpha(y)\xi(t) = \int \alpha_{t^{-1}}(y(s))\xi(s^{-1}t)\, ds$$

where $y \in C_c(G, A)$, $\xi \in L^2(G, \mathscr{H})$ and $t \in G$. If $a \in A$ and $f \in C_c(G)$ we denote $\Pi_\alpha(a, f)$ instead of $\Pi_\alpha(a \otimes f)$. It is known that if G is amenable (abelian groups are) the C*-crossed product $G \underset{\alpha}{\times} A$ is isomorphic to the C*-algebra of operators on $L^2(G, \mathscr{H})$ generated by $\{\Pi_\alpha(y) \mid y \in C_c(G, A)\}$ [9], [10]. If $a \in A$, we denote by $\Pi_\alpha(a)$ the operator on $L^2(G, \mathscr{H})$ defined by:

$$\Pi_\alpha(a)\xi(t) = \alpha_{t^{-1}}(a)\xi(t)$$

where $\xi \in L^2(G, \mathscr{H})$ and $t \in G$. Π_α is a faithful representation of A into the multiplier algebra of $G \underset{\alpha}{\times} A$, denoted by $M(G \underset{\alpha}{\times} A)$ [1], [8], [9]. The action of $\hat{G}$ on $G \underset{\alpha}{\times} A$ defined by ad V is called the dual action and denoted by $\hat{\alpha}$.

In the following sections $[A, G, \alpha]$ will be a C*-dynamical system and $\mathscr{H}$ a Hilbert space so that A acts non-degenerately on $\mathscr{H}$.

2. The C*-algebras $B(A, G)$ and $B_u(A, G)$.

2.1. DEFINITION. We define the C*-algebra $B(A, G)$ as the C*-algebra of operators on $L^2(G,\mathscr{H})$ generated by operators $m_y\lambda_f$ where $y \in C^b(G,A)$ and $f \in C_c(G)$. We denote by $B_u(A,G)$ the C*-subalgebra generated by operators $m_y\lambda_f$ where $y \in C_u^b(G,A)$ and $f \in C_c(G)$.

We first state a relation between the defined C*-algebras and $A \otimes \mathscr{K}(L^2(G))$.

2.2. LEMMA.

$$A \otimes \mathscr{K}(L^2(G)) \subseteqq B_u(A,G) \subseteqq B(A,G) \subseteqq M(A \otimes \mathscr{K}(L^2(G)) .$$

PROOF. If $a \in A$ we denote by y_a the constant function a from G to A. If $f,g \in C_c(G)$, then $a \otimes m_f\lambda_g = m_{y_a f}\lambda_g$ belongs to $B_u(A,G)$. As $A \otimes \mathscr{K}(L^2(G))$ is generated by elements of the form $a \otimes m_f\lambda_g$, the first inclusion is proved. The second inclusion is clear by definition, so it suffices to prove the last one.

For $y \in C^b(G,A)$, $a \in A$ and $f,g,h \in C_c(G)$ one has:

$$m_y\lambda_f(a \otimes m_g\lambda_h) = m_y(a \otimes \lambda_f m\lambda_{gh}) .$$

As $\lambda_f m_g \lambda_g$ can be approximated by elements of the form $\sum_i m_{f_i}\lambda_{g_i}$ ($f_i, g_i \in C_c(G)$), we have that $m_y\lambda_f(a \otimes m_g\lambda_h)$ is approximated by elements of the form $\sum_i m_{y_i}\lambda_{g_i}$ ($y_i \in C_c(G,A)$). As these elements belong to the closure of $(C_c(G,A).\mathrm{C}^*(G)) = A \otimes \mathscr{K}(L^2(G))$ it is proved that $m_y\lambda_f$ is a left multiplier of $A \otimes \mathscr{K}(L^2(G))$. In a similar way $m_y\lambda_f$ will be a right multiplier and as $B(A,G)$ is generated by operators $m_y\lambda_f$ ($y \in C^b(G,A)$, $f \in C_c(G)$), the lemma is proved.

We will now prove that the definition of the C*-algebras $B(A,G)$ and $B_u(A,G)$ is independent of the particular representation of A as an operatoralgebra on a Hilbert space $\mathscr{H}$.

So let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and A_1 and A_2 C*-algebras acting non-degenerately on $\mathscr{H}_1$ and $\mathscr{H}_2$ and suppose that φ is an isomorphism from A_1 onto A_2. Denote by $\varphi \otimes 1$ the isomorphism from $A_1 \otimes \mathscr{K}(L^2(G))$ onto $A_2 \otimes \mathscr{K}(L^2(G))$ so that $(\varphi \otimes 1)(a \otimes T) = \varphi(a) \otimes T$ where $a \in A$ and $T \in \mathscr{K}(L^2(G))$ [12, IV.4.22].

