MATH. SCAND. 54 (1984), 95–100

CONTRACTIVE PROJECTIONS ON JORDAN C*-ALGEBRAS AND GENERALIZATIONS

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Dedicated to Professor Max Koecher on his 60th birthday.

Choi and Effros [2] proved that for every completely positive unital projection P on a C*-algebra A the image $P(A)$ is isometric to a C*-algebra and actually becomes a C*-algebra in the new product $(a,b) \mapsto P(ab)$. This is no longer true if "completely positive" is weakened to "positive", but Effros and Størmer [3] proved that the image $P(A)$ still is a Jordan C*-algebra in the new product $(a,b) \mapsto P(a \circ b)$ with $a \circ b = (ab+ba)/2$ the Jordan product on A. Again this is no longer true if the condition "unital" is dropped and "positive" is replaced by "contractive" (which is equivalent to "positive" in the unital case). Quite recently Friedman and Russo [5; 6] proved that for every contractive projection P on A the image $P(A)$ is a Jordan triple system in the new triple product $(a,b,c) \mapsto P\{ab^*c\}$, where $\{ab^*c\} = (ab^*c + cb^*a)/2$ is the Jordan triple product on A. Actually their result says, that the class of J*-algebras in the sense of Harris (these are the closed linear subspaces of C*-algebras invariant under the Jordan triple product $\{ab^*c\}$ — compare [7]) is stable under contractive projections. In this paper we extend this result to a certain class of hermitian Jordan triple systems (called JB*-triples for short) which contains in particular all J*-algebras and also all Jordan C*-algebras (= JB*-algebras). Our proof is very short. It may be considered as an example, how holomorphy can be used in functional analysis.

We recall from [10] the definition of a JB*-*triple* (called C*-triple system in [9]): This is a complex Banach space U together with a sesquilinear map

$$U \times U \to \mathscr{L}(U)$$

$$(x,y) \mapsto x \,\square\, y^*$$

($\mathscr{L}(U)$ = Banach algebra of all bounded linear operators on U) such that for every $u,v,x,y,z \in U$ the following conditions (i)–(iv) hold:

(i) the triple product

Received March 14, 1983.

$$\{xy^*z\} := x \square y^*(z)$$

on $U \times U \times U$ is symmetric, bilinear in x, z (*symmetry*),

(ii) $[x \square y^*, u \square v^*] = \{xy^*u\} \square v^* - u \square \{vx^*y\}^*$ (*Jordan triple identity*),

(iii) $z \square z^*$ is a hermitian operator on U with spectrum $\geqq 0$ (*positivity*),

(iv) $\|z \square z^*\| = \|z\|^2$ (C^*-*condition*).

In [10] it has been shown, that (iv) can be replaced by the condition

$$\|\{zz^*z\}\| = \|z\|^3 . \tag{iv a}$$

Every C^*-algebra A is a JB^*-triple in the triple product

$$\{xy^*z\} = (xy^*z + zy^*x)/2 , \tag{1}$$

i.e. $x \square y^* = (L(xy^*) + R(y^*x))/2$ with L and R left- and right-multiplication. Hence also every J^*-algebra in particular is a JB^*-triple with respect to (1). Further examples of JB^*-triples are obtained by JB^*-algebras (=Jordan C^*-algebras, for the definition compare [13; 1]) with Jordan product $a \circ b$, if the triple product is defined by

$$\{xy^*z\} = x \circ (y^* \circ z) + z \circ (y^* \circ x) - (x \circ z) \circ y^* . \tag{2}$$

The JB^*-triple of lowest dimension, which does not come from a J^*-algebra nor from a JB^*-algebra is the JB^*-triple U of 1×2-matrices over the complex Cayley numbers; it has dimension 16 and corresponds to the exceptional bounded symmetric domain in $\mathbf{C}^{16}$ [12]. More generally, the JB^*-triples classify all bounded symmetric domains in complex Banach spaces [10].

The JB^*-triples form a category—the JB^*-morphisms are the bounded linear maps $\lambda: U \to \tilde{U}$ such that

$$\lambda\{xy^*z\} = \{(\lambda x)(\lambda y)^*(\lambda z)\} \tag{3}$$

for all $x, y, z \in U$. Every such λ is contractive (i.e. $\|\lambda\| \leqq 1$), on the other hand every surjective linear isometry between JB^*-triples automatically is a JB^*-isomorphism. Hence the algebraic structure of a JB^*-triple is already uniquely determined by the underlying Banach space.

Our main result now is

THEOREM. *Let U be a JB^*-triple and $P \in \mathscr{L}(U)$ a contractive projection* (i.e. $P^2 = P$ *and* $\|P\| = 1$) *with image V and kernel W in U. Then $\{VW^*V\} \subset W$ holds and V is a JB^*-triple with respect to the new triple product*

$$(x, y, z) \mapsto P\{xy^*z\} .$$

For the proof we use the following notation: By a holomorphic vector field on an open subset $B \subset U$ we understand every holomorphic map $f\colon B \to U$ which we prefer to write symbolically as differential operator $X = f(u)\partial/\partial u$ with u "the coordinate of U". X is called complete (on B) if for every $z \in B$ the ordinary differential equation $y' = f(y)$ on B admits a solution $y(t,z) \in B$ to the initial value $y(0,z) = z$ defined for all real t—then $z \mapsto y(t,z)$ defined a biholomorphic automorphism $\exp(tX)$ of B for every $t \in \mathbf{R}$. The following is easily varified in case B is bounded, convex and f extends holomorphically into a neighborhood of the closure of B: X is complete on B if and only if for every boundary point $u_0 \in \partial B$ and every continuous $\mathbf{R}$-linear form $\lambda\colon U \to \mathbf{R}$ with $\lambda(B) > \lambda(u_0)$ the inequality $\lambda(f(u_0)) \geqq 0$ holds.

