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OUTER AUTOMORPHISMS OF INJECTIVE C*-ALGEBRAS

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Introduction.

Let A be a C*-subalgebra of a C*-algebra B and let β be an inner automorphism of B which leaves A invariant. When is the restriction of β to A an inner automorphism of A? That is, when is β implemented by a unitary in A, if A is unital, or by a unitary in $M(A)$, the multiplier algebra of A, if A is not unital?

A deep theorem of Kishimoto [10], which builds on the important earlier work of Elliott [3] and Lance [11], shows that when A is separable and simple and when B is the second dual of A then the answer is "always". We proved in [17] that when A is simple and B is the regular completion of A then the answer is also "always". We shall prove a much stronger result than we did in [17]. Let α be an outer *-automorphism of A, where A is α-simple. Let B be the injective envelope of A (see below for definitions). Then Theorem 3.6 implies that α has a unique extension to an outer *-automorphism of B.

The following elementary example illustrates what can go wrong. Let H be an infinite dimensional Hilbert space, let B be the algebra of all bounded operators on H and let A be the subalgebra of B generated by the identity of B and the algebra of compact operators on H. Then each unitary in B induces an automorphism of A which, in general, will not be inner.

We shall only consider automorphisms which are *-automorphisms.

1. Preliminaries.

We recall that a C*-algebra B is said to be *injective* when it is unital and if, whenever A is a unital C*-algebra and S is a self-adjoint subspace of A containing the unit, then each completely positive map from S into B which maps the unit of A to the unit of B can be extended to a completely positive map from A into B (see, for example, Choi and Effros in [2]). Arveson [1] proved that, for each Hilbert space H, the algebra of bounded operators on H is injective. So each C*-algebra is a subalgebra of an injective algebra.

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Each C*-algebra A can be embedded canonically in a larger C*-algebra Inj A, called the injective envelope of A. The injective envelope is characterized by the following two properties. First, Inj A is injective. Secondly, if φ is a completely positive map from Inj A to Inj A such that $\varphi(a+\lambda 1)=a+\lambda 1$ for all $a \in A$ and all $\lambda \in \mathbf{C}$ then φ is the identity map on Inj A. The existence of injective envelopes is a deep result of Hamana [4] who also established their fundamental properties. For each C*-algebra A, its injective envelope is a monotone complete AW*-algebra which need not be a von Neumann algebra [5].

In all that follows, A is a C*-algebra with injective envelope Inj A and α is a *-automorphism of A. By Corollary 4.2 in [4], α has a unique extension to a *-automorphism $\tilde{\alpha}$ of Inj A. This implies, see Corollary 4.3 in [4], that the relative commutant of A in Inj A is the centre of Inj A.

LEMMA 1.1. *Let B be a C*-subalgebra of* Inj A *such that B contains A. Let J be a closed two-sided ideal of B such that $J \cap A=\{0\}$. Then $J=\{0\}$.*

This is Lemma 3.2 [7].

A subset S of A is said to be *α-invariant* if $\alpha[S]\subset S$ and $\alpha^{-1}[S]\subset S$.

When D is any hereditary C*-subalgebra of A, we define D_0^+ to be the set of all positive elements of D with norm less than one, that is, $\{d \in D\colon \|d\|<1$ and $d \geqq 0\}$. Then, see Theorem 1.4.2 in [15], D_0^+ is an upward directed, approximate unit for D. Since D_0^+ is upward directed it has a supremum p in Inj A. It is straightforward to show that p is a projection and that $pd=dp=d$ for each $d \in D$. It follows from Theorem 6.5 [6] that Inj D can be identified with $p(\text{Inj } A)p$. This fact will be used extensively. When D is α-invariant then D_0^+ is also α-invariant. Hence $\tilde{\alpha}(p)=p$.

LEMMA 1.2. *Let A be a non-unital* C*-*algebra. Then the supremum of A_0^+ in* Inj A *is the unit of* Inj A.

