MATH. SCAND. 54 (1984), 101–113

APPLICATIONS OF SET-VALUED RADON–NIKODYM THEOREMS TO CONVERGENCE OF MULTIVALUED L^1-AMARTS

ĐINH QUANG LUU

Introduction.

The theory of integrals, conditional expectations and martingales of multifunctions has been developed, recently, by Hiai and Umegaki ([6], [7]). Costé [4], Luu ([9], [10]), among others. The class of multi-valued L^1-asymptotic martingales (L^1-amarts) is here introduced and considered. It is shown that this class contains multi-valued martingales [7], quasi-martingales and uniform amarts [10]. The main purpose of this paper is to give some characterization and convergence theorems for multi-valued L^1-amarts.

In Section 1, after stating definitions and notations we shall give some basic properties of integrals, conditional expectations and the Pettis distance. In Section 2, we shall consider set-valued $\dot{\sum}$-measures and prove a set-valued Radon-Nikodym theorem which can be regarded as a multi-valued version of the vector-valued Radon-Nikodym theorem, given by Uhl ([11, Proposition 1.1.]). In Section 3, we shall introduce the class of multi-valued L^1-amarts. Some characterization and convergence theorems for multi-valued L^1-amarts are established. Finally, in Section 4, we shall give some related counter-examples.

1. Notations and definitions.

Let $(\Omega, \mathscr{A}, P)$ be a probability space, $\mathscr{B}$ a sub σ-field of $\mathscr{A}$ and $\boldsymbol{B}$ a separable real Banach space. By $L^1(\boldsymbol{B}, \mathscr{A})$ we mean the Banach space of all (equivalence classes of) Bochner integrable functions $f\colon \Omega \to \boldsymbol{B}$ with

$$\|f\|_1 = \int_\Omega \|f\| \, dP$$

and

$$|||f||| = \sup_{x^* \in U^0} \int_\Omega |\langle x^*, f\rangle| \, dP$$

Received January 8, 1982; in revised form May 19, 1982.

where $U^0 = \{x^* \in \boldsymbol{B}^*;\ \|x^*\| \leqq 1\}$. Thus $|||f|||$ is the Pettis norm of f.

We shall also consider the following classes:

$$\boldsymbol{K} = \{X \in \boldsymbol{B}\ ;\ X \text{ is closed bounded non-empty}\}$$
$$\boldsymbol{K}_c = \{X \in \boldsymbol{K}\ ;\ X \text{ is convex}\}$$
$$\boldsymbol{K}_{cc} = \{X \in \boldsymbol{K}_c\ ;\ X \text{ is compact}\}\ .$$

Therefore these classes become complete metric spaces with the Hausdorff's metric $h(\cdot,\cdot)$, defined by

$$\text{(1.1.)} \qquad h(X, Y) = \max\left\{\sup_{x \in X} d(x, Y),\ \sup_{y \in Y} d(y, X)\right\}, \qquad (X, Y \in \boldsymbol{K})\ .$$

A multi-function $X: \Omega \to \boldsymbol{K}$ is called weakly $\mathscr{A}$-measurable, if the set $\{\omega, X(\omega) \cap V \neq \varnothing\} \in \mathscr{A}$ for each open subset V of $\boldsymbol{B}$. Such a multi-function X will be called integrably bounded, if the real-valued function $\omega \mapsto \|X(\omega)\|$ is integrable, where given $Z \in \boldsymbol{K}$, $\|Z\|$ is defined by

$$\|Z\| = \sup\{\|z\|\ ,\ z \in Z\}\ .$$

If this occurs, then we write $X \in L^1(\boldsymbol{K}, \mathscr{A})$, where two multi-functions $Y_1, Y_2 \in L^1(\boldsymbol{K}, \mathscr{A})$ are considered to be identical, if $Y_1(\omega) = Y_2(\omega)$, a.e. Now let

$$L^1(\boldsymbol{K}_c, \mathscr{A}) = \{X \in L^1(\boldsymbol{K}, \mathscr{A})\ ;\ X(\omega) \in \boldsymbol{K}_c,\ \text{a.e.}\}$$
$$L^1(\boldsymbol{K}_{cc}, \mathscr{A}) = \{X \in L^1(\boldsymbol{K}, \mathscr{A})\ ;\ X(\omega) \in \boldsymbol{K}_{cc},\ \text{a.e.}\}\ .$$

Then according to [7], these classes become complete metric spaces with metric H, defined by

$$H(X, Y) = \int_\Omega h(X(\omega), Y(\omega))\, dP, \qquad (X, Y \in L^1(\boldsymbol{K}, \mathscr{A}))$$

where $h(\cdot,\cdot)$ is given by (1.1).