If B is a C*-algebra, the left strict topology on $M(B)$ is the topology induced by the maps $x \mapsto \|xb\|$ where $x \in M(B)$, $b \in B$. The strict topology is the topology induced by the maps $x \mapsto \|xb\| + \|bx\|$ where $x \in M(B)$, $b \in B$. It was proved in [2] that $M(B)$ is the strict completion of B. The following lemma is well-known [2], [9, 3.12.3].

2.3. LEMMA. *If $\mathscr{K}_1$ and $\mathscr{K}_2$ are Hilbert spaces and B_1 and B_2 C*-algebras acting non-degenerately on $\mathscr{K}_1$ and $\mathscr{K}_2$ and if p is a surjective morphism from B_1 onto B_2, then, by defining $\tilde{p}(x) = \lim p(xu_\lambda)$ (the limit is taken in the weak* *operator topology) where* $x \in M(B_1)$ *and* $\{u_\lambda\}$ *is an approximate unit in* B_1, $\tilde{p}$ *becomes a morphism from* $M(B_1)$ *into* $M(B_2)$. *Furthermore* $\tilde{p}$ *is continuous for the left strict and the strict topology. If* p *is an isomorphism, then also* $\tilde{p}$.

We denote by $\tilde{\varphi}$ the extension of $\varphi \otimes 1$ to an isomorphism from $M(A_1 \otimes \mathscr{K}(L^2(G)))$ onto $M(A_2 \otimes \mathscr{K}(L^2(G)))$.

2.4. PROPOSITION. $\tilde{\varphi}$ *is an isomorphism from* $B(A_1, G)$ $(B_u(A_1, G))$ *onto* $B(A_2, G)$ $(B_u(A_2, G))$ *so that* $\tilde{\varphi}(m_y \lambda_f) = m_{\varphi \circ y} \lambda_f$, *where* $y \in C^b(G, A)$ $(C_u^b(G, A))$ *and* $f \in C_c(G)$.

PROOF. Take $y \in C^b(G, A)$ and $f \in C_c(G)$. If $\{T_\lambda\}$ is an approximate unit in $A_1 \otimes \mathscr{K}(L^2(G))$, then $m_y \lambda_f T_\lambda$ converges to $m_y \lambda_f$ in the left strict topology on $M(A_1 \otimes \mathscr{K}(L^2(G)))$; so $\tilde{\varphi}(m_y \lambda_f T_\lambda)$ converges to $\tilde{\varphi}(m_y \lambda_f)$ in the left strict topology on $M(A_2 \otimes \mathscr{K}(L^2(G)))$ (Lemma 2.3). As $(\varphi \otimes 1)(T_\lambda)$ is an approximate unit in $A_2 \otimes \mathscr{K}(L^2(G))$ we have that $m_{\varphi \circ y} \lambda_f (\varphi \otimes 1)(T_\lambda)$ converges to $m_{\varphi \circ y} \lambda_f$ in the left strict topology on $M(A_2 \otimes \mathscr{K}(L^2(G)))$. So it suffices to prove that for $T \in A \otimes \mathscr{K} L^2(G))$

$$\tilde{\varphi}(m_y \lambda_f T) = m_{\varphi \circ y} \lambda_f (\varphi \otimes 1)(T) .$$

If T is of the form $a \otimes m_g \lambda_h$, where $a \in A$, $g, h \in C_c(G)$, then by Lemma 2.2 we know that for each $\varepsilon > 0$ there exists an element of the form $\sum_{i=1}^n b_i \otimes S_i$ with $b_i \in A$, and $S_i \in \mathscr{K}(L^2(G))$ so that

$$\left\| m_y \lambda_f (a \otimes m_g \lambda_h) - \sum_{i=1}^n b_i \otimes S_i \right\| < \varepsilon$$

and

$$\left\| m_{\varphi \circ y} \lambda_f (\varphi(a) \otimes m_g \lambda_h) - \sum_{i=1}^n \varphi(b_i) \otimes S_i \right\| < \varepsilon .$$

As

$$\tilde{\varphi}\left(\sum_{i=1}^n b_i \otimes S_i \right) = \sum_{i=1}^n \varphi(b_i) \otimes S_i ,$$

the proposition is proved.

Let θ denote the action $\alpha \otimes \mathrm{ad}\, \tilde{\lambda}$ of G on $M(A \otimes \mathscr{K}(L^2(G)))$. By an analogous argument as in the preceeding proposition one can prove the following lemma

2.5. LEMMA. *For each* $t \in G$, *the restriction of* θ_t *to* $B(A, G)$ $(B_u(A, G))$ *is an automorphism of* $B(A, G)$ $(B_u(A, G))$ *so that*

$$\theta_t(m_y\lambda_f\xi)(s) = \alpha_t(y(st))\lambda_f\xi(s)$$

where $y \in C^b(G,A)$ $(C_u^b(G,A))$, $f \in C_c(G)$, $\xi \in L^2(G,\mathscr{H})$ *and* $s \in G$.