Let p be the projection in Inj A which is the supremum of A_0^+. Then p is in the commutant of A and so p is in the centre of Inj A. So $1-p$ is a central projection. Thus $(1-p)$Inj A is a closed two-sided ideal of Inj A whose intersection with A is the zero ideal. So, by Lemma 1.1, $p=1$.

For any C*-algebra A, we recall that its multiplier algebra, $M(A)$, is defined to be $\{m \in A^{**}\colon mA\subset A$ and $Am\subset A\}$ which is a C*-subalgebra of A^{**}. Clearly, when A is unital, $M(A)$ coincides with A. When A is not unital, $M(A)$ can be much larger than the algebra obtained from A by adjoining a unit. For example, the multiplier algebra of $C_0(\mathbf{R})$ is the algebra of all bounded continuous functions on $\mathbf{R}$. The following lemma shows that we may regard $M(A)$ as being naturally embedded in Inj A.

LEMMA 1.3. *Let A be a non-unital* C*-*algebra with multiplier algebra $M(A)$. Let φ be the canonical embedding of A in* Inj A, *the injective envelope of A. Then there exists an isometric *-isomorphism φ: $M(A) \to \operatorname{Inj} A$ which extends φ. Moreover,*

$$\Phi[M(A)] = \{z \in \operatorname{Inj} A : zA \subset A \text{ and } Az \subset A\} .$$

Let B be the smallest C*-subalgebra of Inj A which contains A and the unit and is such that, whenever (b_j) is an upward directed net in B with supremum b in Inj A, then b is in B. Then B is the regular completion of A [5, 18].

By the proof of Corollary 2.2 in [16], there is an extension of φ to an isometric *-isomorphism Φ: $M(A) \to B$ such that

$$\Phi[M(A)] = \{z \in B : zA \subset A \text{ and } Az \subset A\} .$$

Let

$$M = \{z \in \operatorname{Inj} A : zA \subset A \text{ and } Az \subset A\} .$$

To establish the lemma it suffices to show that the C*-algebra M is contained in B.

By Lemma 1.2, the upward directed set A_0^+ has 1 as its supremum in Inj A. So, whenever $m \in M$, mm^* is the supremum in Inj A of $\{m a m^* : a \in A_0^+\}$. Since $m a m^* \in A$ for each $a \in A_0^+$, it follows that $mm^* \in B$. Hence $M \subset B$.

Let B be any C*-algebra. We shall define $C^*(B, 1)$ to be the algebra formed by adjoining a unit to B, if B is not unital, and define $C^*(B, 1)$ to be B whenever B is unital.

2. Cross-products by discrete groups.

We recall some basic properties of cross-products which will be needed later.

Let G be a discrete group and let β be a homomorphism of G into the group of *-automorphisms of A. We recall that a (non-degenerate) covariant representation of the system (A, G, β) is a triple (π, H, u), where H is a Hilbert space, (π, H) is a (non-degenerate) representation of A, and u is a unitary representation of G on H such that

$$\pi(\beta_\gamma(a)) = u_\gamma \pi(a) u_\gamma^*$$

for each $a \in A$ and each $\gamma \in G$. We refer the reader to [15], for a lucid account of C*-dynamical systems and cross-products. Corresponding to the system (A, G, β) there can be constructed the (*universal*) *cross-product* $A \times_\beta G$. There exists a unitary representation U of G in $M(A \times_\beta G)$ such that

$$U_\gamma \, a \, U_\gamma^* = \beta_\gamma(a)$$

for each $a \in A$ and each $\gamma \in G$. Also, $A \times_\beta G$ is the closure of the sub-algebra whose elements are all finite sums of the form $\sum a_\gamma U_\gamma$, where each a_γ is in A. Further, $A \times_\beta G$ has the following "universal" property, given a non-degenerate covariant representation (π, H, u) of (A, G, β), there exists a representation (Π, H) of $A \times_\beta G$ such that

$$\Pi(aU_\gamma) = \pi(a)u_\gamma$$

for each $a \in A$ and every $\gamma \in G$.