It is interesting to note that if

$$X \in L^1(\boldsymbol{K}, \mathscr{A}), \qquad Y \in L^1(\boldsymbol{K}_c, \mathscr{A}), \qquad \text{and} \qquad Z \in L^1(\boldsymbol{K}_{cc}, \mathscr{A})\ ,$$

then

$$\mathscr{E}(X, \mathscr{B}) \in L^1(\boldsymbol{K}, \mathscr{B}), \qquad \mathscr{E}(Y, \mathscr{B}) \in L^1(\boldsymbol{K}_c, \mathscr{B}), \qquad \text{and} \qquad \mathscr{E}(Z, \mathscr{B}) \in L^1(\boldsymbol{K}_{cc}, \mathscr{B})$$

where given $M \in L^1(\boldsymbol{K}, \mathscr{A})$, $\mathscr{E}(M, \mathscr{B})$ denotes the $\mathscr{B}$-conditional expectation of M (cf. [7]).

In connection with the Pettis norm, we present here the Pettis distance $H_w(X, Y)$, defined for any two elements $X, Y \in L^1(\boldsymbol{K}_c, \mathscr{A})$ as follows

$$H_w(X, Y) = \sup_{x^* \in U^0} \int_\Omega |\delta^*(x^*, X(\omega)) - \delta^*(x^*, Y(\omega))| \, dP ,$$

where given $Z \in \boldsymbol{K}_c$ and $x^* \in U^0$, $\delta^*(x^*, Z) = \sup\{\langle x^*, z\rangle, \ z \in Z\}$.

The proof of the following result is similar to that of the vector-valued case.

PROPERTY 1.1. *Let* $X, Y \in L^1(\boldsymbol{K}_c, \mathscr{A})$ *and* $X_1, Y_1 \in L^1(\boldsymbol{K}_c, \mathscr{B})$, *then*

$$H_w(X, Y) \leqq H(X, Y) \tag{1.2}$$

$$H_w[\mathscr{E}(X, \mathscr{B}), \mathscr{E}(X, \mathscr{B})] \leqq H_w(X, Y) \tag{1.3}$$

$$\sup_{A \in \vee_n \mathscr{A}_n} h\left(\text{cl} \int_A X_1 \, dP, \text{cl} \int_A Y_1 \, dP\right) \leqq H_w(X_1, Y_1) \tag{1.4}$$

$$\leqq 2 \sup_{A \in \vee_n \mathscr{A}_n} h\left(\text{cl} \int_A X_1 \, dP, \text{cl} \int_A Y_1 \, dP\right)$$

For futher information, we refer to [6] and [7].

2. A Radon-Nikodym theorem for set-valued $\dot{\sum}$-measures.

Throughout this paper, let $2^{\boldsymbol{B}}$ denote the class of all non-empty subsets of $\boldsymbol{B}$. Following Hiai [6], call $M: \mathscr{A} \to 2^{\boldsymbol{B}}$ a set-valued measure, if $M(\emptyset) = \{0\}$ and

$$M\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} M(A_n)$$

for every sequence $\langle A_n \rangle$ of pairwise disjoint elements of $2^{\boldsymbol{B}}$ where given a sequence $\langle X_n \rangle$ of $2^{\boldsymbol{B}}$, the sum $\sum_{n=1}^{\infty} X_n$ is defined by

$$\sum_{n=1}^{\infty} X_n = \left\{x \in \boldsymbol{B}, \ x = \sum_{n=1}^{\infty} X_n \text{ (unconditionally convergent), each } x_n \in X_n\right\}.$$

For such a set-valued measure M and for each $A \in \mathscr{A}$, we define

$$|M|(A) = \sup \sum_{n=1}^{k} \|M(A_n)\| ,$$

where the sup is taken over all $\mathscr{A}$-measurable finite partitions $\langle A_n \rangle_{n=1}^k$ of A. If $|M|(\Omega) < \infty$, then M is said to be of bounded variation. Thus according to [6, Proposition 1.1.] $|M|$ becomes a positive measure. Similarly, following Costé [2] call $M: \mathscr{A} \to \boldsymbol{K}$ a set-valued $\dot{\sum}$-measure, if

$$M(\emptyset) = \{0\} \quad \text{and} \quad M\left(\bigcup_{n=1}^{\infty} A_n\right) = \dot{\sum_{n=1}^{\infty}} M(A_n)$$

for every sequence $\langle A_n \rangle$ of pairwise disjoint elements of $\mathscr{A}$, where given a sequence $\langle X_n \rangle$ of $\boldsymbol{K}$, the sum $\dot{\sum}_{n=1}^{\infty} X_n$ is defined by

