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ALMOST ISOMETRIC EMBEDDINGS OF l_1 IN PRE-DUALS OF VON NEUMANN ALGEBRAS

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Abstract.

There exists a constant $1<\lambda_0<\infty$ having the following property: If X is a subspace of a predual M_* of a von Neumann algebra M, and if X is isomorphic to l_1 with constant $\lambda<\lambda_0$, then there exists a projection P from M_* onto X with $\|P\|\leqq a(\lambda)$, where $a(\lambda)\to 1$ as $\lambda\to 1$. This extends a theorem of L. E. Dor for the case of commutative M. Consequently, every $\mathscr{L}_{1,\lambda}$-subspace of M_*, with $\lambda<\lambda_0$, is $c(\lambda)$-complemented in M_*.

1. Introduction.

It is well known that if M is a semifinite von Neumann algebra with a canonical trace τ, and if $\{x_j\}\subset L_1(M,\tau)$ are isometrically equivalent to the standard basis of l_1, then $\{x_j\}$ are mutually two-sided disjointly supported (i.e., $x_ix_j^*=0=x_i^*x_j$ for $i\neq j$), see [12] and [20]. This implies immediately the existence of a norm-one projection P from $L_1(M,\tau)$ onto $\overline{\text{span}}\{x_j\}$ (defined by $Px=\sum_j\tau(v_j^*x)x_j$, where v_j are the partial isometries in the polar decompositions of the x_j). In [3] and [20] this disjointness result is used to obtain a complete description of isometries T from $L_1(N,\sigma)$ into $L_1(M,\tau)$, where N is another semifinite von Neumann algebra with a canonical trace σ. It follows, in particular, that for every such T, $T(L_1(N,\sigma))$ is one-complemented in $L_1(M,\tau)$.

We are interested here in the almost-isometric version of this result, namely in the following problem

PROBLEM A. Does there exists a constant $\lambda_0>1$ so that if M and N are von Neumann algebras with preduals M_* and N_* and if T is an isomorphism from N_* into M_* with constant $\|T\|\cdot\|T^{-1}\|=\lambda<\lambda_0$, then there exists a bounded projection P from M_* onto $T(N_*)$ (with $\|P\|\leqq c(\lambda)$, where $c(\lambda)\to 1$ as $\lambda\to 1$)?

The answer to Problem A is known to be affirmative in case both M and N are commutative, namely of the form $L_\infty(\mu)$ and $L_\infty(\nu)$ respectively (and thus

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$M_* = L_1(\mu)$ and $N_* = L_1(\nu)$). Indeed, in [6] L. E. Dor proved that if $\{f_j\}$ are normalized functions in $L_p(0,1)$, $1 \leqq p < \infty$, $p \neq 2$, which are equivalent to the standard l_p-basis with a constant λ, then there exist disjoint measurable subsets A_j of $[0,1]$ so that $\|f_{j|A_j}\|_p \geqq \lambda^{-2/(p-2)}$. From this it follows in the case $p=1$ and $\lambda < 2^{\frac{1}{2}}$ that there exists a projection P from $L_1(0,1)$ onto $\overline{\text{span}}\{f_j\}$ with $\|P\| \leqq (2\lambda^{-2}-1)^{-1}$. By a direct-limit argument and the fact that $L_1(\nu)$ is one-complement in its second dual, it follows that if T is an isomorphism from $L_1(\nu)$ into $L_1(\mu)$ with $\|T\| \cdot \|T^{-1}\| = \lambda < 2^{\frac{1}{2}}$ then there exists a projection P from $L_1(\mu)$ onto $T(L_1(\nu))$ with $\|P\| \leqq (2\lambda^{-2}-1)^{-1}$. Recently, D. Alspach and W. B. Johnson [1] generalize this by replacing $L_1(\nu)$ with a general $\mathscr{L}_{1,\lambda}$-space (which need not be complemented in its second dual) for λ close enough to 1. In the present work we obtain a further generalization, by replacing $L_1(\mu)$ with a general predual of von Neumann algebra. In particular, it solves Problem A affirmatively for the case where N is commutative and M is arbitrary. We conjecture that answer to Problem A is affirmative without any restrictions on N or M.

We remark that the isomorphic analog of Problem A is known to have a negative answer even in the commutative case. Indeed, there exists an uncomplemented subspace of $L_1[0,1]$ which is isomorphic to an $L_1(\mu)$-space, see [4]. We thank the referee for bringing this point to our attention.

Let us recall briefly some basic facts which are necessary for the statement of our eesults; see [7], [13], [17], and [19] for more information on non-commutative L_p-spaces, [5], [14], and [18] for von Neumann algebras, and [11] for Banach space theiry.

Let M be a semi-finite von Neumann algebra, and let τ be a normal, faithful, semi-finite (n.f.s., in short) trace on M. Then for $1 \leqq p < \infty$, $L_p(M,\tau)$ denotes the completion of $\{x \in M;\ \tau(|x|^p) < \infty\}$ with respect to the norm $\|x\|_p = [\tau(|x|^p)]^{1/p}$ (here $|x| = (x^*x)^{\frac{1}{2}}$ is the usual modulus). It is known that $L_p(M,\tau)$ has a realization as a space of all (possibly unbounded) operators x affiliated with M for which $\|x\|_p = [\tau(|x|^p)]^{1/p} < \infty$. The dual of $L_1(M,\tau)$ is $M \equiv L_\infty(M)$, where the duality is given by

$$\langle x, y \rangle = \tau(xy); \quad x \in L_1(M,\tau),\ y \in M\ .$$

Moreover, the Hölder inequality holds: if $1/r = 1/p + 1/q$ then

$$\|xy\|_r \leqq \|x\|_p \cdot \|y\|_q; \quad x \in L_p(M,\tau),\ y \in L_q(M,\tau)\ ,$$

and the dual of $L_p(M,\tau)$ is $L_q(M,\tau)$, $1/p + 1/q = 1$, where the duality is given via the trace of the composition.