Proof Theorem. Denote by B the open unit ball of U. Then $D := P(B) = B \cap V$ is the open unit ball of V. Fix an element $b \in U$ and consider the holomorphic vector field

$$X := f(u)\frac{\partial}{\partial u}$$

on U, where the polynomial map $f\colon U \to U$ is defined by $f(u) = b - \{ub^*u\}$. By [9] X is complete on B. Consider further the holomorphic vector field

$$Y = g(v)\frac{\partial}{\partial v} = (P(b - \{vb^*v\}))\frac{\partial}{\partial v}$$

on $D \subset V$, where $g := P \circ f \mid D$. Let $v_0 \in \partial D$ be a boundary point and $\lambda\colon \mathrm{V} \to \mathbf{R}$ a continuous $\mathbf{R}$-linear form with $\lambda(D) > \lambda(v_0)$. Then λ can be extended to a continuous $\mathbf{R}$-linear form on U with $\lambda(W) = 0$ and $\lambda(B) > \lambda(v_0)$. This implies $\lambda(g(v_0)) = \lambda(f(v_0)) \geqq 0$ and consequently Y is a complete holomorphic vector field on D. The same argument shows that for $\tilde{b} := P(b)$ also

$$\tilde{Y} = (\tilde{b} - P\{v\tilde{b}^*v\})\frac{\partial}{\partial v}$$

is complete on D. Since Y, $\tilde{Y}$ have the same value and derivative at the origin, Cartan's uniqueness theorem [11; 1.2] implies $Y = \tilde{Y}$ and hence

$$P\{P(a)b^*P(c)\} = P\{P(a)P(b)^*P(c)\} \tag{4}$$

for all $a, b, c \in U$, whence $\{VW^*V\} \subset W$. The mapping $V \to D$ defined by

$$b \mapsto \exp\left((b - P\{vb^*v\})\frac{\partial}{\partial v}\right)(0)$$

is differentiable and has invertible derivative at the origin. By the implicit function theorem therefore the orbit $G(0) \subset D$ of the group $G := \mathrm{Aut}\,(D)$ of all biholomorphic automorphisms of D is open in D. By [8, Lemma 2] therefore G is transitive on D, i.e. D is a bounded symmetric domain with corresponding JB*-triple product $P\{xy^*z\}$ [9; 10].

Corollary 1. *Let A be a (unital) JB*-algebra and $P \in \mathscr{L}(A)$ a unital* (i.e. $P(1) = 1$) *contractive projection. Then the image $V = P(A)$ is a JB*-algebra in the product $(a, b) \mapsto P(a \circ b)$ and the involution $a \mapsto P(a^*)$. The kernel W of P satisfies*

$$V \circ W^* \subset W.$$

Proof. For every $a, b \in U$ we have $a \circ b^* = \{ab^*1\}$ and hence

$$P(P(a) \circ b^*) = P(P(a) \circ P(b)^*) \tag{5}$$

as a consequence of (4).

Corollary 2 (Effros-Størmer). *Let A be a unital C*-algebra and $P \in \mathscr{L}(A)$ a unital positive projection. Then the image V of P is a Jordan algebra in the product*

$$(a, b) \mapsto P(ab + ba)/2\,.$$

Proof. A is a JB*-algebra in the Jordan product $a \circ b := (ab + ba)/2$ and P is contractive by [13, Corollary 1].

We consider some simple examples

Example 1. Let U be a JB*-triple and $e \in U$ a tripotent, i.e. $\{ee^*e\} = e$. Then

$$U = U_1 \oplus U_{\frac{1}{2}} \oplus U_0$$

with U_ν the ν-eigenspace of the hermitian operator $\theta := e \square e^*$ is called the *Peirce-decomposition* with respect to e [12]. $\sigma := \exp\,(2\pi i \theta)$ is an isometry of U and is called the *Peirce reflection* with respect to e. $P := (1 + \sigma)/2 \in \mathscr{L}(U)$ is the projection with image $V := U_1 \oplus U_0$ and kernel $W := U_{\frac{1}{2}}$. Obviously, P is *bicontractive* (i.e. P and $1 - P$ are contractive). V and W are JB*-subtriples of U.

Example 2. Let $\Gamma \subset \mathbf{C}^*$ be a compact subgroup with normalized Haar measure $d\mu$ and let S be a locally compact topological space with a continuous Γ-action. Fix an integer n and define a contractive projection P on the commutative C*-algebra $A = \mathscr{C}_0(S)$ by

$$Pf(s) = \int_\Gamma \gamma^{-n} f(\gamma s)\, d\mu(\gamma)\,.$$

Then the image

$$P(A) = \{f \in A : f(\gamma s) = \gamma^n f(s)\}$$

is a JB*-subtriple but not a subalgebra in general. In the special case where $n = \pm 1$, $\Gamma = \{t \in \mathbf{C} : |t| = 1\}$ and S is a principal Γ-bundle, the image $P(A)$ is isometric to a C*-algebra if and only if S is a trivial Γ-bundle, i.e. $S \approx (S/\Gamma) \times \Gamma$ —compare [10, 1.13 Corollary].

The next example is taken from [4, § 1].

EXAMPLE 3. Let $A = \mathscr{C}([0,1])$ and define a unital contractive projection P on A by

$$Pf(t) = f(0) + (f(1) - f(0))t\,.$$

Then the image $V = P(A)$ is the space of all affine functions. V is not a JB*-subtriple of A but isometric to the C*-algebra $\mathbf{C} \times \mathbf{C}$.

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