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DENSE STRONG CONTINUITY OF MAPPINGS AND THE RADON-NIKODYM PROPERTY

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Summary.

Let $F: X \to E$ $(F: X \to E^*)$ be a weak-continuous (weak*-continuous) mapping from the Baire space X into the Banach space E (dual Banach space E^*). When do there exist a dense G_δ subset of X at the points of which F is norm-continuous?

It turns out that this is the case when $E(E^*)$ has the Radon-Nikodym property. The same holds true for multivalued mappings provided one uses a suitable notion of norm continuity of set-valued maps. An application is given to the theory of weak Asplund spaces. In particular, without using renorming theorems, it is proved that closed linear subspaces of a weakly compactly generated Banach space is weak Asplund.

0. Introduction.

Let $F: X \to E$ be a single-valued mapping from the topological space X into the normed space E. Suppose F is continuous with respect to the weak topology in E. Is it true that at some points of X the map F is continuous with respect to the norm topology in E? In [1] Alexiewicz and Orlicz gave a positive answer to this question for the case when X is of 2nd Baire category and E is a separable normed space. Fort [7] proved a general result which contained the result of Alexiewicz and Orlicz but still remained in the frames of separable normed spaces. Later, under the requirement that X is strongly countably complete and without any restrictions for the Banach space E, Namioka [16] showed that F will be norm-continuous at the points of some dense G_δ-subset of X. Along this line went the papers of Troallic [22], Talagrand [20] and Christensen [4]. In this paper we consider the case when $F: X \to E(E^*)$ is a multivalued mapping of the Baire space X into the normed space E (dual normed space E^*) with the Radon-Nikodym property. We suppose that F: X

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$\to E$ ($F: X \to E^*$) is upper semi-continuous and compact valued with respect to the weak (weak*) topology in E (in E^*) and prove that at each point x_0 of some dense G_δ subset C of X the mapping F has the following continuity property.

There exists such a point $y_0 \in Fx_0$ that for each $\varepsilon > 0$ an open neighbourhood $U \in x_0$, $U \subset X$, exists for which

$$(*) \qquad \sup\{\inf\{\|y-y_0\| : y \in Fx\} : x \in U\} \leqq \varepsilon .$$

Earlier similar results about multivalued mappings $F: X \to E(E^*)$ were considered in [11], [13], and [3]. It turns out that the Banach space E is an Asplund space (the definition is given in section 2) if and only if for every upper semi continuous compact valued $F: X \to (E^*, w^*)$ where X is a Baire space, there exists a dense G_δ subset $C \subset X$ at the points of which (*) is satisfied. If $Q: E \to E_1$ is a continuous linear mapping of the Asplund space E onto a dense subspace of the Banach space E_1, then E_1 is weak Asplund. In addition to this result of Stegall (see [19]) we prove here that every closed linear subspace E_2 of E_1 is also weak Asplund. As a corollary we obtain (without using renorming theorems) that every closed linear subspace E_2 of some weakly compactly generated Banach space E_1 is weak Asplund.

1. Main results.

(1.1) Definition. The correspondence (set-valued map) $F: X \to Y$ from the Hausdorff topological space X into the Hausdorff topological space Y is usco (upper semi-continuous and compact valued) if

i) $Fx \subset Y$ is compact and non-empty for all $x \in X$; and
ii) for every open set $U \subset Y$ the set $\{x \in X: Fx \subseteqq U\}$ is open in X.

Let E be a normed space and E^* be its dual. As usual, we denote by "w^*" the weak* topology in E^* and by "w" the weak topology in E. It is convenient for us to use here the notion "Radon-Nikodym Property" (RNP) in the following form.

(1.2) Definition. (Namioka and Phelps [17], Diestel and Uhl ([6, p. 281)]). The dual Banach space E^* has RNP if for every bounded subset $A^* \subset E^*$ and every number $\varepsilon > 0$ there exist $x \in E$ and a number $t > 0$ such that the diameter of the set

$$S(A^*; x, t) := \{x^* \in A^* : \langle x, x^* \rangle > \sup\{\langle x, y\rangle : y \in A^*\} - t\}$$

(called sometimes w^*-slice of A^*) is smaller than ε.