Let (ϱ, H) be the universal representation of A. Then we define a corresponding covariant representation $(\tilde{\varrho}, l^2(G,H), \lambda)$ as follows. First, $l^2(G,H)$ is the Hilbert space of all square summable H-valued sequences indexed by G, that is, $l^2(G,H) = l^2(G) \otimes H$. Secondly, for each $\underline{\xi} \in l^2(G,H)$ and each $g \in G$

$$(\lambda_g \underline{\xi})(\gamma) = \underline{\xi}(g^{-1}\gamma) .$$

Thirdly, for each $a \in A$ and each $\underline{\xi} \in l^2(G,H)$

$$(\tilde{\varrho}(a)\underline{\xi}(\gamma) = (\beta_\gamma^{-1}(a))\underline{\xi}(\gamma) .$$

Let $\tilde{\varrho} \times \lambda$ be the representation of $A \times_\beta G$ corresponding to the above (non-degenerate) covariant representation of the system (A, G, β). The algebra $(\tilde{\varrho} \times \lambda)[A \times_\beta G]$ is defined to be the reduced cross-product, $A \times_{r\beta} G$. It turns out, see [15], that if (ϱ, H) were replaced by any faithful representation of A, then the corresponding construction would give an algebra isomorphic to $A \times_{r\beta} G$.

Whenever the group G is amenable, in particular when G is abelian, the homomorphism $\tilde{\varrho} \times \lambda$ is faithful.

LEMMA 2.1. *Let G be a discrete group with a representation β in the automorphism group of A. Let $u: G \to \operatorname{Inj} A$ be a unitary representation of G such that $u_\gamma z u_\gamma^* = \beta_\gamma(z)$ for all $z \in A$. Let I be the canonical embedding of A into* Inj A. *Let B be the* C*-*subalgebra of* Inj A *generated by all finite sums of the form $\sum a_\gamma u_\gamma$, where each $a_\gamma \in A$. Then there exists a surjective homomorphism Π from $A \times_\beta G$ onto B such that*

$$\Pi(aU_\gamma) = au_\gamma$$

for all $a \in A$ and all $\gamma \in G$.

Let H_1 be the universal representation space of Inj A.

When $1 \in A$, then A acts non-degenerately on H_1. So (I, H_1, u) is a non-degenerate, covariant representation of the system (A, G, β). By the universal property of the cross-product, Π exists.

Let us now suppose that A is not unital and let $C^*(A, 1)$ be the C*-algebra obtained from A by adjoining a unit. Then Inj A is the injective envelope of

$C^*(A, 1)$. Let H be the closure of $A[H_1]$. Since $u_\gamma A u_\gamma^* = A$ for each $\gamma \in G$, we have $u_\gamma[H] = H$ for each $\gamma \in G$. Thus H is invariant under B. For all $b \in B$ let $\pi(b) = b \mid H$ and let $\tilde{u}$: $G \to \mathscr{L}(H)$ be defined by $\tilde{u}_\gamma = u_\gamma \mid H$. Then $(\pi, H, \tilde{u})$ is a covariant non-degenerate representation of the system (A, G, β). So there exists an homomorphism Π_1 from $A \times_\beta G$ onto $\pi[B]$ such that

$$\Pi_1(aU_\gamma) = \pi(a)\tilde{u}_\gamma ,$$

for each $a \in A$ and each $\gamma \in G$.

Since π is faithful on A, $\pi^{-1}\{0\}$ is an ideal of B which is disjoint from A. So, by Lemma 1.1, $\pi^{-1}\{0\} = 0$. Let $\Pi = \pi^{-1} \circ \Pi_1$. Then Π has the required properties.

Let A, G, β be as above. We shall need the following basic property of reduced cross-products by discrete groups. There exists a completely positive map E from $A \times_{r\beta} G$ onto A such that

(i) $E(\tilde{\varrho}(a)) = a$ for all $a \in A$
(ii) $E(\tilde{\varrho}(a)\lambda_\gamma) = 0$ whenever γ is not the neutral element of G.