So we get the following proposition

2.6. PROPOSITION. θ *is an action of* G *on the* C*-*algebras* $B(A,G)$ *and* $B_u(A,G)$.

2.7. REMARKS.

i) If G is a compact group, then $B_u(A,G) = B(A,G) = A \otimes \mathscr{K}(L^2(G))$. So in the compact case we get the bidual system $[A \otimes \mathscr{K}(L^2(G)), G, \alpha \otimes \mathrm{ad}\,\tilde{\lambda}]$.

ii) If α is a uniformly continuous action (i.e. $s \mapsto \alpha_s(\cdot)$ is continuous uniformly on the unit ball of A), then it is easy to check that θ is a continuous action on $B_u(A,G)$ and so $[B_u(A,G), G, \theta]$ is a C*-dynamical system.

iii) In general, the action θ is not continuous on $B_u(A,G)$ as can be seen from the following example. Take $G = \mathbf{R}$, $R = C_u^b(\mathbf{R})$ and α is translarion. Define for each $t \in \mathbf{R}$ the following function $f_t \in C_u^b(\mathbf{R})$:

$$\text{for } t \geqq 2,\ f_t(s) = \begin{cases} 0 & \text{if } s \in (-\infty, 1-1/t] \\ t(s-1)+1 & \text{if } s \in [1-1/t,\ 1] \\ 1 & \text{if } s \in [1, +\infty) \end{cases}$$

for $t < 2$, $f_t = f_2$.

If $t_1, t_2 \in \mathbf{R}$, then $\|f_{t_1} - f_{t_2}\| \leqq |t_1 - t_2|$, so the function $t \mapsto f_t$ belongs to $C_u^b(\mathbf{R}, C_u^b(\mathbf{R}))$. However the function $t \mapsto \alpha_t f_{st}$ is not continuous uniformly for $s \in \mathbf{R}$.

3. A fixed point theorem for C*-crossed products.

3.1. NOTATION.

i) We denote by $\check{\Pi}$ the faithful representation of A on $L^2(G,\mathscr{H})$ defined by

$$\check{\Pi}_\alpha(a)\xi(t) = \alpha_t(a)\xi(t)\,,$$

where $a \in A$, $\xi \in L^2(G,\mathscr{H})$ and $t \in G$.

ii) If $y \in C^b(G,A)$ and $f \in C_c(G)$ and $f \in C_c(G)$ we denote by $\Pi_1^\alpha(y,f)$ the bounded linear operator on L^2 $(G \times G, \mathscr{H})$ defined by

$$\Pi_1^\alpha(y,f)\chi(s,t) = \alpha_s(y(t)) \int \chi(s, r^{-1}t) f(r)\, dr$$

where $\chi \in L^2(G \times G, \mathscr{H})$ and $s, t \in G$.

3.2. LEMMA. $\check{\Pi}_\alpha$ *induces an isomorphism* $\tilde{\check{\Pi}}_\alpha$ *from* $B(A,G)$ *onto the* C*-*algebra* *of operators on* $L^2(G \times G, \mathscr{H})$ *generated by operators* $\Pi_1^\alpha(y, f)$ *where* $y \in C^b(G, A)$ *and* $f \in C_c(G)$. *Furthermore* $\tilde{\tilde{\Pi}}_\alpha(m_y \lambda_f) = \Pi_1^\alpha(y, f)$.

PROOF. This follows from Proposition 2.4 and the identification of $L^2(G(L^2(G, \mathscr{H})))$ by $L^2(G \times G, \mathscr{H})$.

3.3. NOTATION.

i) We denote by X the linear isometry of $L^2(G \times G, \mathscr{H})$ onto $L^2(G \times G, \mathscr{H})$ defined by

$$X\chi(s,t) = \chi(s, s^{-1}t),$$

where $\chi \in L^2(G \times G, \mathscr{H})$ and $s, t \in G$.

ii) If $y \in C^b(G, A)$ we denote by α^{-1} the function from G to A defined by

$$\alpha^{-1}y(s) = \alpha_{s^{-1}}(y(s))$$

where $s \in G$. Clearly $\alpha^{-1}y \in C^b(G, A)$.

iii) If $y \in Cb(G, A)$ and $f \in C_c(G)$ we denote by $\Pi_2^\alpha(y, f)$ the bounded linear operator on $L^2(G \times G, \mathscr{H})$ defined by

$$\Pi_2^\alpha(y, f)\chi(s,t) = \alpha_{t^{-1}}(y(ts)) \int \chi(s, r^{-1}t) f(r)\, dr,$$

where $\chi \in L^2(G \times G, \mathscr{H})$ and $s, t \in G$.