The Banach space E has RNP if for every bounded subset A of E and every $\varepsilon>0$ there exist $x^* \in E^*$ and a number $t>0$ such that the diameter of the set

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(called sometimes w-slice of A) is smaller than ε.

(1.3) THEOREM. *Let* $F\colon X \to (E^*, w^*)$ $(F\colon X \to (E, w))$ *be an* usco *mapping from the Baire space* X (i.e. *every open* $U \subset X$ *is of* 2nd *Baire category*) *in the dual space* E^* (*space* E) *which has* RNP. *Then there exists a dense* G_δ *subset of* X *at each point* x_0 *of which the following condition is fulfilled*

(*) *there exists a point* $y_0 \in Fx_0$ *such that for every* $\varepsilon>0$ *an open* $U \subset X$, $U \ni x_0$, *exists with* $\inf\{\|y_0 - y\| : y \in Fx\} \leqq \varepsilon$ *whenever* $x \in U$.

In the proof of this theorem we will make essential use of a general concept introduced by Christensen in [3]. For any topological spaces X and Y denote by USCO (X, Y) the set of all correspondences $F\colon X \to Y$ which are usco. By $G(F)$, as usual, we will denote the graph of the correspondence $F\colon X \to Y$. The set USCO (X, Y) is naturally ordered by inclusion of the graphs. Since the graph of any usco correspondence is automatically closed and since being usco is preserved by taking correspondence whose graph is closed and contained in the graph of a given usco correspondence (see the discussions in [3]), the order in USCO (X, Y) is inductive and Zorn's lemma can be applied. Therefore every $F \in$ USCO (X, Y) contains a minimal correspondence $\tilde{F} \in$ USCO (X, Y). Minimal elements of USCO (X, Y) have some interesting properties.

(1.4) LEMMA. *If* F *is minimal in* USCO (X, Y) *and* U *is an open subset of* X, *then the restriction of* F *to* U *is minimal in* USCO (U, Y).

(1.5) LEMMA. *Let* F *be minimal in* USCO (X, Y) *and* Y_1 *be a closed subset of* Y *for which* $F(x) \cap Y_1 \neq \emptyset$ *for every* $x \in X$. *Then* $F(x) = U\{Fx : x \in X\} \subset Y_1$.

PROOF OF THEOREM (1.3). It is clear that theorem (1.3) can be derived form the following statement.

(1.6) LEMMA. *Let* $X, F, E^*(E)$ *be as in Theorem* (1.3). *Suppose, in addition, that* F *is minimal* usco. *Then there is a dense* G_δ *set* $C \subset X$ *such that for each* $x_0 \in C$ *the set* Fx_0 *contains only one point and* $F\colon X \to E^*$ $(F\colon X \to E)$ *is upper semi-continuous at* x_0 *with respect to the norm in* E^* (*in* E).

We will prove only the first half of the theorem. The second half (contained in brackets) can be proved similarly and we omit the proof.

Let B^* be the unit ball of E^*. For every positive integer n put

$$V_n = \bigcup \left\{ \operatorname{int}\left\{ x \in X : Fx \subset y + \frac{1}{n} B^* \right\} : y \in E^* \right\},$$

where $\operatorname{int} S$ stands for the interior of the set S in X. As union of open sets V_n is an open subset of X. We will show now that it is dense in X. Take some nonempty open $U \subset X$. We have to see that $U \cap V_n \neq \emptyset$. Put, for each integer $k > 0$, $F_k = \{x \in X : Fx \cap kB^* \neq \emptyset\}$. Since $F\colon X \to (E^*, w^*)$ is upper semicontinuous Fx, $k = 1, 2, 3, \ldots$, are closed sets which cover X. As $U \subset X$ is a Baire space, there is some integer k for which $U_1 = (\operatorname{int} F_k) \cap U \neq \emptyset$. Having in mind (1.4) and (1.5) we see that $F(U_1) = \bigcup \{Fx\colon x \in U_1\} \subset kB^*$. In particular $F(U_1)$ is a bounded subset of E^*. By the Radon-Nikodym property there exists some w^*-open set $W \subset E^*$ such that $W \cap F(U_1) \neq \emptyset$ and $\operatorname{diam}(W \cap F(U_1)) \leqq 1/2n$. Using minimality of $F\colon U_1 \to E^*$ once more we see that for some $x_1 \in U_1$, $Fx_1 \subset W$. Since $F\colon X \to E^*$ was w^*-upper semi-continuous at x_1 we will find some open $U_2 \subset U_1$, $U_2 \ni x_1$, such that $F(U_2) \subset W \cap F(U_1)$. Hence $U_2 \subset V_n \cap U_1 \subset V_n \cap U$. Therefore V_n is dense in X. To prove (1.6) (and Theorem (1.3)) it remains to put