1) For each k, $\dot{\sum}_{n=1}^{k} X_n = X_1 \dot{+} X_2 \dot{+} \dots \dot{+} X_k = \mathrm{cl}\,(X_1 + \dots + X_k)$,
2) $\dot{\sum}_{n=1}^{\infty} X_n \in \boldsymbol{K}$,
3) $\lim_{k\to\infty} h(\dot{\sum}_{n=1}^{k} X_n, \dot{\sum}_{n=1}^{\infty} X_n) = 0$.

In connection with [6, Theorem 1.3.], we present here the following result.

PROPERTY 2.1. *Let $M: \mathscr{A} \to 2^{\boldsymbol{B}}$ be a set-valued set-function then,*

1) *If M is a set-valued measure of bounded variation, then* $\mathrm{cl}\, M$ *and* $\overline{\mathrm{co}}\, M$ (defined in [6]), *are both $\dot{\sum}$-measures with*

$$|M|(A) = |\mathrm{cl}\, M|(A) = |\overline{\mathrm{co}}\, M|(A) \qquad (A \in \mathscr{A}).$$

2) *If M is a set-valued measure of bounded variation with $M(\Omega)$ relatively weakly compact, then $\bar{M}$ is also a $\dot{\sum}$-measure with*

$$|M|(A) = |\bar{M}|(A) \qquad (\text{cf. } [6]).$$

3) *If $M(A)$ is weakly compact for each $A \in \mathscr{A}$, then M is a set-valued measure of bounded variation if and only if M is a $\dot{\sum}$-measure of bounded variation.*

A $\dot{\sum}$-measure $M: \mathscr{A} \to \boldsymbol{K}$ is said to satisfy the Uhl's condition, if given $\varepsilon > 0$ there is some $C \in \boldsymbol{K}_{cc}$ such that for any but fixed $\delta > 0$ one can choose some $A_\delta \in \mathscr{A}$ with $P(A_\delta) \geqq 1 - \varepsilon$ and such that: $\forall A \in \mathscr{A}$ if $A \subset A_\delta$ then $M(A) \subset P(A)C + \delta U$, where U denotes the unit ball of $\boldsymbol{B}$.

The main purpose of this section is to prove the following Radon-Nikodym theorem for set-valued $\dot{\sum}$-measures which is a multi-valued version of the vector-valued Radon-Nikodym theorem given by Uhl ([11, Proposition 1.1.]).

THEOREM 2.2. *Let $M: \mathscr{A} \to \boldsymbol{K}_c$ be a $\dot{\sum}$-measure. Then M has a Radon-Nikodym derivative, contained (uniquely) in $L^1(\boldsymbol{K}_{cc}, \mathscr{A})$, if and only if the following conditions are statisfied:*

1) *M is P-continuous*, i.e. *if $P(A) = 0$, then $M(A) = \{0\}$,*
2) $|M|(\Omega) < \infty$,
3) *M satisfies the Uhl's condition.*

PROOF. Let $M: \mathscr{A} \to \boldsymbol{K}_c$ be a $\dot{\sum}$-measure. Suppose first that M has a Radon-Nikodym derivative, take, $X \in L^1(\boldsymbol{K}_{cc}, \mathscr{A})$, that is

$$M(A) = \int_A X\, dP \qquad (A \in \mathscr{A}).$$

Hence by [6, Corollary 5.4], $M(A) \in \boldsymbol{K}_{cc}$ for each $A \in \mathscr{A}$. Further, by virtue of (3) in Property 2.1., M is even a set-valued measure. Therefore, Theorem 5.2.

in [6] implies that M satisfies conditions (1–3). Conversely, suppose that conditions (1–3) are satisfied. We shall show that M satisfies even condition (iii) of Theorem 5.2. in [6]. Indeed, let $\varepsilon>0$ be any but fixed. Take the set $C \in \boldsymbol{K}_{cc}$ which exists in condition 3) for this ε. Thus, one can choose a sequence $\langle A_n \rangle$ in $\mathscr{A}$ with $P(A_n) \geqq 1-\varepsilon$ $(n \geqq 1)$ and

$$M(A) \subset P(A)C + n^{-2}U \qquad (n \geqq 1,\ A \in \mathscr{A} \text{ and } A \subset A_n)\,.$$