3.4. LEMMA. $X^* \tilde{\tilde{\Pi}}_\alpha(B(A, G))X$ *is the* C*-*algebra of operators on* $L^2(G \times G, \mathscr{H})$ *generated by operators* $\Pi_2^\alpha(y, f)$, *where* $y \in C^b(G, A)$ *and* $f \in C_c(G)$.

PROOF. Take $y \in C^b(G, A)$, $f \in C_c(G)$, $\chi \in L^2(G \times G, \mathscr{H})$, and $s, t \in G$. Then

$$\begin{aligned} X^*\Pi_1^\alpha(\alpha^{-1}y, f)X\chi(s,t) &= \alpha_s(\alpha^{-1}y(ts)) \int X\chi(s, r^{-1}st) f(r)\, dr \\ &= \alpha_{t^{-1}}(y(ts)) \int \chi(s, r^{-1}t) f(r)\, dr \\ &= \Pi_2^\alpha(y, f)\chi(s,t). \end{aligned}$$

So $X^*\Pi_1^\alpha(\alpha^{-1}y, f)X = \Pi_2^\alpha(y, f)$ and as $y \mapsto \alpha^{-1}y$ is an isomorphism of $C^b(G, A)$ the lemma is proved.

3.5. NOTATION. i) We denote by U the linear isometry of $L^2(\hat{G} \times G, \mathscr{H})$ onto $L^2(G \times G, \mathscr{H})$ defined by

$$U\chi(s,t) = \int \chi(\tau, t)\langle s, \tau\rangle\, d\tau,$$

where $\chi \in L^1(\hat{G} \times G, \mathscr{H}) \cap L^2(\hat{G} \times G, \mathscr{H})$ and $s, t \in G$.

ii) We denote by W the linear isometry from $L^2(\hat{G} \times G, \mathscr{H})$ onto $L^2(\hat{G} \times G, \mathscr{H})$ defined by

$$W\chi(\tau, t) = \langle t, \tau \rangle \chi(\tau, t),$$

where $\chi \in L^2(\hat{G} \times G, \mathscr{H})$ and $\tau \in \hat{G}$, $t \in G$.

iii) If $y \in C^b(G, A)$ and $f \in C(G)$ we denote by $\Pi_3^\alpha(y, f)$ the bounded linear operator on $L^2(\hat{G} \times G, \mathscr{H})$ defined by

$$\Pi_3^\alpha(y, f)\chi(\tau, t) = \iint \alpha_{t^{-1}}(y(sr))U\chi(s, r^{-1}t)f(r)\overline{\langle sr, \tau \rangle}\, drds,$$

where $\chi \in L^2(\hat{G} \times G, \mathscr{H})$ so that $U\chi \in C_c(G \times G, \mathscr{H})$ and $\tau \in \hat{G}$, $t \in G$. This is well-defined as can be seen from the next lemma; also $\|\Pi_3^\alpha(y, f)\| \leqq \|y\|_\infty \|f\|_1$.

(iv) We denote by $B(A, G, \alpha)$ the C*-algebra of operators on $L^2(\hat{G} \times G, \mathscr{H})$ generated by operators $\Pi_3^\alpha(y, f)$ where $y \in c^b(G, A)$ and $f \in C_c(G)$.

3.6. Lemma. *If $y \in C^b(G, A)$ and $f \in C_c(G)$ then*

$$W^*U^*\Pi_2^\alpha(y, f)UW = \Pi_3^\alpha(y, f).$$

So

$$W^*U^*X^*\tilde{\tilde{\Pi}}_\alpha(B(A, G))XUW = B(A, G, \alpha).$$

Proof. Take $y \in C^b(G, A)$, $f \in C_c(G)$, $\chi \in C_c(G \times G, \mathscr{H})$ and $\tau \in \hat{G}$, $t \in G$. We get

$$\begin{aligned} W^*U^*\Pi_2^\alpha(y, f)\chi(\tau, t) &= \overline{\langle t, \tau \rangle} \iint \alpha_{t^{-1}}(y(ts))\chi(s, r^{-1}t)f(r)\overline{\langle s, \tau \rangle}\, drds \\ &= \overline{\langle t, \tau \rangle} \iint \alpha_{t^{-1}}(y(sr))\chi(st^{-1}r, r^{-1}t)f(r)\overline{\langle st^{-1}r, \tau \rangle}\, drds \\ &= \iint \alpha_{t^{-1}}(y(sr))UW^*U^*\chi(s, r^{-1}t)f(r)\overline{\langle sr, \tau \rangle}\, drds \\ &= \Pi_3^\alpha(y, f)W^*U^*\chi(\tau, t). \end{aligned}$$

So $W^*U^*\Pi_2^\alpha(y, f)UW = \Pi_3^\alpha(y, f)$, and the lemma is proved.