A general von Neumann algebra M has a unique (up to an isometry) predual, denoted by M_*, which consists of all normal (i.e., σ-weakly continuous) linear functionals on M, see [15] or [18, Theorem III.3.5]. One denotes $L_1(M)=M_*$ and $L_\infty(M)=M$.

Our main results are the following.

THEOREM 1.1. *Let M be a semifinite von Neumann algebra with a* n.f.s. *trace τ. Let $\{x_j\}$ be normalized elements of $L_1(M,\tau)$ satisfying for some $0<\theta\leqq 1$ and all finite sequences of scalars $\{a_j\}$*

$$\theta \sum_j |a_j| \leqq \left\| \sum_j a_j x_j \right\|_1 .$$

Then there exist sequences $\{u_j\}$ and $\{w_j\}$ of positive elements of M satisfying $\sum_j u_j=\sum_j w_j=I$ and

$$\theta^2 \leqq \tau(|x_j|u_j), \qquad \theta^2 \leqq \tau(w_j|x_j^*|)$$

for $j=1,2,\ldots$.

Moreover, there exists an absolute constant $0<\theta_0<1$, so that if $\theta_0<\theta\leqq 1$ then there exists a projection P from $L_1(M,\tau)$ onto $\overline{\text{span}}\{x_j\}$ with

$$\|P\| \leqq [1-2(1-\theta^2)^{\frac{1}{2}}-\theta(1-\theta^2)]^{-1} \leqq [1-2(1-\theta)^{\frac{1}{2}}]^{-2} .$$

THEOREM 1.2. *There exists a constant $\lambda_0>1$ so that if M is any von Neumann algebra and if T is any isomorphism from l_1 into $L_1(M)=M_*$ with $\|T\|\cdot\|T^{-1}\| =\lambda<\lambda_0$, then there exists a projection P from $L_1(M)$ onto $T(l_1)$ with $\|P\|\leqq \lambda[1-2(1-\lambda^{-1})^{\frac{1}{2}}]^{-2}$.*

Our proof of Theorem 1.1 in section 2 below is modelled after Dor's work [6], but we use the "square function" $(\sum_j |x_j|^2)^{\frac{1}{2}}$ rather than Dor's "maximal function". We use also some operator inequalities generalizing Jensen's inequality.

Theorem 1.2 is proved in section 3, using Theorem 1.1 and a structure theorem, due to U. Haagerup (see [7] and [8]) which says that if M is countably decomposable then $L_1(M)=M_*$ is isometric to a one-complemented subsapce of a direct limit of $L_1(N_k,\tau_k)$-spaces, where τ_k are finite traces.

In section 4 we generalize Theorem 1.2 to almost isometric embeddings of $\mathscr{L}_{1,\lambda}$-spaces (with λ close to 1) in preduals of von Neumann algebras, using the result of Alspach and Johnson [1], and discuss also some open problems.

2. Some operator inequalities and the proof of Theorem 1.1.

The proof of Theorem 1.1 is based on some operator inequalities, which have an independent interest (they can be used in studying Problem C; see section 4 below).

Recall first the following known fact (see [2] and [9]).

PROPOSITION 2.1. *Let* $0<\alpha$ *and let* $\varphi_\alpha\colon B(H)^+ \to B(H)^+$ *be defined by* $\varphi_\alpha(x) = x^\alpha$.

(i) *If* $0<\alpha\leqq 1$, *then* φ_α *is operator-monotone*, i.e., $x, y \in B(H)^+$ *and* $0\leqq x\leqq y$ *implies* $x^\alpha \leqq y^\alpha$.
(ii) *If* $1\leqq\alpha\leqq 2$, *then* φ_α *is operator-convex*, i.e., *if* $x, y \in B(H)^+$ *and* $0\leqq\lambda\leqq 1$ *then*

$$((1-\lambda)x+\lambda y)^\alpha \leqq (1-\lambda)x^\alpha+\lambda y^\alpha .$$

PROPOSITION 2.2. *Let* (Ω, F, μ) *be a probability space, and let* $f\colon \Omega \to B(H)^+$ *be a simple measurable function. For each* $0<p<\infty$ *consider*

$$F(p) = \left(\int_\Omega f(\omega)^p \, d\mu(\omega)\right)^{1/p} .$$

Then $F(p)$ *is a non-decreasing function of* p *in the interval* $[1,\infty)$.

PROOF. It is clearly enough to prove that if $1\leqq p\leqq r\leqq 2p$, then $F(p)\leqq F(r)$. By applying the operator-convex function $\varphi_{r/p}$ we get first that

$$F(p)^r = \left(\int_\Omega f(\omega)^p \, d\mu(\omega)\right)^{r/p} \leqq \int_\Omega f(\omega)^r \, d\mu(\omega) = F(r)^r .$$

Next, by applying the operator-monotone function $\varphi_{1/r}$, we get $F(p)\leqq F(r)$ as desired.

REMARK. Proposition 2.2 can be generalized to non-simple functions assuming as values even unbounded positive operators, which obey some integrability restrictions.

For the rest of this section, let M be a semi-simple von Neumann algebra with a normal, faithful, semi-finite (n.f.s.) trace τ, and let $|x| = (x^*x)^{\frac{1}{2}}$.