Now put

$$A_\varepsilon = \lim_n \sup A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\,.$$

Then it is clear that $A_\varepsilon \in \mathscr{A}$ with $P(A_\varepsilon) \geqq 1-\varepsilon$. Given $A \in \mathscr{A}$ with $A \subset A_\varepsilon$, one has $A \subset \bigcup_{m=n}^{\infty} A_m$ $(n \geqq 1)$. Hence, if $n \in \mathbf{N}$ is any but fixed and if we define

$$S_n = A_n;\ S_{n+1} = A_{n+1} \setminus A_n;\ \ldots;\ S_{n+k+1} = A_{n+k+1} \setminus \bigcup_{m=n}^{n+k} A_m\,,$$

then it is obvious that $\langle S_m \rangle_{m \geqq n}$ is disjoint and

$$A \subset \bigcup_{m=n}^{\infty} A_m = \bigcup_{m=n}^{\infty} S_m\,.$$

Consequently,

$$\begin{aligned} M(A) &= \dot{\sum_{m=n}^{\infty}} M(A \cap S_m) \\ &\subset \dot{\sum_{m=n}^{\infty}} [P(A \cap S_m)C + m^{-2}U] \\ &= P(A)C + \left(\sum_{m=n}^{\infty} m^{-2}\right)U\,. \end{aligned}$$

This follows that $M(A) \subset P(A)C$. Further, put $\varepsilon_n = 1/n$ $(n \geqq 1)$. Take the sets C_n and A_n as above for each ε_n. Then

$$M(\Omega) = M(A_n) + M(A_n^c)\,.$$

It implies that:

$$h(M(A_n), M(\Omega)) \leqq |M|(A_n^c) \to 0, \qquad \text{as } n \to \infty\,.$$

Consequently, $M(\Omega) \in \boldsymbol{K}_{cc}$, hence $M(A) \in \boldsymbol{K}_{cc}$ for each $A \in \mathscr{A}$. Therefore the measure M satisfies conditions (i)–(iii) required in Theorem 5.2 in [6]. Thus, M has a Radon-Nikodym derivative contained (uniquely) in $L^1(\boldsymbol{K}_{cc}, \mathscr{A})$. This completes the proof of Theorem 2.2.

COROLLARY 2.3. *Let $X \in L^1(\boldsymbol{K}_c, \mathscr{A})$, then the set-valued set-function M: $\mathscr{A} \to \boldsymbol{K}_c$, defined by $M(A) = \mathrm{cl}\int_A X\,dP$, is a set-valued $\dot{\sum}$-measure with $|M|(A) = \int_A \|X\|\,dP$. Furthermore, $X \in L^1(\boldsymbol{K}_{cc}, \mathscr{A})$ if and only if M satisfies the Uhl's condition.*

PROOF. This follows from [6, Proposition 4.1], Property 2.1, and the above theorem.

3. Characterization and convergence theorems for multi-valued L^1-amarts.

Throughout this section, we fix an increasing sequence $\langle \mathscr{A}_n \rangle$ of sub σ-fields of $\mathscr{A}$ with $\mathscr{A}_n \uparrow \mathscr{A}$. A sequence $\langle X_n \rangle$ of multi-functions is said to be adapted to $\langle \mathscr{A}_n \rangle$, if each X_n is weakly $\mathscr{A}_n$-measurable. Unless otherwise mentioned all our considered sequences are assumed adapted to $\langle \mathscr{A}_n \rangle$ and taken from $L^1(\boldsymbol{K}_c, \mathscr{A})$. Call $\langle X_n \rangle$ a martingale (cf. [7]), if $X_n = X_n(m)$ for all $m \geqq n \in \mathsf{N}$ where given $m \geqq n \in \mathsf{N}$, $X_n(m) = \mathscr{E}(X_m, \mathscr{A}_n)$. Equivalently, $\langle X_n \rangle$ satisfies the equality $H(X_n, X_n(m)) = 0$ for all $m \geqq n \in \mathsf{N}$. We call $\langle X_n \rangle$ an L^1-amart, if

$$\lim_{m \geqq n \to \infty} H(X_n, X_n(m)) = 0 \tag{3.1}$$

equivalently,

$$\forall \varepsilon > 0\ \exists n_0\ \forall m \geqq n \geqq n_0 \qquad H(X_n, X_n(m)) \leqq \varepsilon\,. \tag{3.2}$$

REMARK 3.1. As in [10], call $\langle X_n \rangle$ a uniform amart if

$$\lim_{\tau \geqq \eta \in T} H(X_\eta, X_\eta(\tau)) = 0\,,$$

where T denotes the set of all bounded stopping times and

$$X_\eta(\tau) = \mathscr{E}(X_\tau, \mathscr{A}_\eta) \qquad (\tau \geqq \eta \in T)\,.$$

Thus, by (3.1), every uniform amart (hence by [10], every quasi-martingale, martingale) is an L^1-amart.