3.7. Remark. As we have used the same isomorphisms as in the duality theory of Takai [9], [10] we have that $B(A, G, \alpha) \subseteq M(\hat{G} \times_{\hat{\alpha}} (G \times_\alpha A))$ by Lemma 2.2. The action θ on $B(A, G)$ is transformed into the bidual action $\hat{\hat{\alpha}} = \operatorname{ad} V$. If $y \in C^b(G, A)$, $f \in C_c(G)$ and $t \in G$ we have that

$$\hat{\hat{\alpha}}_t \Pi_3^\alpha(y, f) = \Pi_3^\alpha(y_t, f),$$

where $y_t \in C^b(G, A)$ is defined by $y_t(s) = y(st)$ if $s \in G$.

It is now easily seen by the following lemma that $B(A, G, \alpha)$ contains $G \underset{\alpha}{\times} A$.

3.8. LEMMA. *If $a \in A$ and $f \in C_c(G)$ then*

$$W^* U^* X^* \tilde{\tilde{\Pi}}_\alpha(m_{\alpha^{-1} y_a} \lambda_f) X U W = \Pi_{\hat{\alpha}}(\Pi_\alpha(a, f)).$$

PROOF. If $\chi \in L^2(\hat{G} \times G, \mathscr{H})$ so that $U\chi \in C_c(G \times G, \mathscr{H})$ and $\tau \in \hat{G}$, $t \in G$, then $W^* U^* X^* \tilde{\tilde{\Pi}}_\alpha(m_{\alpha^{-1} y_a} \lambda_f) X U W \chi(\tau, t)$

$$= \Pi_3^\alpha(y_a, f)\chi(\tau, t) = \alpha_{t^{-1}}(a) \iint U\chi(s, r^{-1}t) f(r) \overline{\langle sr, \tau \rangle}\, dr ds$$

$$= \alpha_{t^{-1}}(a) \int \chi(\tau, r^{-1}t) f(r) \overline{\langle r, \tau \rangle}\, dr = \Pi_{\hat{\alpha}}(\Pi_\alpha(a, f))\chi(\tau, t).$$

We will now prove that $G \underset{\alpha}{\times} A$ is the fixed point algebra in $B(A, G, \alpha)$ for the action $\hat{\hat{\alpha}}$.

3.9. LEMMA. *If $y \in C^b(G, A)$, $f \in C_c(G)$ and $\chi \in L^2(\hat{G} \times G, \mathscr{H})$ so that $U\chi \in C_c(G \times G, \mathscr{H})$, then*

$$\iint \left\| \iint \alpha_{t^{-1}}(y(s)) U\chi(s, r^{-1}t) f(r) \overline{\langle sr, \tau \rangle}\, dr ds \right\|^2 dt d\tau \leqq \|y\|_\infty^2 \|f\|_1^2 \|\chi\|^2.$$

PROOF.

$$\iint \left\| \iint \alpha_{t^{-1}}(y(s)) U\chi(s, r^{-1}t) f(r) \overline{\langle sr, \tau \rangle}\, dr ds \right\|^2 dt d\tau$$

$$= \iint \|(U^* m_{y_t} \lambda_f U\chi)(\tau, t)\|^2\, d\tau\, dt \quad (y_t \text{ is defined by } y_t(s) = \alpha_{t^{-1}}(y(s)) \ (s \in G))$$

$$= \int \|(U^* m_{y_t} \lambda_f U\chi)(\cdot, t)\|_2^2\, dt$$

$$= \int \|(m_{y_t} \lambda_f \chi)(\cdot, t)\|_2^2\, dt$$

$$\leqq \|y\|_\infty^2 \|f\|_1^2 \|\chi\|^2.$$

3.10. NOTATION. If $y \in C^b(G, A)$ and $f \in C_c(G)$ we denote by $\Pi_4^\alpha(y, f)$ the bounded linear operator on $L^2(G \times G, \mathscr{H})$ defined by

$$\Pi_4^\alpha(y,f)\chi(\tau,t) = \iint \alpha_{t^{-1}}(y(s))U\chi(s,r^{-1}t)\overline{\langle sr,\tau\rangle}\,drds$$

where $\chi \in L^2(\hat{G}\times G,\mathscr{H})$ so that $U\chi \in C_c(G\times G,\mathscr{H})$ and $\tau \in \hat{G}$, $t \in G$. The preceeding lemma shows that this operator is well defined and that $\|\Pi_4^\alpha(y,f)\| \leqq \|y\|_\infty \|f\|_1$.