PROPOSITION 2.3. *Let* $x_1, x_2, \ldots, x_n \in L_p(M, \tau)$, $1\leqq p<\infty$. *Let* $0<\theta$ *and suppose that for all scalars* $a_1, \ldots, a_n$

$$\theta\left(\sum_{j=1}^n |a_j|^p\right)^{1/p} \leqq \left\|\sum_{j=1}^n a_j x_j\right\|_p, \quad \text{if } 1\leqq p\leqq 2 \tag{2.1}$$

or

$$\left\|\sum_{j=1}^n a_j x_j\right\|_p \leqq \theta\left(\sum_{j=1}^n |a_j|^p\right)^{1/p}, \quad \text{if } 2\leqq p\leqq \infty . \tag{2.2}$$

Then for every choice of scalars $a_1,\ldots,a_n$ we have

$$\theta\left(\sum_{j=1}^{n}|a_j|^p\right)^{1/p} \leqq \left\|\left(\sum_{j=1}^{n}|a_j|^2|x_j|^2\right)^{\frac{1}{2}}\right\|_p, \quad \textit{if } 1\leqq p\leqq 2 \tag{2.3}$$

or

$$\left\|\left(\sum_{j=1}^{n}|a_j|^2|x_j|^2\right)^{\frac{1}{2}}\right\|_p \leqq \theta\left(\sum_{j=1}^{n}|a_j|^p\right)^{1/p}, \quad \textit{if } 2\leqq p\leqq \infty\,. \tag{2.4}$$

PROOF. Since $\{x \in M;\ \tau(|x|^p<\infty\}$ is dense in $L_p(M,\tau)$ in case $1\leqq p<\infty$, it is clearly enough to assume that $x_j \in M$. Assume first that $1\leqq p\leqq 2$. Let $r_j(t)$, $j=1,2,\ldots,n$, denote the Rademacher functions. Then for every choice of scalars $a_1,\ldots,a_n$ and every $t \in [0,1]$ we get by (2.1)

$$\theta^p \sum_{j=1}^{n}|a_j|^p \leqq \left\|\sum_{j=1}^{n} r_j(t)a_jx_j\right\|_p^p = \tau\left|\sum_{j=1}^{n} r_j(t)a_jx_j\right|^p.$$

Integrating this inequality over $[0,1]$, we get by Proposition 2.2

$$\begin{aligned}\theta^p \sum_{j=1}^{n}|a_j|^p &\leqq \int_0^1\left(\tau\left|\sum_{j=1}^{n} r_j(t)a_jx_j\right|^p\right)dt\\ &= \tau\left(\int_0^1\left|\sum_{j=1}^{n} r_j(t)a_jx_j\right|^p dt\right)\\ &= \left\|\left(\int_0^1\left|\sum_{j=1}^{n} r_j(t)a_jx_j\right|^p dt\right)^{1/p}\right\|_p^p\\ &\leqq \left\|\left(\int_0^1\left|\sum_{j=1}^{n} r_j(t)a_jx_j\right|^2 dt\right)^{\frac{1}{2}}\right\|_p^p\\ &= \left\|\left(\sum_{j=1}^{n}|a_j|^2|x_j|^2\right)^{\frac{1}{2}}\right\|_p^p.\end{aligned}$$

where the last step follows by the orthogonality of the $\{r_j\}_{j=1}^n$. The proof in the case $2\leqq p<\infty$ is similar, but all inequalities are reversed.

If $p=\infty$, then for all scalars $a_1,\ldots,a_n$ and all $t \in [0,1]$ we have

$$\left\|\sum_{i,j=1}^{n} r_i(t)r_j(t)\bar{a}_ia_jx_i^*x_j\right\| = \left\|\sum_{j=1}^{n} r_j(t)a_jx_j\right\|^2 \leqq \theta^2 \max_{1\leqq j\leqq n}|a_j|\,.$$

So

$$\left\|\sum_{j=1}^{n} |a_j|^2|x_j|^2\right\| = \left\|\int_0^1 \left|\sum_{j=1}^{n} r_j(t)a_jx_j\right|^2 dt\right\|$$
$$\leqq \int_0^1 \left\|\left|\sum_{j=1}^{n} r_j(t)a_jx_j\right|^2\right\| dt$$
$$\leqq \int_0^1 \theta^2 \max_{1\leqq j\leqq n} |a_j|^2\, dt = \theta^2 \max_{1\leqq j\leqq n} |a_j|^2 .$$

In what follows we denote by $s(x)$ the support projection of the selfadjoint operator x, i.e., the projection onto $(\ker x)^{\perp}$. Notice that $s(x)$ belongs to the von Neumann algebra generated by x, in fact $s(x)=\chi_{\mathbf{R}\setminus\{0\}}(x)$.

PROPOSITION 2.4. *Let M be a von Neumann algebra, let $x_1,\ldots,x_n \in M$ and let $x=(\sum_{j=1}^n |x_j|^2)^{\frac{1}{2}}$. Then there exists a unique sequence $y_1,\ldots,y_n \in M$ so that*

(i) $$y_jx = x_j, \quad j=2,2,\ldots,n;$$

(ii) $$\sum_{j=1}^{n} y_j^*y_j = s(x);$$

(iii) $$\sum_{j=1}^{n} y_j^*x_j = x .$$

PROOF. Suppose that M acts on a Hilbert space H. For every j and every $\xi \in H$ we have

$$\|x_j\xi\|^2 = (x_j^*x_j\xi,\xi) \leqq (x^2\xi,\xi) = \|x\xi\|^2 .$$

Therefore there exists a unique operator y_j on H satisfying $y_j(x\xi)=x_j\xi$ for all $\xi \in H$ and $y_j|_{\ker x}=0$. Clearly, $y_j=s-\lim_{\varepsilon\downarrow 0}x_j(x+\varepsilon I)^{-1}$, so $y_j \in M$. From $y_jx=x_j$ we get $xy_j^*=x_j^*$, and thus

$$x^2 = \sum_{j=1}^{n} x_j^*x_j = \sum_{j=1}^{n} xy_j^*y_jx = x\left(\sum_{j=1}^{n} y_j^*y_j\right)x .$$

This, together with $y_j|_{\ker x}=0$ for $j=1,2,\ldots,n$ imply that $\sum_{j=1}^n y_j^*y_j=s(x)$. Also,

$$x^2 = \sum_{j=1}^{n} x_j^*x_j = \sum_{j=1}^{n} xy_j^*x_j = x\sum_{j=1}^{n} y_j^*x_j .$$

So

$$x = s(x)\sum_{j=1}^{n} y_j^*x_j = \sum_{j=1}^{n} y_j^*x_j .$$

This proves the existence of $\{y_j\}_{j=1}^n \subseteq M$ which satisfy (i), (ii), and (iii). For the uniqueness, notice that $\sum_{j=1}^n y_j^* y_j = s(x)$ implies that $y_j|_{\ker(x)} = 0$ for all j. So $y_i x = x_j$ completely determines y_j, since $x(H)$ is dense in $(\ker(x))^{\perp}$.