When A is not unital, neither is $A \times_{r\beta} G$. Then E can be extended to a completely positive map from $C^*(A \times_{r\beta} G, 1)$ onto $C^*(A, 1)$ where $E1 = 1$. To see this let P be the projection from $l^2(G, H)$ onto H defined by $P\xi = \xi(0)$, where 0 is the neutral element of G. Then the compression $z \to PzP$ is a completely positive linear map whose restriction to $C^*(A \times_{r\beta} G, 1)$ has the required properties.

When G is amenable, in particular, when G is abelian then we may identify the cross-product and the reduced cross-product. So there exists a completely positive projection E from $C^*(A \times_\beta G, 1)$ onto $C^*(A, 1)$, such that $E(aU_\gamma) = 0$ whenever γ is not the neutral element of G.

3. Automorphisms.

When α is an automorphism of a C^*-algebra A, then A is said to be *α-simple* if the only α-invariant, closed, proper, two-sided ideal of A is 0.

We shall need the following notation. Let $H^\alpha(A)$ be the family of all non-zero, closed, α-invariant, hereditary C^*-sub-algebras of A. For each $B \in H^\alpha(A)$ we define $\mathrm{Sp}(\alpha \mid B)$ to be the spectrum of the operator α, restricted to B, regarded as an operator on the Banach space B. We define the *Connes spectrum* of α to be

$$\Gamma(\alpha) = \bigcap \{\mathrm{Sp}(\alpha \mid B) : B \in H^\alpha(A)\} .$$

Let $(A, \mathbf{Z}, \langle\alpha\rangle)$ be the dynamical system, where $\langle\alpha\rangle$ is the action of $\mathbf{Z}$ defined by $n \to \alpha^n$. Then the Connes spectrum of the dynamical system, as defined by

Olesen, coincides with $\Gamma(\alpha)$, see page 340 in [15]. Provided that A is α-simple, $\Gamma(\alpha)$ also coincides with the Borchers spectrum of the system $(A, \mathbf{Z}, \langle\alpha\rangle)$.

Let $\Gamma(\alpha)^{\perp} = \{n \in \mathbf{Z} : \lambda^n = 1 \text{ for all } \lambda \in \Gamma(\alpha)\}$.

LEMMA 3.1. (Olesen-Pedersen) *Let α be a *-automorphism of A such that A is α-simple. Let n be a positive integer. Then the following statements are equivalent*:

(i) *The integer n is an element of $\Gamma(\alpha)^{\perp}$.*

(ii) *There exists $B \in H^{\alpha}(A)$ and a *-derivation δ on B such that $\alpha^n | B = \exp\delta$ and $\alpha\circ\delta = \delta\circ\alpha$.*

(iii) *For each $\varepsilon > 0$ there can be found $B \in H^{\alpha}(A)$ and a *-derivation δ on B, commuting with α, such that $\alpha^n | B = \exp\delta$ and $\|\exp\delta - I\| < \varepsilon$.*

(iv) *There exists $B \in H^{\alpha}(A)$ such that $\|(\alpha^n - I) | B\| < 2$.*

The equivalence of (i) and (ii) is a consequence of Theorem 4.3 in [14]. It follows from Lemma 4.1 in [14] that (ii) implies (iii). Trivially (iii) implies (iv). To complete the circle of implications, we observe that, by the Kadison-Ringrose Theorem, (iv) implies the existence of a derivation δ on B such that $\alpha^n | B = \exp\delta$. Moreover δ is the limit of a sequence of polynomials in α and hence commutes with α. That is, (iv) implies (ii).

We come now to the first key theorem.