The following result gives us a characterization of L^1-amarts.

THEOREM 3.2. *A sequence $\langle X_n \rangle$ is an L^1-amart if and only if there is a unique martingale $\langle M_n \rangle$ in $L^1(\boldsymbol{K}_c, \mathscr{A})$ such that*

$$\lim_{n \to \infty} H(X_n, M_n) = 0\,. \tag{3.3}$$

PROOF. ($\Rightarrow$) Let $\langle X_n \rangle$ be an L^1-amart. Then by (3.2) and [7, Theorem 5.2

(1)], the sequence $\langle X_n(m)\rangle_{m=n}^{\infty}$ is Cauchy in metric H for each $n \in \mathbf{N}$. Consequently, there is a sequence $\langle M_n \rangle$, adapted to $\langle \mathscr{A}_n \rangle$ such that

$$\lim_{m\to\infty} H(X_n(m), M_n) = 0 \quad (n \geqq 1)\,. \tag{3.4}$$

We claim that $\langle M_n \rangle$ is a martingale. Indeed, let $m \geqq n \in \mathbf{N}$ be any but fixed. By (3.4), one has

$$\lim_{k\to\infty} H(X_m(m+k), M_m) = 0\,.$$

Hence, by [7, Theorem 5.2 (1)], we obtain

$$\lim_{k\to\infty} H[\mathscr{E}(X_m(m+k), \mathscr{A}_n), M_n(m)] = 0\,.$$

Therefore, in view of [7, Theorem 5.3 (3)], we get

$$\lim_{k\to\infty} H[X_n(m+k), M_n(m)] = 0\,.$$

Consequently, by (3.4) one has $M_n(m) = M_n$ a.e. This proves the above assertion. Further, since for all $m \geqq n \in \mathbf{N}$

$$H(X_n, M_n) \leqq H[X_n, X_n(m)] + H[X_n(m), M_n]$$

then by (3.1) and (3.4) we have

$$\lim_{n\to\infty} H(X_n, M_n) = 0\,.$$

This proves (3.3).

($\Leftarrow$) Conversely, suppose that there is a martingale $\langle M_n \rangle$ which satisfies (3.3). Hence for all $m \geqq n \in \mathbf{N}$, we get

$$H[X_n, X_n(m)] \leqq H[X_n(m), M_n] + H[M_n, X_n]\,.$$

Consequently, again, by [7, Theorem 5.2 (1)]

$$H[X_n, X_n(m)] \leqq H[X_m, M_m] + H[M_n, K_n]\,.$$

Therefore, condition (3.3) implies that

$$\lim_{m\geqq n\to\infty} H(X_n(m), X_n) = 0\,.$$

This proves (3.1), hence $\langle X_n \rangle$ must be a L^1-amart.

We show now that the martingale satisfying (3.3) is unique. Otherwise, there are two martingale $\langle M_n^1 \rangle$ and $\langle M_n^2 \rangle$ such that

$$\lim_{n\to\infty} H(X_n, M_n^i) = 0 \quad (i=1,2)\,.$$

Hence by [7, Theorem 5.2 (1)], for each n and for all k of N one has

$$\begin{aligned} H[M_n^1, M_n^2] &\leqq H[M_{n+k}^1, M_{n+k}^2] \\ &\leqq H[X_{n+k}, M_{n+k}^1] + H[X_{n+k}, M_{n+k}^2] . \end{aligned}$$

Consequently, $H(M_n^1, M_n^2)=0$, by letting $k\uparrow\infty$. Equivalently, $M_n^1=M_n^2$ a.e. $(n\geqq 1)$. This completes the proof of the theorem.

COROLLARY 3.3. *An L^1-amart $\langle X_n\rangle$ in $L^1(\boldsymbol{K}_c, \mathscr{A})$ is H-convergent (hence) to some element of $L^1(\boldsymbol{K}_c, \mathscr{A})$, if and only if the martingale associated with $\langle X_n\rangle$ is H-convergent.*

The following result generalizes Theorem 6.5 in [7].