3.11. LEMMA. *If* $y \in C_0(G,A)$ *and* $f \in C_c(G)$ *then* $\Pi_4^\alpha(y,f)$ *belongs to* $\hat{G} \underset{\hat{\alpha}}{\times} G \underset{\alpha}{\times} A$.

PROOF. If $y \in C_0(G,A)$, the y can be approximated (in the sup-norm) by elements of the form $\sum_{i=1}^n a_i\otimes\hat{h}_i$, where $a_i \in A$, $h_i \in L^1(\hat{G})$. By Lemma 3.9, $\Pi_4^\alpha(y,f)$ can be approximated by elements of the form $\sum_{i=1}^n \Pi_4^\alpha(a_i\otimes\hat{h}_i,f)$. If $a \in A$, $h \in L^1(\hat{G})$, $\chi \in L^2(\hat{G}\times G,\mathscr{H})$ so that $U\chi \in C_c(G\times G,\mathscr{H})$ and $\tau \in \hat{G}$, $t \in G$ then we get

$$\begin{aligned}\Pi_4^\alpha(a\otimes\hat{h},f)\chi(\tau,t) &= \alpha_{t^{-1}}(a)\iint \hat{h}(s)U\chi(s,r^{-1}t)f(r)\overline{\langle sr,\tau\rangle}\,drds \\ &= \alpha_{t^{-1}}\iint \chi(\sigma^{-1}\tau,r^{-1}t)h(\sigma)\,d\sigma\overline{\langle r,\tau\rangle}f(r)\,dr \\ &= \Pi_{\hat{\alpha}}(\Pi_\alpha(a,f),h)\chi(\tau,t)\,.\end{aligned}$$

So each $\Pi_4^\alpha(a_i\otimes\hat{h}_i,f)$ belongs to $\hat{G} \underset{\hat{\alpha}}{\times} G \underset{\alpha}{\times} A$ and this proves the lemma.

3.12. LEMMA. *If* $\tilde{x} \in B(A,G,\alpha)$ *and* $g \in L^1(\hat{G})$, *then we have that* $\tilde{x}\lambda_g$ *and* $\lambda_g\tilde{x}$ *belong to* $\hat{G} \underset{\hat{\alpha}}{\times} G \underset{\alpha}{\times} A$.

PROOF. Take $y \in C^b(G,A)$, $f \in C_c(G)$ and $g \in L^1(\hat{G})$ so that $\hat{g} \in C_c(G)$. Denote by K_1 and K_2 the compact supports of f and $\hat{g}$. Let $\varepsilon>0$ and take $\delta>0$ so that $\delta<\varepsilon/\|\hat{g}\|_\infty\|f\|_1$.

By continuity of y, for each $r \in G$ there exists an open neighbourhood V_r of r so that $\|y(sr)-y(sr')\|<\delta$ for $r' \in V_r$ and $s \in K_2$. By compactness of K_1 there exist $r_1,\ldots,r_n \in K_1$ such that $K_1 \subset \bigcup_{i=1}^n V_{r_i}$. Let $\{h_i\}_{i=1}^n$ be positive functions in $C_c(G)$ which form a partition of unity subordinated to the covering $\{V_{r_i}\}_{i=1}^n$. Denote by y_i the function from G to A defined by $y_i(s)=y(sr_i)$. By the preceeding lemma we have that $\sum_{i=1}^n \Pi_4^\alpha(y_i\hat{g},h_if)$ belongs to $\hat{G} \underset{\hat{\alpha}}{\times} G \underset{\alpha}{\times} A$. If $\chi \in L^1(\hat{G}\times G,\mathscr{H})$ so that $U\chi \in C_c(G\times G,\mathscr{H})$ we get:

$$\begin{aligned}&\left\|\Pi_3^\alpha(y,f)\lambda_g\chi-\sum_{i=1}^n \Pi_4^\alpha(y_i\hat{g},h_if)\chi\right\|^2 \\ &= \iint\left\|\iint \alpha_{t^{-1}}\left(y(sr)-\sum_{i=1}^n h_i(r)y_i(s)\right)\hat{g}(s)U\chi(s,r^{-1}t)f(r)\overline{\langle sr,\tau\rangle}\,drds\right\|^2 dtd\tau\end{aligned}$$

$$\leqq \|f\|_1 \iiint_{K_1} \left\| \int \alpha_{t^{-1}}\left(y(sr) - \sum_{i=1}^{n} h_i(r) y_i(s) \right) \hat{g}(s) U\chi(s, r^{-1}t) \overline{\langle s, \tau\rangle}\, ds \right\|^2 |f(r)|\, dr\, dt\, d\tau$$

$$\leqq \|f\|_1 \iint \sup_s \left\| \alpha_{t^{-1}}\left(y(sr) - \sum_{i=1}^{n} h_i(r) y_i(s) \right) \right\|^2 |g(s)|^2 \int \|\chi(\tau, r^{-1}t)\|^2 d\tau |f(r)|\, dr dt$$

$$\leqq \delta^2 \|f\|_1^2 \|g\|_\infty^2 \|\chi\|^2 \leqq \varepsilon^2 \|\chi\|^2 .$$