THEOREM 3.2. *Let A be a non-zero C*-algebra. Let α be an automorphism of A, not the identity automorphism, such that A is α-simple. Further, for each integer n, either $\alpha^n = I$ or else, for every α-invariant, non-zero, hereditary C*-subalgebra D,*

$$\|(\alpha^n - I) | D\| = 2 .$$

Then, $\tilde{\alpha}$, the unique extension of α to an automorphism of Inj A, *is an outer automorphism.*

If there is no positive integer n for which $\alpha^n = I$, let $G = \mathbf{Z}$. Otherwise, let k be the smallest positive integer for which $\alpha^k = I$ and let $G = \mathbf{Z}_k$.

By Lemma 3.1, $\Gamma(\alpha)^{\perp} = \{0\}$ and hence $\Gamma(\alpha)$ is the full circle group. So, by a theorem of Olesen and Pedersen [15], the reduced cross-product $A \times_{r\alpha} G$ is simple. Since G is abelian, it is an amenable group and so the canonical homomorphism from $A \times_{\beta} G$ onto $A \times_{r\alpha} G$ is an isomorphism.

We shall assume that $\tilde{\alpha}$ is not an outer automorphism of Inj A and then derive a contradiction. By our assumptions there is a unitary u in Inj A which implements $\tilde{\alpha}$.

When $G = \mathbf{Z}_k$, we have $u^k(a + \lambda 1)u^{-k} = a + \lambda 1$ for $a \in A$ and $\lambda \in G$. So, by the fundamental property of the injective envelope, $u^k z u^{-k} = z$ for each x in Inj A, that is u^k is in the centre of Inj A. Since $\tilde{\alpha}(u^k) = u^k$, either u^k is a scalar multiple of the identity or, by spectral theory, there exists a non-trivial central projection q such that $\tilde{\alpha}(q) = q$. If such a q exists then $q(\text{Inj})A$ is a non-zero, proper, closed two-sided ideal of Inj A which is $\tilde{\alpha}$-invariant. So, by Lemma 1.1, $q(\text{Inj } A) \cap A$ is a non-zero ideal of A. But $q(\text{Inj } A \cap A)$ is α-invariant and A is α-simple. So $q(\text{Inj } A) \cap A = A$. Similarly, $(1-q)(\text{Inj } A) \cap A = A$. This is impossible, so u^k is a scalar multiple of the identity. We may suppose that $u^k = 1$.

Let B be the closed subspace of Inj A generated by all sums of the form $\sum_{j \in S} a_j u^j$, where S is a finite subset of G and $a_j \in A$ for each $j \in S$. Then B is a C*-subalgebra of Inj A and, by Lemma 2.1, there exists a surjective *-homomorphism Π from $A \times_\beta G$ onto B such that $\Pi(aU^j) = au^j$. Since G is abelian, $A \times_\beta G$ may be identified with $A \times_{r\alpha} G$ which is simple. So we may regard Π as an isomorphism from $A \times_{r\alpha} G$ onto B.

From the basic properties of reduced cross-products by discrete groups, discussed in section 2, it follows that there exists a completely positive projection E from $C^*(B, 1)$ onto $C^*(A, 1)$ such that $E(au^j) = 0$ for $a \in A$ and $j \in G \setminus \{0\}$. Since Inj A is an injective C*-algebra, E can be extended to a completely positive map $\tilde{E}$ from Inj A to Inj A. Since the restriction of $\tilde{E}$ to $C^*(A, 1)$ is the identity map it follows, by the fundamental property of the injective envelope, that $\tilde{E}$ is the identity map on Inj A.

Let a be any non-zero element of A. Then

$$au = \tilde{E}(au) = E(au) = 0 .$$

So

$$a = auu^* = 0 .$$

This is impossible. So the assumption that $\tilde{\alpha}$ was implemented by a unitary in Inj A is false, that is, $\tilde{\alpha}$ is an outer automorphism.

LEMMA 3.3. *Let α be an automorphism of a non-zero C*-algebra A such that A is α-simple. Let H be any non-zero, α-invariant hereditary C*-subalgebra of A. Then H is also α-simple.*

Let J be any proper, closed α-invariant ideal of H. Then there exists a primitive ideal of H, Q, such that $J \subset Q$. Since H is an hereditary C*-subalgebra of A, there exists a primitive ideal P in A such that $Q = P \cap H$.