COROLLARY 3.4. (see [9] and [4]). *A Banach space $\boldsymbol{B}$ has the Radon-Nikodym property* w.r.t. $(\Omega, \mathscr{A}, P)$ *if and only if every uniformly integrable (equivalently, L^1-bounded and equicontinuous) L^1-amart in $L^1(\boldsymbol{K}_c, \mathscr{A})$ is regular* i.e. *there is some $X_\infty \in L^1(\boldsymbol{K}_c, \mathscr{A})$ such that*

$$\lim_{n\to\infty} H(X_n, X_n(\infty)) = 0 .$$

PROOF. This follows from Theorem 3.2 and Corollary 3.5 in [9], where it was shown that a Banach space $\boldsymbol{B}$ has the Radon-Nikodym property if and only if every uniformly integrable martingale in $L^1(\boldsymbol{K}_c, \mathscr{A})$ is regular.

Now let $\langle X_n\rangle$ be a L^1-amart and $\langle M_n\rangle$ the martingale satisfying (3.3). Define $F\colon \bigvee_n \mathscr{A}_n \to \boldsymbol{K}_c$ by $F(A)=\mathrm{cl}\int_A M_n\,dP$ $(A\in\mathscr{A}_n)$ then by Corollary 2.3, F is a finitely additive $\dot{\sum}$-measure on $\bigvee_n \mathscr{A}_n$. Furthermore, by (3.3) we get

$$\forall\varepsilon>0\ \exists n_0\ \forall n\geqq n_0\ \sup_{A\in\mathscr{A}_n} h\left[\mathrm{cl}\int_A X_n\,dP,\ F(A)\right]\leqq\varepsilon . \tag{3.5}$$

In the sequel, F will be called the limit $\dot{\sum}$-measure associated with $\langle X_n\rangle$.

LEMMA 3.5. *Let*

$$H_w[\boldsymbol{K}_c, \langle\mathscr{A}_n\rangle] = \left\{X\in L^1(\boldsymbol{K}_c,\mathscr{A}), \lim_{n\to\infty} H_w[\mathscr{E}(X,\mathscr{A}_n), X]=0\right\}$$

and $\langle X_n\rangle$ a L^1-amart with its limit $\dot{\sum}$-measure F.

Suppose that F has a generalised Radon-derivative, take

$$X_\infty\in H_w[\boldsymbol{K}_c,\langle\mathscr{A}_n\rangle] \text{ that is } F(A) = \mathrm{cl}\int_A X_\infty\,dP \quad (A\in\mathscr{A}) .$$

Then $\langle X_n \rangle$ *is* H_w*-convergent to* X_∞.

PROOF. Under the above assumptions, it follows from (3.5), (1.4), and [7, Theorem 5.4 (2)] that

$$\lim_{n\to\infty} H_w[\mathscr{E}(X_\infty, \mathscr{A}_n), X_n] = 0 .$$

On the other hand since, by definition of $H_w[\boldsymbol{K}_c, \langle \mathscr{A}_n \rangle]$, we have

$$\lim_{n\to\infty} H_w[\mathscr{E}(X_\infty, \mathscr{A}_n), X_\infty] = 0 ,$$

then

$$\lim_{n\to\infty} H_w[X_n, X_\infty] = 0 .$$

This completes the proof of the lemma.

THEOREM 3.6. *Let* $\boldsymbol{B}$ *be a Banach space with the Radon-Nikodym property and* $\langle X_n \rangle$ *a uniformly integrable* L^1*-amart with* $F(\Omega)$ *compact, where* F *is the limit* $\dot{\sum}$*-measure associated with* $\langle X_n \rangle$. *Then* $\langle X_n \rangle$ *is* H_w*-convergent to some element of* $L^1(K_c, \mathscr{A})$.

PROOF. We call $\langle X_n \rangle$ uniformly integrable, if so is $\langle \|X_n(\cdot)\| \rangle$. Let $\langle X_n \rangle$ be a uniformly integrable L^1-amart and F the limit $\dot{\sum}$-measure associated with $\langle X_n \rangle$. Then also the martingale $\langle M_n \rangle$ satisfying (3.3) is uniformly integrable. Therefore, F can be extended to a $\dot{\sum}$-measure $F\colon \mathscr{A} \to \boldsymbol{K}_c$ which is P-continuous and of bounded variation. Further, since $F(\Omega)$ is compact, then by [6, Corollary 2.4] and Property 2.1 (3), F is also a set-valued measure taking values in $\boldsymbol{K}_{cc}$. Hence by virtue of [6, Theorem 4.3], F has a generalized Radon-Nikodym derivative, take $X_\infty \in L^1(\boldsymbol{K}_c, \mathscr{A})$, that is