So $\Pi_3^\alpha(y, f)\lambda_g$ belongs to $\hat{G} \times_{\hat{\alpha}} G \times_\alpha A$.
We have also

$$\lambda_g \Pi_3^\alpha(y, f)\chi(\tau, t) = \iiint \alpha_{t^{-1}}(y(sr)) U\chi(s, r^{-1}t) f(r) g(\sigma) \overline{\langle sr, \sigma^{-1}\tau\rangle}\, dr ds d\sigma$$

$$= \iint \alpha_{t^{-1}}(y(sr)) \hat{g}(sr) U\chi(s, r^{-1}t) f(r) \overline{\langle sr, \tau\rangle}\, dr ds .$$

So we can use an analogous argument to prove that $\lambda_g \Pi_3^\alpha(y, f)$ belongs to $\hat{G} \times_{\hat{\alpha}} G \times_\alpha A$.

As the functions $g \in L^1(\hat{G})$ with $\hat{g} \in C_c(G)$ are dense in $L^1(\hat{G})$ we have that $\Pi_3^\alpha(y, f)$ satisfies the statements of the lemma and as $B(A, G, \alpha)$ is generated by operators of the form $\Pi_3^\alpha(y, f)$ where $y \in C^b(G, A)$ and $f \in C_c(G)$, the lemma is proved.

3.13. Lemma. *If $y \in C^b(G, A)$ and $f \in C_c(G)$ then*

i) $\lambda_\sigma \Pi_3^\alpha(y, f)\lambda_{\sigma^{-1}} = \Pi_3^\alpha(y, f_\sigma)$ *for each* $\sigma \in G$.
ii) *the map* $\sigma \mapsto \lambda_\sigma \Pi_3^\alpha(y, f)\lambda_{\sigma^{-1}}$ *is (norm) continuous.*

So for each $\tilde{x} \in B(A, G, \mathscr{H})$ the map $\sigma \mapsto \lambda_\sigma \tilde{x} \lambda_{\sigma^{-1}}$ is norm continuous.

Proof. It suffices to prove the first statement. So take $\chi \in L^2(\hat{G} \times G, \mathscr{H})$ so that $U\chi \in C_c(G \times G, \mathscr{H})$ and $\tau \in \hat{G}$, $t \in G$. Then we get

$$\lambda_\sigma \Pi_\alpha(y, f)\lambda_{\sigma^{-1}}\chi(\tau, t) = \iint \alpha_{t^{-1}}(y(sr)) U\lambda_{\sigma^{-1}}\chi(s, r^{-1}t) f(r) \overline{\langle sr, \sigma^{-1}\tau\rangle}\, dr ds$$

$$= \iint \alpha_{t^{-1}}(y(sr)) U\chi(s, r^{-1}t) \langle r, \sigma\rangle f(r) \overline{\langle sr, \tau\rangle}\, dr ds$$

$$= \Pi_3^\alpha(y, f_\sigma)\chi(\tau, t) .$$

3.14. Lemma. *$G \times_\alpha A$ is the fixed point algebra in $B(A, G, \alpha)$ for the action $\hat{\hat{\alpha}}$.*

Proof. If $[B, G, \beta]$ is a C*-dynamical system, then M. B. Landstad [8, Theorem 4] proved a theorem which characterizes the C*-algebra B in the multiplier algebra of the crossed product by the following three conditions: $\tilde{x} \in M(G \times_\beta B)$ belongs to B if and only if:

i) $\tilde{x}\lambda_f$ and $\lambda_f\tilde{x}$ belong to $G \times_\beta B$ for $f \in L^1(G)$,
ii) $s \mapsto \lambda_s\tilde{x}\lambda_{s^{-1}}$ is (norm) continuous,
iii) $\hat{\beta}_\sigma(\tilde{x}) = \tilde{x}$ for all $\sigma \in \hat{G}$.

We can apply the theorem on the dual system $[G \times_\alpha A, \hat{G}, \hat{\alpha}]$. The lemma follows then from 3.12 and 3.13.