$$C = \bigcap \{V_n : n = 1, 2, 3, \ldots\}.$$

(1.7) Remarks. As stated in Theorems 1 and 2 in the paper of Christensen [3], the "bracket part" of our Theorem (1.3) remains valid even for arbitrary Banach space E (not necessarily having RNP), provided some mild requirements are imposed on X. Nothing similar is to be expected for mappings $F\colon X \to E^*$ in dual spaces E^*. The presence of RNP in E^* is not only a sufficient but also necessary condition for the Theorem (1.3) to be true. Indeed, suppose the conclusion of (1.3) is valid for some dual space E^*. Take some w^*-closed bounded subset $X \subset E^*$. As any other compact space the set X endowed with the w^*-topology will be a Baire space. Consider the identity map $F\colon (X, w^*) \to (E^*, w^*)$ and apply (1.3). This yields a lot of points in (X, w^*) at which the identity is w^*-to-norm continuous. Thus X contains nonempty relatively w^*-open sets of arbitrary small diameter. As shown in the paper of Namioka and Phelps [17] this implies RNP.

For the particular case when F is a monotone operator the construction from the proof of (1.6) was used in [12].

(1.8) Theorem. *Let $F\colon X \to (E, w)$ be a minimal* usco *correspondence from the Baire space X into the Banach space E having* RNP. *Suppose that X has the countable chain condition on the open sets,* i.e. *each disjoint family of open subsets of X is countable. Then $F(x) = \bigcup \{Fx\colon x \in X\}$ is a norm separable subset of E.*

PROOF. The proof is almost identical to the proof of Theorem 3 from [3].

2. Asplund and weak Asplund spaces.

We apply here our main result in order to give an improved version of a theorem of Stegall.

(2.1) DEFINITION. (Asplund [2], Namioka and Phelps [17]). The complete normed space E is called Asplund (weak Asplund) if every continuous real-valued convex function $f: E \to R$ is Frechet (Gâteaux) differentiable at the points of some dense G_δ subset of E.

Evidently, every Asplund space E is weak Asplund. Mazur [15] proved that every separable complete normed space E is weak Asplund. The well known separable space l_1 is an example of a weak Asplund space which is not Asplund. At present there are many nice characterizations of Asplund spaces (see Stegall [19], Namioka and Phelps [17]; some of the results are gathered in the book by Diestal and Uhl [6]). Most important for us is the fact that E is an Asplund space if and only E^* has the Radon-Nikodym property.

The situation with weak-Asplund spaces is not so fortunate (see Larman and Phelps [14], Talagrand [21] and Hagler and Sullivan [8]).

No characterization of weak Asplund spaces seems to be known at the moment. We discuss here the following interesting connection between Asplund and weak Asplund spaces.

(2.2) THEOREM. (Stegall [19]). *Let E be an Asplund space and $Q: E \to E_1$ be a continuous linear map of E onto a dense subset of the Banach space E_1. Then E_1 is weak Asplund.*

As mentioned in the manuscript of Phelps ([18, p. 3–12]) it was not clear from the proof of Stegall whether every closed linear subspace E_2 of E_1 is also weak Asplund. We give here an affirmative answer to this question.