$$F(A) = \mathrm{cl} \int_A X_\infty dP \quad (A \in \mathscr{A}) .$$

Again, since $F(\Omega)$ is compact then in view of Corollary 2.4 and [6, Lemma 5.1], the class $\mathscr{E} = \{F(A);\ A \in \vee_n \mathscr{A}_n\}$ is relatively compact w.r.t. the Hausdorff's metric $h(\cdot,\cdot)$, given by (1.1). Now let $\hat{\boldsymbol{B}}$ be the space of all real-valued functions on $\boldsymbol{B}^*$, positively homogeneous, whose restrictions to equicontinuous sets of $\boldsymbol{B}^*$, are bounded and strongly continuous. Then by the remark of Theorem II-19 ([1, p. 51]), $\hat{\boldsymbol{B}}$ becomes a Banach space with the norm

$$\|\varphi\| = \sup\{|\varphi(x^*)|;\ \|x^*\| \leqq 1\} \qquad (\varphi \in \hat{\boldsymbol{B}}) .$$

Moreover, one can embed $\boldsymbol{K}_c$ (hence $\boldsymbol{K}_{cc}$) into a closed convex cone in $\hat{\boldsymbol{B}}$ in such a way that conditions (i)–(iii), mentioned in [7, Theorem 3.6 (1)] are satisfied (see, Theorem II-18 and II-19, pp. 49–51 in [1]). Therefore, as a $\hat{\boldsymbol{B}}$-valued measure, F has a relatively compact range. Thus by ([8, Theorem 9]), given $\varepsilon > 0$, there is some $\vee_n \mathscr{A}_n$-simple function X_ε in $L^1(\hat{\boldsymbol{B}}, \mathscr{A})$ such that

$$\sup_{A \in \vee_k \mathscr{A}_k} \left\| \int_A X_\varepsilon \, dP - F(A) \right\| \leqq \frac{\varepsilon}{6}.$$

Equivalently,

$$\sup_{A \in \vee_k \mathscr{A}_k} h\left[\int_A X_\varepsilon \, dP, \; F(A) \right\| \leqq \frac{\varepsilon}{6}.$$

Since X_ε is $\vee_n \mathscr{A}_n$-simple, then X_ε is weakly $\mathscr{A}_{n_0}$-measurable for some $n_0 \in \mathbf{N}$. We infer that by Property 1.1. if $n \geqq n_0$ then

$$\begin{aligned} H_w[X_n, X_\infty] &\leqq 2 \sup_{A \in \vee_k \mathscr{A}_k} h\left[\mathrm{cl} \int_A X_n \, dP, \int_A X_\varepsilon \, dP \right] + 2\frac{\varepsilon}{6} \\ &\leqq 2 \sup_{A \in \mathscr{A}_n} h\left[\mathrm{cl} \int_A X_n \, dP, \int_A X_\varepsilon \, dP \right] + \frac{\varepsilon}{3} \\ &\leqq 2 \sup_{A \in \mathscr{A}_n} h\left[\mathrm{cl} \int_A X_n \, dP, \; F(A) \right] + 2 \cdot \frac{\varepsilon}{6} + \frac{\varepsilon}{3}. \end{aligned}$$

But in view of (3.5), one can suppose, without loss of generality, that for $n \geqq n_0$ the following inequality holds

$$\sup_{A \in \mathscr{A}_n} h\left(\mathrm{cl} \int_A X_n \, dP, \; F(A) \right) \leqq \frac{\varepsilon}{6}.$$

Therefore, if $n \geqq n_0$ one has

$$H_w[X_n, X_\infty] \leqq 2 \cdot \frac{\varepsilon}{6} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

It follows that $\langle X_n \rangle$ is H_w-convergent to X_∞. This completes the proof of Theorem 3.6.

Note that neither in Lemma 3.5 nor in Theorem 3.6, the word "H_w-convergent" cannot be replaced by "H-convergent" (see Example 4.3, below). We obtain however the following result which generalizes [7, Theorem 6.3].