REMARK. Let $x_1, \ldots, x_n$; $y_1, \ldots, y_n \in B(H)$, then

$$\left|\sum_{j=1}^{n} y_j^* x_j\right|^2 \leqq \left\|\sum_{j=1}^{n} y_j^* y_j\right\| \cdot \sum_{j=1}^{n} x_j^* x_j \tag{2.5}$$

and

$$\left|\sum_{j=1}^{n} y_j^* x_j\right| \leqq \left\|\left(\sum_{j=1}^{n} y_j^* y_j\right)^{\frac{1}{2}}\right\| \left(\sum_{j=1}^{n} x_j^* x_j\right)^{\frac{1}{2}}. \tag{2.6}$$

("generalized Cauchy–Schwartz inequalities").

Indeed, let $X = \sum_{i=1}^n x_i \otimes e_{i,1}$, $Y = \sum_{i=1}^n y_i \otimes e_{i,1}$, and notice that (2.5) is equivalent to

$$(Y^*X)^*(Y^*X) = X^*YY^*X \leqq \|Y^*\|^2 X^*X = \|Y^*Y\| X^*X$$

and that (2.6) follows from (2.5) by applying the operator-monotone function $x \mapsto x^{\frac{1}{2}}$. Proposition 2.4 says that the inequalities (2.5) and (2.6) become equalities for a special choice of $\{y_i\}_{i=1}^n$, depending on the $\{x_j\}_{j=1}^n$.

The following lemma is central in the proof of Theorem 1.1. Here M is again semi-finite, and τ is a n.f.s. trace on M.

LEMMA 2.5. *Let $\{x_j\}_{j=1}^n \subseteq L_1(M,\tau)^+$, $\|x_j\|_1 = 1$, $j = 1, 2, \ldots, n$ $(n \leqq \infty)$. Let $0 < \theta \leqq 1$ and suppose that for all non-negative scalars $\alpha_1, \ldots, \alpha_k$ $(k \leqq n)$*

$$\theta \sum_{j=1}^{k} \alpha_j \leqq \left\|\left(\sum_{j=1}^{k} \alpha_j^2 x_j^2\right)^{\frac{1}{2}}\right\|_1. \tag{2.7}$$

Then there exists a sequence $\{u_j\}_{j=1}^n \subseteq M^+$ so that $\sum_{j=1}^n u_j = I$ and $\tau(x_j u_j) \geqq \theta^2$ for $j = 1, 2, \ldots$.

PROOF. We proceed like in [6, Proposition 2.2]. In the space $(M \times M \times \ldots)$ with the product w*-topology we consider the set

$$D = \left\{(y_j)_{j=1}^n;\ y_j \in M,\ \sum_{j=1}^{n} y_j^* y_j \leqq I\right\}.$$

D is convex and compact. The easiest way to see this is to notice that the map

$$\varphi\left((y_j)_{j=1}^n\right) = \sum_{j=1}^{n} y_j \otimes e_{j,1}$$

establishes an affine homeomorphism between D and the unit ball of $(M\otimes B(l_2^n))I\otimes e_{1,1}$, taken with the w*-topology.

CLAIM, There exists a $(y_i)_{i=1}^n \in D$ so that $\operatorname{Re}\tau(y_ix_i)\geqq\theta$ for all i.

Indeed, if this is not the case we consider the map T from D into real l_∞^n, equipped with the w*-topology, defined by

$$T(y_i)_{i=1}^n = (\operatorname{Re}\tau(y_ix_i))_{i=1}^n .$$

T is affine and continuous, so $T(D)$ is convex and compact, and by assumption it is disjoint from the w*-closed, convex set $A=\{(c_i)_{i=1}^n \in l_\infty^n;\ c_i\geqq\theta,\ \forall i\}$. So by the separation theorem there exists an element $\alpha=(\alpha_i)_{i=1}^n$ in real l_1^n so that for some $\theta_1<\theta$ we have

$$\theta \leqq \sum_{i=1}^n \alpha_i c_i, \quad \text{for every } (c_i)_{i=1}^n \in A \tag{2.8}$$

and

$$\sum_{i=1}^n \alpha_i \operatorname{Re}\tau(y_ix_i)\leqq\theta_1, \quad \text{for every } (y_i)_{i=1}^n \in D . \tag{2.9}$$

Clearly, (2.8) implies that $\alpha_i\geqq 0$ for all i and $\sum_{i=1}^n \alpha_i\geqq 1$. Since $\theta_1/\theta<1$, there is a finite $k\leqq n$ so that $\theta_1<\theta\sum_{i=1}^k \alpha_i$.