(2.3) DEFINITION. The multivalued map $T: E \to E^*$ is said to be monotone if $\langle x_1 - x_2, x_1^* - x_2^* \rangle \geqq 0$ whenever $x_i^* \in Tx_i$, $i = 1, 2$. T is said to be maximal monotone if its graph is not properly contained into the graph of any other monotone mapping. By Zorn's lemma for every monotone $T: E \to E^*$ there exists a maximal monotone map $\hat{T}: E \to E^*$ such that $Tx \subset \hat{T}x$ for each $x \in E$.

The subdifferential of a continuous real-valued convex function $f: E \to R$ is a multivalued mapping $\partial: E \to E^*$ defined by the formula

$$\partial(x_0) = \{x^* \in E^* : f(x) - f(x_0) \geqq \langle x - x_0, x^* \rangle \quad \text{for every } x \in E\} .$$

It is easily seen that every subdifferential $\partial: E \to E^*$ is a monotone mapping. Since the real valued continuous convex function $f: E \to R$ is Gâteaux differentiable at some $x_0 \in E$ if and only if the set $\partial(x_0)$ consists of only one point, the next result is an improvement of (2.2).

(2.4) THEOREM. *Let E be an Asplund space and $Q: E \to E_1$ be a continuous linear map of E onto a dense subset of the complete normed space E_1. Let E_2 be a closed linear subspace of E_1 and $T: E_2 \to E_2^*$ be a multivalued monotone mapping. Then there exists a dense G_δ subset C of E_2 such that Tx is a singleton for each $x \in C$. In particular, E_2 is weak Asplund.*

PROOF. Without loss of generality we could think that $T: E_2 \to E_2^*$ is a maximal monotone mapping. It is known (Kenderov [9]) that in this case T is norm-to-w^* upper semicontinuous. Also important for us is the theorem of Rockafellar [18] about the local boundedness of monotone mappings. I.e. to each $x_0 \in E$ there corresponds a norm-open $U \subset E$, $x_0 \in U$, such that $T(U) = \bigcup\{Tx: x \in U\}$ is a bounded subset of E^*.

After these preliminary remarks let us consider the diagram

$$\begin{array}{ccccc} E & \xrightarrow{Q} & E_1 & \xleftarrow{\text{id}} & E_2 \\ & & & & \downarrow T \\ E^* & \xleftarrow{Q^*} & E_1^* & \xrightarrow{(\text{id})^*} & E_2^* \end{array}$$

where id is the identity inclusion of E_2 into E_1 and Q^*, (id)* are the mappings conjugate to Q and id correspondingly. We have to prove that the set $D = \{x \in E_2$: Tx has more than one point$\}$ is of the first Baire category in E_2. According to a known result it is enough to show that this set D is of the first category locally. I.e. for each $x \in E_2$, there exists an open (in the norm of E_2) set $U \subset E_2$, $U \ni x$, such that $U \cap D$ is of the first Baire category in E_2. Having in mind the above mentioned result of Rockafellar it suffices to show that $U \cap D$ is of 1st category whenever U is a norm-open subset of E_2 for which $T(U)$ is a bounded subset of E_2. Take such an $U \subset E_2$. It is no harm to think that $T(U) \subset B_2^*$, where B_2^* is the unit ball of E_2^*. Define the mapping $T_1: U \to E_1^*$ by the formula: for $x \in U$

$$T_1 x = ((\text{id}^*)^{-1} Tx) \cap B_1^* .$$

Here B_1^* denotes the w^*-compact unit ball od E_1^*. Since the restriction of id* on B_1^* is w^*-continuous and, therefore, maps w^*-closed subsets of B_1^* into w^*-closed subsets of B_2^*, the map $T_1: (U, \|\cdot\|) \to (B_1^*, w^*)$ is usco. The map $Q^*: E_1^* \to E^*$ is continuous with respect to w^*-topologies in E_1^* and E^*. Since $Q(E)$ is dense in E_1, Q^* is one-to-one. Therefore the map $F: (U, \|\cdot\|) \to (E^*, w^*)$ defined by the formula $F = Q^* \circ T$ will also be usco. Since E is

Asplund, and therefore E^* has RNP, we can apply Theorem (1.3). Hence a dense G_δ subset $C \subset (U, \|\cdot\|)$ exists at each point of which the condition (*) is fulfilled. The rest of the proof is contained into the following result.