Let Λ be the collection of all primitive ideals of A, L, such that $J \subset L$. By the preceding paragraph, Λ is not empty. Since J is α-invariant, $L \in \Lambda$ if, and only if $\alpha[L] \in \Lambda$. Let M be the intersection of all the ideals in Λ. Then M is an α-invariant closed ideal of A. Each ideal in Λ is primitive and hence proper. So M

is a proper, α-invariant, closed ideal of A. Since A is α-simple, M must be the zero ideal of A. Hence J is the zero ideal of H. So H is α-simple.

LEMMA 3.4. *Let β be a *-automorphism of a C*-algebra B such that B is β-simple. Let there exist a positive integer k such that β^k is a derivable automorphism. Then, given any primitive ideal J, the primitive ideal space of B,* Prim B, *is the finite set*

$$\{\beta^n[J] : n=0,1,\ldots,k-1\}$$

equipped with the discrete topology.

Let E be any primitive ideal of B. Then $\beta^k[E]=E$ because β^k is derivable. Since B is β-simple, it follows that

$$\bigcap_{n=0}^{k-1} \beta^n[E] \quad \text{is the zero ideal .}$$

In particular,

$$\bigcap_{n=0}^{k-1} \beta^n[E] \subset J .$$

Since J is primitive, it is a prime ideal. So, for some positive integer n, $\beta^n[E] \subset J$. Similarly, for some positive integer m,

$$\beta^m[J] \subset E .$$

So

$$\beta^{m+n}[E] \subset \beta^m[J] \subset E .$$

Hence

$$E \supset \beta^{m+n}[E] \supset \beta^{2(m+n)}[E] \supset \ldots \supset \beta^{k(m+n)}[E] .$$

Since β^k is derivable, we have $E=\beta^{k(m+n)}[E]$.

Hence $\beta^m[J]=E$. So every primitive ideal of B is in the finite set $\{\beta^\alpha[J]: \alpha =0,1,\ldots,k-1\}$.

Since each closed ideal of B is the intersection of the primitive ideals which contain it and since Prim B is finite, one of the primitive ideals must be a maximal ideal. Since β is an automorphism, it follows that each of the primitive ideals is maximal and hence corresponds to a closed point in the hull-kernel topology. In other words, Prim B has the discrete topology.

COROLLARY 3.5. *Let β be a *-automorphism of a C*-algebra B such that B is β-simple and, for some positive integer k, $\|\beta^k - 1\| < 2$. Then there exists a unitary v*

in $M(B)$, *the multiplier algebra of* B, *such that* $\tilde{\beta}(v)=v$ *and* $\beta^k=\operatorname{Ad} v^k$ *where* $\tilde{\beta}$ *is the unique extension of* β *to an automorphism of* $\operatorname{Inj} B$.

Since $\|\beta^k-1\|<2$ it follows from the Kadison-Ringrose Theorem that $\beta^k=\exp\delta_1$ for some *-derivation δ_1. Moreover, δ_1 is the norm limit of a sequence of polynomials in powers of β. So δ_1 commutes with β. Let $\delta=(1/k)\delta_1$. Then $\beta\circ\delta=\delta\circ\beta$.

Let h be the minimal positive generator of δ in B^{**}.

By Lemma 3.3, every real valued function on $\operatorname{Prim} B$ is continuous. So, by Theorem 8.6.9 in [15], h is in $M(B)$, the multiplier algebra of B. Because δ commutes with β and β^{-1}, trivial algebraic manipulation shows that $\tilde{\beta}(h)$ and $\tilde{\beta}^{-1}(h)$ are also positive generators of δ. So $h\leqq\tilde{\beta}(h)$ and $h\leqq\tilde{\beta}^{-1}(h)$. Thus $h=\tilde{\beta}(h)$.