Let $x=(\sum_{i=1}^k \alpha_i^2x_i^2)^{\frac12}$. By Proposition 4, there exists a sequence $y_1,\ldots,y_k \in M$ so that $\sum_{i=1}^k y_i^*y_i=s(x)\leqq I$, and $y_ix=\alpha_ix_i$ for every i. By the assumption (2.7) we get

$$\begin{aligned}\theta\sum_{i=1}^k \alpha_i &\leqq \tau\left(\sum_{i=1}^k \alpha_i^2x_i^2\right)^{\frac12} = \tau(x)\\ &= \tau\left(\sum_{i=1}^k y_i^*y_ix\right)\\ &= \tau\left(\sum_{i=1}^k y_i^*\alpha x_i\right)\\ &= \sum_{i=1}^k \alpha_i\tau(y_i^*x_i) .\end{aligned}$$

Since $\tau(x)\geqq 0$ we get that

$$\begin{aligned}\tau(x) &= \sum_{i=1}^k \alpha_i\tau(y_i^*x_i) = \sum_{i=1}^k \alpha_i\overline{\tau(y_i^*x_i)}\\ &= \sum_{i=1}^k \alpha_i\tau(x_iy_i) = \sum_{i=1}^k \alpha_i\tau(y_ix_i) .\end{aligned}$$

It follows from this and the above inequality that

$$\theta \sum_{i=1}^{k} \alpha_i \leqq \sum_{i=1}^{k} \alpha_i \operatorname{Re} \tau(y_i x_i)$$

$$\leqq \theta_1 < \theta \sum_{i=1}^{k} \alpha_i$$

by (2.9) and the definition of k. This contradiction proves the claim.

Let $(y_i)_{i=1}^n \in D$ be so that $\operatorname{Re} \tau(y_i x_i) \geqq \theta$ for all i. Then

$$(2.10) \qquad \theta \leqq \operatorname{Re} \tau(y_i x_i) \leqq |\tau(y_i x_i)| = |\tau(y_i x_i^{\frac{1}{2}} x_i^{\frac{1}{2}})|$$
$$\leqq [\tau(x_i^{\frac{1}{2}} y_i^* y_i x_i^{\frac{1}{2}})]^{\frac{1}{2}} \cdot \tau(x_i^{\frac{1}{2}} x_i^{\frac{1}{2}})^{\frac{1}{2}} = [\tau(y_i^* y_i x_i)]^{\frac{1}{2}},$$

by the Cauchy–Schwartz inequality and the fact that $\|x_i\|_1 = \tau(x_i) = 1$. Define $u_1 = I - \sum_{i=2}^n y_i^* y_i$ and $u_i = y_i^* y_i$ for $1 < i \leqq n$. Clearly, $u_i \geqq 0$ and $\sum_{i=1}^n u_i = I$. Since $u_1 \geqq y_1^* y_1$ it follows from (2.10) that $\theta^2 \leqq \tau(u_i x_i)$ for all $i = 1, 2, \dots, n$. This completes the proof of the Lemma.

Proof of Theorem 1.1. Let $\{x_j\}_{j=1}^n$ $(n \leqq \infty)$ be normalized elements of $L_1(M, \tau)$ satisfying

$$\theta \sum_{j=1}^{k} |a_j| \leqq \left\| \sum_{j=1}^{k} a_j x_j \right\|_1$$

for every choice of scalars $\{a_j\}_{j=1}^k$ $(k \leqq n)$. By Proposition 2.3 we have

$$\theta\left(\sum_{j=1}^{k} |a_j|\right) \leqq \left\| \left(\sum_{j=1}^{k} |a_j|^2 |x_j|^2\right)^{\frac{1}{2}} \right\|_1$$

and

$$\theta\left(\sum_{j=1}^{k} |a_j|\right) \leqq \left\| \left(\sum_{j=1}^{k} |a_j|^2 |x_j^*|^2\right)^{\frac{1}{2}} \right\|_1$$

for every choice of scalars $\{a_j\}_{j=1}^k$ $(k \leqq n)$. By Lemma 2.5 there exist two sequences $\{u_j\}_{j=1}^n$ and $\{w_j\}_{j=1}^n$ in M^+ so that

$$\sum_{j=1}^{n} u_j = I = \sum_{j=1}^{n} w_j$$

and

$$\theta^2 \leqq \tau(u_j |x_j|), \qquad \theta^2 \leqq \tau(w_j |x_j^*|)$$

for all j. It follows that

$$\begin{aligned}\|x_j - x_j u_j\|_1 &\leqq \| |x_j|^{\frac{1}{2}} |x_j|^{\frac{1}{2}} (1-u_j)^{\frac{1}{2}} (1-u_j)^{\frac{1}{2}} \|_1 \\ &\leqq \| |x_j|^{\frac{1}{2}} \|_2 \cdot \| |x_j|^{\frac{1}{2}} (1-u_j)^{\frac{1}{2}} \|_2 \cdot \| (1-u_j)^{\frac{1}{2}} \|_\infty \\ &\leqq 1 \cdot [\tau(|x_j|^{\frac{1}{2}} (1-u_j) |x_j|^{\frac{1}{2}})]^{\frac{1}{2}} \cdot 1 \\ &= [\tau(|x_j| - |x_j| u_j)]^{\frac{1}{2}} \leqq (1-\theta^2)^{\frac{1}{2}} .\end{aligned}$$

Similarly, $\|x_j - w_j x_j\| \leqq (1-\theta^2)^{\frac{1}{2}}$. It follows that

$$\begin{aligned}\|x_j - w_j x_j u_j\|_1 &\leqq \|x_j - x_j u_j\|_1 + \|x_j u_j - w_j x_j u_j\|_1 \\ &\leqq 2(1-\theta^2)^{\frac{1}{2}} ,\end{aligned}$$

and thus

$$c_j = \|w_j x_j u_j\|_1 \geqq 1 - \|x_j - w_j x_j u_j\|_1 \geqq 1 - 2(1-\theta^2)^{\frac{1}{2}} = c .$$