(2.5) LEMMA. *Let $x_0 \in U$ be a point at which (*) is fulfilled. Then Tx_0 has only one point.*

PROOF OF THE LEMMA. Take the corresponding point $y_0^* \in Fx_0$ (from (*)) and put $z_0^* = (Q^*)^{-1} y_0^* \in T_1 x_0$. We will show that $T_1 x_0 \subset z_0^* + E_2^\perp$, where

$$E_2^\perp = \{x^* \in E_1^* : \langle x, x^* \rangle = 0 \quad \text{for every } x \in E_2\} .$$

This is equivalent to Tx_0 $(= (\mathrm{id})^* T_1 x_0) = (\mathrm{id})^* z_0^*$. Hence it is enough to prove that

$$(**) \qquad \langle x, z^* - z_0^* \rangle = 0 \quad \text{whenever } x \in E_2 \text{ and } z^* \in T_1 x_0 .$$

Suppose (**) were not fulfilled and take some $e_2 \in E_2$ and $z_1^* \in T_1 x_0$ for which $\langle e_2, z_1^* - z_0^* \rangle = \varepsilon > 0$. By the monotonicity of $T\colon U \to E_2^*$, for every positive number t such that $x = x_0 + te_2 \in U$ and $x^* \in Tx$ we have $\langle e_2, x^* - (\mathrm{id})^* z_1^* \rangle \geqq 0$. Then, for each $z^* \in T_1 x$, we have

$$\begin{aligned}
&\langle e_2, z^* - z_0^* \rangle \\
&\quad = \langle e_2, z^* - z_1^* \rangle + \langle e_2, z_1^* - z_0^* \rangle \\
&\quad \geqq \langle (\mathrm{id}) e_2, z^* - z_1^* \rangle + \varepsilon = \langle e_2, (\mathrm{id})^* (z^* - z_1^*) \rangle + \varepsilon \geqq \varepsilon .
\end{aligned}$$

Therefore, for $t > 0$,

$$T_1(x) = T_1(x_0 + te_2) \subset A := \{z^* \in E_1^* : \langle e_2, z^* \rangle \geqq \varepsilon + \langle e_2, z_0^* \rangle\} .$$

Note that $z_0^* \notin A \cap B_1^*$. Since $Q(E)$ is dense in E_1 and since $A \cap B_1^*$ is a w^*-compact and convex set which does not contain z_0^*, there will exist some $b \in E$ for which $\langle Qb, z^* - z_0^* \rangle \geqq \delta > 0$ whenever $z^* \in A \cap B_1^*$. In particular, for $y^* \in F(x_0 + te_2) = Q^* \circ T_1(x_0 + te_2)$ and $z^* \in T_1(x_0 + te_2)$, $y^* = Q^* z^*$, we have

$$\|b\| \, \|y^* - y_0^*\| \geqq \langle b, y^* - y_0^* \rangle = \langle b, Q^*(z^* - z_0^*) \rangle = \langle Q(b), z^* - z_0^* \rangle \geqq \delta > 0 .$$

Therefore $\|y^* - y_0^*\| \geqq \delta / \|b\| > 0$ for every $y^* \in F(x_0 + te_2)$. Since t may be arbitrarily small positive number this contradicts the condition (*).

(2.6) COROLLARY. (Asplund [2]). *Let E_2 be a closed linear subspace of the weakly compactly generated Banach space E_1* (i.e. *in E_1 a w-compact set K exists whose close linear span is all of E_1*). *Then E_2 is weak Asplund.*

PROOF. We remind the factorization theorem of Davis, Figel, Johnson, and Pelczynski [5] according to which for every weakly compactly generated Banach space E_1 there exists a reflexive space E and a continuous linear map $Q: E \to E_1$ such that $Q(E)$ is dense in E_1. As noted by Stegall [19] this result, together with theorem (2.2) and the fact that reflexive spaces are Asplund, imply that E_1 is weak Asplund. Using theorem (2.4) instead of (2.2) one gets that E_2 is weak Asplund.

(2.7) REMARK. In contrast to Asplund's original proof of (2.6) the one given here does not use renorming results.

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