Let $v=\exp ih$, so that v is a unitary in $M(B)$ with the required properties.

We come now to the main theorem.

THEOREM 3.6. *Let* A *be a non-zero* C*-*algebra. Let* α *be a* *-*automorphism of* A *such that* A *is* α-*simple. Let* $\tilde{\alpha}$ *be the unique extension of* α *to a* *-*automorphism of* $\operatorname{Inj} A$, *the injective envelope of* A. *If* $\tilde{\alpha}$ *is an inner automorphism of* $\operatorname{Inj} A$ *then* α *is also an inner automorphism of* A, *being implemented by a unitary in* $M(A)$, *the multiplier algebra of* A.

Let u be a unitary in $\operatorname{Inj} A$ which implements $\tilde{\alpha}$. By Theorem 3.2 there exists a positive integer k and some $B\in H^\alpha(A)$ such that $\|(\alpha^k-I)\mid B\|<2$. Equivalently, by Lemma 3.1, $k\in\Gamma(\alpha)^\perp$. Let us suppose k to be the smallest positive integer in $\Gamma(\alpha)^\perp$ and let $B\in H^\alpha(A)$ such that $\|(\alpha^k-I)\mid B\|<2$.

By Corollary 3.5, there exists a unitary v in $M(B)$ for which $\alpha(v)=v$ and such that $\alpha^k\mid B=\operatorname{Ad} v$. Let $\gamma=\operatorname{Ad} v^*\circ(\alpha\mid B)$. Then the dynamical systems $(B,\mathbf{Z},\langle\gamma\rangle)$ and $(B,Z,\langle\alpha\mid B\rangle)$ are exterior equivalent. So, by Proposition 8.11.5 in [15], $\Gamma(\gamma)=\Gamma(\alpha\mid B)$.

Let q be any positive integer in $\Gamma(\gamma)^\perp$. Then $q\in\Gamma(\alpha\mid B)^\perp$. Since $\Gamma(\alpha)\subset\Gamma(\alpha\mid B)$, $q\in\Gamma(\alpha)^\perp$. Hence $q\geqq k$.

We shall now assume that $k>1$ and deduce a contradiction. For $1\leqq r<k$ we have that $r\notin\Gamma(\gamma)^\perp$ and so, by Theorem 3.1 applied to B and γ, γ is not implemented by a unitary in $\operatorname{Inj} B$. Hence $\alpha\mid B$ is not implemented by a unitary in $\operatorname{Inj} B$. But, see section 1, there is a projection p from $\operatorname{Inj} A$ such that $\operatorname{Inj} B$ may be identified with $p(\operatorname{Inj} A)p$. Since B is α-invariant we also have $\tilde{\alpha}(p)=p$. Thus p commutes with u, the unitary in $\operatorname{Inj} A$ which implements $\tilde{\alpha}$. So pu is a unitary in $\operatorname{Inj} B$ which implements $\alpha\mid B$. This is a contradiction. So $k=1$.

Since $\alpha\mid B$ is a derivable automorphism, each ideal of B is α-invariant. By Lemma 3.2, each closed α-invariant ideal of B is either B or the zero ideal. So B is simple. It then follows from Lemma 1.1 that the centre of $\operatorname{Inj} B$ is trivial.

We have

$$(up)x(up)^* = vxv^*$$

for all x in B and so all x in $\operatorname{Inj} B$. So $(up)^*v$ is in the centre of $\operatorname{Inj} B$. Thus up is a scalar multiple of v. So up is a multiplier of B.

Let $J = \{x \in A : ux \in A\}$. Then J is a closed, α-invariant ideal. By the preceding paragraph, J contains B, so that J is not the zero ideal. Since A is α-simple, J must be the whole of A. So, for all $x \in A$, $ux \in A$. Whenever $y \in A$,

$$yu = uu^*yu = u\alpha^{-1}(y) .$$

So $yu \in A$. Thus u is a multiplier of A.

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