Let v_j be the partial isometry in the polar decomposition of $w_j x_j u_j$, that is $w_j x_j u_j = v_j |w_j x_j u_j|$, $j = 1, 2, \ldots$. Let $X = \overline{\text{span}}\, \{x_j\}_{j=1}^n$ and define a map P_0 from $L_1(M, \tau)$ into X by

$$P_0 x = \sum_{j=1}^n c_j^{-1} \tau(v_j^* w_j x u_j) x_j .$$

$P_0 x$ is well-defined, since the series converges absolutely. Indeed, if $x \in L_1(M, \tau)$ with polar decomposition $x = v|x|$, then

$$\begin{aligned}\sum_{j=1}^n \|c_j^{-1} \tau(v_j^* w_j x u_j) x_j\|_1 &\leqq c^{-1} \sum_{j=1}^n \| (v_j^* w_j^{\frac{1}{2}})(w_j^{\frac{1}{2}} v |x|^{\frac{1}{2}})(|x|^{\frac{1}{2}} u_j^{\frac{1}{2}}) u_j^{\frac{1}{2}} \|_1 \\ &\leqq c^{-1} \sum_{j=1}^n \|v_j^* w_j^{\frac{1}{2}}\|_\infty \|w_j^{\frac{1}{2}} v |x|^{\frac{1}{2}}\|_2 \| |x|^{\frac{1}{2}} u_j^{\frac{1}{2}} \|_2 \|u_j^{\frac{1}{2}}\|_\infty \\ &\leqq c^{-1} \left(\sum_{j=1}^n \|w_j^{\frac{1}{2}} v |x|^{\frac{1}{2}}\|_2^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n \| |x|^{\frac{1}{2}} u_j^{\frac{1}{2}} \|_2^2 \right)^{\frac{1}{2}} \\ &= c^{-1} \left(\sum_{j=1}^n \tau(w_j^{\frac{1}{2}} v |x| v^* w_j^{\frac{1}{2}}) \right)^{\frac{1}{2}} \left(\sum_{j=1}^n \tau(u_j^{\frac{1}{2}} |x| u_j^{\frac{1}{2}}) \right)^{\frac{1}{2}} \\ &= c^{-1} \left(\tau\left(\sum_{j=1}^n w_j |x^*| \right) \right)^{\frac{1}{2}} \left(\tau\left(\sum_{j=1}^n u_j |x| \right) \right)^{\frac{1}{2}} \\ &= c^{-1} (\tau|x^*|)^{\frac{1}{2}} (\tau|x|)^{\frac{1}{2}} = c^{-1} \|x^*\|_1^{\frac{1}{2}} \|x\|_1^{\frac{1}{2}} \\ &= c^{-1} \|x\|_1 .\end{aligned}$$

So P_0 is bounded and $\|P_0\| \leqq c^{-1}$. Next, if $x = \sum_{j=1}^n t_j x_j \in X$, then using the above estimate we get

$$
\begin{aligned}
\|P_0x-x\|_1 &\leqq \sum_{j=1}^{n} |c_j^{-1}\tau(v_j^* w_j x u_j)-t_j| \\
&= \sum_{j=1}^{n} \left|c_j^{-1} \sum_{i=1}^{n} t_i\tau(v_j^* w_j x_i u_j)-t_j\right| \\
&= \sum_{j=1}^{n} \left|c_j^{-1} \sum_{\substack{i=1\\ i\neq j}}^{n} t_i\tau(v_j^* w_j x_i u_j)\right| \\
&\leqq c^{-1} \sum_{i=1}^{n} |t_i| \sum_{\substack{j=1\\ j\neq i}}^{n} \|v_j^* w_j x_i u_j\|_1 \\
&\leqq c^{-1} \sum_{i=1}^{n} |t_i| \left(\tau\left(\sum_{\substack{j=1\\ j\neq i}}^{n} w_j|x_i^*|\right)\right)^{\frac{1}{2}} \left(\tau\left(\sum_{\substack{j=1\\ j\neq i}}^{n} u_j|x_i|\right)\right)^{\frac{1}{2}} \\
&= c^{-1} \sum_{i=1}^{n} |t_i| (\tau((I-w_i)|x_i^*|))^{\frac{1}{2}} (\tau((I-u_i)|x_i|))^{\frac{1}{2}} \\
&\leqq c^{-1} \sum_{i=1}^{n} |t_i| (1-\theta^2)^{\frac{1}{2}} (1-\theta^2)^{\frac{1}{2}} \quad . \\
&\leqq c^{-1}\theta^{-1}(1-\theta^2)\|x\|_1 \, .
\end{aligned}
$$

It follows that

$$
\|P_{0|X}-I_X\| \leqq c^{-1}\theta^{-1}(1-\theta^2) = (1-2(1-\theta^2)^{\frac{1}{2}})^{-1}\cdot\theta^{-1}(1-\theta^2) = \alpha \, .
$$

If $1 \geqq \theta > \theta_0 = 0.91273\ldots$, then $0 \leqq \alpha < 1$ and $P_{0|X}$ is invertible. Let $T = (P_{0|X})^{-1}$ and define $P\colon L_1(M,\tau) \to X$ by $P = TP_0$. Clearly, P is a projection, $P(L_1(M,\tau)) = X$, and

$$
\|P\| \leqq \|T\| \, \|P_0\| \leqq (1-\alpha)^{-1}c^{-1} = (1-2(1-\theta^2)^{\frac{1}{2}}-\theta^{-1}(1-\theta^2))^{-1} \, .
$$

Putting $\theta = 1/\lambda$, we obtain the estimate

$$
\|P\| \leqq 1+2\sqrt{2}(\lambda-1)^{\frac{1}{2}}+0(\lambda-1) \, .
$$

This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2.

Let $0<\theta_0<1$ be the constant from Theorem 1.1, put $\lambda_0 = \theta_0^{-1}$, and for every $1<\lambda$

$$
f(\lambda) = [1-2(1-\lambda^{-1})^{\frac{1}{2}}]^{-2} \, .
$$

STEP 1. Reduction to the case where M is countably decomposable.

We say that a projection p in von Neumann algebra M is countably decomposable if it can be written as the sum of at most countably many disjoint, non-zero projections from M. M is said to be countably decomposable if the identity projection $I \in M$ (hence, every projection $p \in M$) is countably decomposable.