We then have our main theorem

3.15. THEOREM.
i) *$G \times_\alpha A$ is the fixed point algebra in $B(A,G)$ for the action θ.*
ii) *$G \times_\alpha A$ is the fixed point algebra in $B_u(A,G)$ for the action θ.*

PROOF. The first statement follows from the preceeding lemma, Remark 3.7 and Lemma 3.8. As $B_u(A,G)$ is contained in $B(A,G)$ and $G \times_\alpha A$ is contained in $B_u(A,G)$ the second statement follows from the first one.

3.16. REMARKS.
i) If G is compact we get the known fixed point theorem of Landstad–Takai [8], [10] and Wassermann [14].
ii) Although Theorem 3.15 holds with both the C*-algebras $B(A,G)$ and $B_u(A,G)$, the C*-algebra $B_u(A,G)$ is not nicely transformed by the isomorphisms used in this section. This is due to the fact that the isomorphism $y \mapsto \alpha y^{-1}$ (Lemma 3.4) of $C^b(G,A)$ does not restrict to an isomorphism of $C^b_u(G,A)$. This can be seen from the same example as in Remark 2.7(iii).

We end this section by showing that $B_u(A,G)$ it self is a crossed product.

3.17. PROPOSITION. *$B_u(A,G)$ is isomorphic to $G \times_\lambda C^b_u(G,A)$.*

PROOF. We represent $B_u(A,G)$ faithfully on $L^2(G \times G, \mathscr{H})$ by defining

$$m_y\lambda_g\chi(s,t) = y(t)\int \chi(s,r^{-1}t)g(r)\,dr\,,$$

where $y \in C^b_u(G,A)$, $g \in C_c(G)$ $\chi \in L^2(G \times G, \mathscr{H})$ and $s,t \in G$. Then

$$\begin{aligned} X^*m_y\lambda_g X\chi(s,t) &= y(st)\int X\chi(s,r^{-1}st)g(r)\,dr \\ &= (\lambda_{t^{-1}}y)(s)\int \chi(s,r^{-1}t)g(r)\,dr \end{aligned}$$

$$= \Pi_\lambda(y, g)\chi(s, t) .$$

So $X^* B_u(A, G) X = G \underset{\lambda}{\times} C_u^b(G, A)$.

3.18. REMARK. There is a class of C*-dynamical systems (A, G, α) for which the crossed product $G \underset{\alpha}{\times} A$ is contained in $A \otimes B_u(G, \mathbf{C})$. For these C*-dynamical systems the analogy to the W*-case is alsmost complete. This class of C*-dynamical systems is treated in [6].

I am very greatful to the referee for suggestions on the proofs of Lemmas 2.2 and 3.9 and on Remark 3.18.

REFERENCES

1. C. A. Akeman, G. K. Pedersen, and J. Tomiyama, *Multipliers of* C*-*algebras*, J. Funct. Anal. 13 (1973), 277–301.
2. R. C. Busby, *Double centralizers and extensions of* C*-*algebras*, Trans. Amer. Math. Soc. 132 (1968), 79–99.
3. T. Digernes, *Poids dual sur un produit croisé* C. R. Acad. Sci. Paris, Sér. A 278 (1974), 937–940.
4. T. Digernes, *Duality for weights on covariant systems and its applications*, Thesis. University of California at Los Angeles, 1975.
5. U. Haagerup, *On the dual weights for crossed products of Von Neumann algebras* I, II Math. Scand. 43 (1978), 99–118; 119–140.
6. G. Henrard, *Duality and a fixed point theorem for almost periodic* C*-*crossed products*, preprint, Leuven, 1983.
7. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, I, II (Grundlehren Math. Wiss. 115, 152) Springer-Verlag, Berlin - Heidelberg - New York, 1963, 1970.
8. M. B. Landstad, *Duality theory of covariant systems*, Trans. Amer. Math. Soc. 248 (1979), 223–267.
9. G. K. Pedersen, C*-*algebras and their automorphism groups* (London Math. Soc. Monographs 14), Academic Press, London, New York, San Francisco, 1979.
10. H. Takai, *On a duality for crossed products of* C*-*algebras*, J. Funct. Anal. 19 (1975), 25–39.
11. M. Takesaki, *Duality for crossed products and the structure of Von Neumann algebras of type* III, Acta Math. 131 (1973), 249–310.
12. M. Takesaki, *Theory of operator-algebras*, I, Springer-Verlag, Berlin - Heidelberg - New York, 1979.
13. A. van Daele, *Continuous crossed products and type* III *Von Neumann algebras*, (London Math. Soc. Lecture Note Ser. 31), Cambridge University Press, Cambridge, 1978.
14. A. J. Wassermann, *Automorphic actions of compact groups on operator algebras*, Thesis, University of Pennsylvania, 1981.

KATHOLIEKE UNIVERSITEIT LEUVEN
DEPARTEMENT OF MATHEMATICS
CELESTIJNENLAAN 200 B
B-3030 HEVERLEE
BELGIUM