Suppose that the statement of Theorem 1.2 holds whenever M is countably decomposable. Let N be a von Neumann algebra and let $T: l_1 \to N_*$ be an isomorphism with $\|T\| . \|T^{-1}\| = \lambda < \lambda_0$. Recall that every $\varphi \in N_*$ has a canonical polar decomposition (see [18, Theorem III.4.2]) $\varphi = u|\varphi|$, with u a partial isometry from N and $|\varphi|$ a positive element of N_*. Let M be the von Neumann subalgebra of N generated by all the partial isometries u in the polar decompositions of elements $\varphi \in T(l_1)$, as well as the subprojections of uu^* and u^*u. We claim that M is countably decomposable. Indeed this follows easily from the separability of $T(l_1)$ and from the fact that the support projection of every $\varphi \in (N_*)^*$ is countably decomposable (in N).

Let $j: M \to N$ be the inclusion map. By normality, $j = Q^*$ where $Q: N_* \to M_*$ is a quotient map (the restriction map). By the construction, $Q_1 = Q|_{T(l_1)}$ is an isometry. Consider the isomorphism $S = Q_1 T$ from l_1 into M_*. Clearly, $\|S\| \, \|S^{-1}\| \leqq \|T\| \, \|T^{-1}\| \leqq \lambda$. Since M is countably decomposable and $\lambda < \lambda_0$, we get by our previous hypothesis the existence of a projection P from M_* onto $S(l_1)$ with $\|P\| \leqq \lambda \cdot f(\lambda)$. Define $R = Q_1^{-1} P Q$. Then R is a projection from N_* onto $T(l_1)$ and $\|R\| \leqq \|P\|$.

STEP 2. Proof of Theorem 1.2 in the countably-decomposable case.

Let M be a countably decomposable von Neumann algebra and let T: $l_1 \to M_*$ be an into-isomorphism so that $\|T\| \cdot \|T^{-1}\| = \lambda < \lambda_0$. By a result of U. Haagerup and A. Connes (see [7, § 7], and [8]), M is isomorphic as a von Neumann algebra to a subalgebra of a von Neumann algebra N (the crossed product of M with the group of dyadic rationals), which is the direct limit of an increasing sequence $\{N_k\}_{k=1}^{\infty}$ of *finite* von Neumann subalgebras. Moreover, there exist normal conditional expectations from N onto each of the N_k and onto the image of M in N. Using the fact that a normal map is a dual map (being w*-continuous) we may assume that M_* is a 1-complemented subspace of the Banach space $X = N_*$ which has the following structural properties (see [8]):

(i) There exists an increasing sequence $\{X_k\}_{k=1}^{\infty}$ of subspaces of X so that $\bigcup_{k=1}^{\infty} X_k$ is norm-dense in X;

(ii) X_k is isometric to $(N_k)_* = L_1(N_k, \tau_k)$, $k = 1, 2, \ldots$, where τ_k is a finite n.f.s. trace on N_k;

(iii) There exist norm-one projections P_k from X onto X_k, $k = 1, 2, \ldots$, so that $P_k P_n = P_{\min\{k,n\}}$ for all $k, n = 1, 2, \ldots$.

Let $\{e_i\}_{i=1}^{\infty}$ be the canonical basis of l_1 and let $l_1^n = \text{span}\,\{e_i\}_{i=1}^n$. By a standard approximation argument, there exist an increasing sequence $\{k(n)\}$ of positive integers and one-to-one operators S_n: $l_1^n \to X_{k(n)}$ so that

$$\|S_n - T|_{l_1^n}\| \to 0, \qquad \|S_n\| \cdot \|S_n^{-1}\| = \lambda_n < \lambda_0,\ \lambda_n \to \lambda\,.$$

By Theorem 1.1 there exist projections Q_n from $X_{k(n)}$ onto $S_n(l_1^n)$ with $\|Q_n\| \leqq f(\lambda_n)$. It follows that $\tilde{R}_n = Q_n P_{k(n)}$, $n = 1, 2, \ldots$, are projections from X onto $S_n(l_1^n)$. By a standard perturbation argument, there exist projections R_n from X onto $T(l_1^n)$, with

$$\lim_{n\to\infty} \|R_n\| \leqq f(\lambda)\,.$$

By a standard compactness argument there exist a subnet $\{R_{n(\alpha)}\}$ of the sequence $\{R_n\}$ and an operator R: $X \to T(l_1)^{**}$ so that $Rx = \lim_\alpha R_{n(\alpha)}x$ in the $\sigma(T(l_1)^{**}, T(l_1)^*)$ topology for every $x \in X$. Clearly, $RTy = Ty$ for every $y \in l_1$, and $\|R\| \leqq f(\lambda)$. Since l_1 is 1-complemented in l_1^{**}, there exists a projection Q from $T(l_1)^{**}$ onto $T(l_1)$ with $\|Q\| \leqq \lambda$. It follows that QR: $X \to X$ is a projection with range $T(l_1)$. Thus $P = QR|_{M_*}$ is a projection from M_* onto $T(l_1)$ and $\|P\| \leqq \|Q\| \cdot \|R\| \leqq \lambda f(\lambda)$. This completes the proof of Theorem 1.2.

4. Almost isometric embeddings of $\mathscr{L}_{1,\lambda}$ spaces in preduals of von Neumann algebras.

Recall [10] that a Banach space X is an $\mathscr{L}_{1,\lambda}$-space if every finite dimensional subspace E of X is contained in a further finite dimensional subspace F of X which is λ-isomorphic to l_1^n, where $n = \dim F$ (i.e., X is a direct limit of a family of finite dimensional subspaces, λ-isomorphic to l_1^n-spaces). The class of $\mathscr{L}_{1,\lambda}$-spaces is quite large; let us just mention that every $L_1(\nu)$-space is an $\mathscr{L}_{1,\lambda}$-space for every $\lambda > 1$, and that there exist $\mathscr{L}_{1,\lambda}$-spaces which are uncomplemented in their second duals.

COROLLARY 4.1. *There exists* $1 < \lambda_1$ *and a function* b: $[1, \lambda_1) \to [1, \infty)$ *satisfying* $\lim_{\lambda\to 1^+} b(\lambda) = 1$, *so that if X is an* $\mathscr{L}_{1,\lambda}$*-subspace of a predual* M_* *of a von Neumann algebra M with* $\lambda < \lambda_1$ *then there exists a projection R from* M_* *onto X with* $\|R\| \leqq b(\lambda)$. *In particular, this holds if X is λ-isomorphic to* $L_1(\nu)$ *for some measure* ν.

We need the following known fact (see [18, Theorem III.2.14]).

PROPOSITION 4.2. *Let M be a von Neumann algebra. Then every* $f \in M^*$ *has a unique representation* $f = g + h$, *where g is a normal functional* (i.e., $g \in M_*$) *and h is a singular functional, and* $\|f\| = \|g\| + \|h\|$.

It follows that $M^* \equiv M_* \oplus_1 S$, where S is the subspace of M^* consisting of all singular functionals. We shall need also the following result whose proof is contained in [1].

LEMMA 4.3. *There exists a function* φ: $(1,2) \to [1,\infty)$ *satisfying* $\lim_{\lambda \to 1^+} \varphi(\lambda) = 1$, *for which the following is true. Suppose that* Y *is a Banach space having a subspace* X *so that*

(i) *There exist a constant* $1 < A \leqq 2$ *and a projection* P *from* Y^{**} *onto* Y *so that for every* $y^{**} \in Y^{**}$

$$A \cdot \min\{\|Py^{**}\|, \|(I-P)y^{**}\|\} \leqq \|y^{**}\| .$$

(ii) *There exists an operator* Q: $Y \to X^{**}$ *so that*

$$Q|_X = I_X \quad \text{and} \quad \max\{\|Q\|, \|PQ\|\} = \lambda < A .$$

Then there exists a projection R *from* Y *onto* X *so that* $\|R\| \leqq \varphi(\lambda)$.

PROOF OF COROLLARY 4.1. Let $a(\lambda) = \lambda[1-2(1-\lambda^{-1})^{\frac{1}{2}}]^{-2}$ and put $\lambda_1 = \min\{\lambda_0, a^{-1}(2)\}$, where λ_0 is the constant from Theorem 1.2. Clearly, $1 < \lambda_1$. Let $b(\lambda) = \varphi(a(\lambda))$, where $1 \leqq \lambda < \lambda_1$ and φ is as in Lemma 4.3. Clearly $b(\lambda) \to 1$ as $\lambda \to 1+$.

Suppose now that M is a von Neumann algebra and X is a subspace of M_* which is an $\mathscr{L}_{1,\lambda}$-space for some $\lambda < \lambda_1$. By definition, there exists a directed family $\{X_j\}_{j \in J}$ of subspaces of X with $X = \overline{\bigcup_{j \in J} X_j}$, $n_j = \dim X_j < \infty$, and X_j is λ-isomorphic to $l_1^{n_j}$ for every $j \in J$. By Theorem 1.2 there exists for each $j \in J$ a projection Q_j from M_* onto X_j with $\|Q_j\| \leqq a(\lambda)$. As in the proof of Theorem 1.2, there exists an operator Q: $M_* \to X^{**}$ so that $Qy = \lim_j Q_j y$ in the $\sigma(X^{**}, X^*)$-topology for every $y \in M_*$. It follows that $Q|_X = I_X$ and $\|Q\| \leqq a(\lambda)$.

Let P be the projection from M^* onto M_* given by Proposition 4.2, and put $Y = M_*$. Clearly, $\|P\| = 1$ and (i) of Lemma 4.3 is satisfied with $A = 2$. Since $a(\lambda) < A = 2$, we get by Lemma 4.3 that there exists a projection R from $M_* = Y$ onto X with $\|R\| \leqq \varphi(a(\lambda)) = b(\lambda)$.

Corollary 4.1 gives an affirmative answer to Problem A in the case where N is commutative (i.e., $N_* = L_1(\nu)$) and M is general. As for non-commutative N's, the most interesting case seems to be $N = B(l_2^n)$, $n \geqq 1$ (so $N_* = B(l_2^n)_* = C_1^n$, the $n \times n$ complex matrices with the trace-norm):

PROBLEM B. Does there exist a constant $\lambda_2 > 1$ and a function c: $[1, \lambda_2) \to [1, \infty)$ so that if M is a semifinite von Neumann algebra and X is a subspace of M_* which is λ-isomorphic to C_1^n for some $n \geqq 2$ and $\lambda < \lambda_2$, then there exists a projection P from M_* onto X with $\|P\| \leqq c(\lambda)$?

It is natural to extend Problem A to non-commutative L_p-spaces.

PROBLEM C. Does there exist for every $1<p<\infty$, $p\neq 2$, a constant $\lambda(p)>1$ and a function ψ_p: $[1,\lambda(p))\to[1,\infty)$ (possibly with $\lim_{\lambda\to 1+}\psi_p(\lambda)=1$) so that if M and N are semifinite von Neumann algebras with n.f.s. traces τ and σ respectively, and if $L_p(N,\sigma)$ is λ-isomorphic to a subspace X of $L_p(M,\tau)$, then there exists a projection P from $L_p(M,\tau)$ onto X with $\|P\|\leqq\psi_p(\lambda)$?

Of course, the most important cases are $N=l_\infty^n$ (with $L_p(N,\sigma)=l_p^n$) and $N=B(l_2^n)$ (with $L_p(N,\sigma)=C_p^n$). We remark that Problem C has an affirmative solution in case both M and N are commutative, i.e., $L_p(M,\tau)=L_p(\mu)$ and $L_p(N,\sigma)=L_p(\nu)$ for some measures μ and ν, see [16]. Also, by [3] and [20], any subspace of $L_p(M,\tau)$ which is isometric to $L_p(N,\sigma)$ is 1-complemented.

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