THEOREM 3.7. *An L^1-amart $\langle X_n \rangle$ (in $L^1(\boldsymbol{K}_c, \mathscr{A})$) is H-convergent to some $X_\infty \in L^1(\boldsymbol{K}_{cc}, \mathscr{A})$ if and only if it is uniformly integrable and satisfies the Uhl's condition, i.e. given $\varepsilon > 0$, there is some $C \in \boldsymbol{K}_{cc}$ such that for any but fixed $\delta > 0$ one can choose some $n_0 \in \mathbf{N}$, $A_0 \in \mathscr{A}_{n_0}$ with $P(A_0) \geqq 1 - \varepsilon$ and such that*

$$\forall n \geqq n_0 \; \forall A \in \mathscr{A}_n \textit{ if } A \subset A_0, \textit{ then } \int_A X_n dP \subset P(A)C + \delta U \, .$$

PROOF. Let $\langle X_n \rangle$ be a L^1-amart in $L^1(\boldsymbol{K}_c, \mathscr{A})$ and $\langle M_n \rangle$ the martingale satisfying (3.3). Suppose first that $\langle X_n \rangle$ is H-convergent to some $X_\infty \in L^1(\boldsymbol{K}_{cc}, \mathscr{A})$. Thus, if we define $M'_n = \mathscr{E}[X_\infty, \mathscr{A}_n]$ for each n then by [7, Theorem 6.1], oen has also

$$\lim_{n \to \infty} H(M'_n, X_\infty) = 0 \, .$$

Hence

$$\lim_{n \to \infty} H(X_n, M'_n) = 0 \, .$$

Consequently, by Theorem 3.2, the uniqueness of $\langle M_n \rangle$ implies that $M_n = M'_n$ $(n \geqq 1)$, thus $\lim_{n \to \infty} H(M_n, X_n) = 0$. Applying [7, Theorem 6.3] to the martingale $\langle M_n \rangle$, we infer that $\langle M_n \rangle$ is uniformly integrable and satisfying the Uhl's condition, hence by (3.3) so is $\langle X_n \rangle$. Conversely, suppose that $\langle X_n \rangle$ is uniformly integrable and satisfies the Uhl's condition then by (3.5) the limit $\dot{\sum}$-measure F, associated with $\langle X_n \rangle$ is P-continuous of bounded variation and satisfying the Uhl's condition. Therefore, by Theorem 2.2, F has a Radon-Nikodym derivative, take X_∞ contained (uniquely) in $L^1(\boldsymbol{K}_{cc}, \mathscr{A})$. Now, define $M'_n = \mathscr{E}(X_\infty, \mathscr{A}_n)$ $(n \geqq 1)$. Again, by [7, Theorem 6], $\langle M'_n \rangle$ is H-convergent to X_∞. Thus, by Property 1.1 and Lemma 3.5, we have

$$\lim_{n \to \infty} H_w[X_n, M'_n] = 0 \, .$$

But $\lim_{n \to \infty} H(X_n, M_n) = 0$ then it is easy to check that in the case, one has $H_w(M_n, M'_n) = 0$ $(n \geqq 1)$.

Equivalently, by Property 1.1 and [6, Corollary 5.4], we obtain

$$\mathrm{cl} \int_A M_n dP = \int_A M'_n dP \qquad (A \in \mathscr{A}_n) \, .$$

Consequently, in view of [7, Lemma 4.4], we get

$$M_n \subset M'_n \text{ a.e.} \qquad (n \geqq 1) \, .$$

Hence, $M_n \in L^1(\boldsymbol{K}_{cc}, \mathscr{A}_n)$ $(n \geqq 1)$. It implies that

$$M_n = M'_n \text{ a.e.} \qquad (n \geqq 1) \, .$$

Therefore, by (3.3) and Corollary 3.3, $\langle X_n \rangle$ is H-convergent to $X_\infty \in L^1(\boldsymbol{K}_{cc}, \mathscr{A})$. This completes the proof of Theorem 3.7.

4. Some counter examples.

EXAMPLE 4.1. (See [6, Example 1.4 (2)]). Let $\boldsymbol{B}$ be a nonreflexive Banach space. Hence by ([6, Example 1.4]), $\boldsymbol{B}$ contains two disjoint closed bounded convex sets, X and Y which cannot be separated. Therefore the set

$$X-Y = \{x-y;\ x \in X;\ y \in Y\}$$

is not closed. Let $\Omega=[0,1)$, $\mathscr{A}=\mathscr{B}_{[0,1)}$ and P the Lebesgue measure on $\mathscr{B}_{[0,1)}$. Define $M\colon \mathscr{A} \to 2^{\boldsymbol{B}}$ by

$$M(A) = P[A \cap [0,\tfrac{1}{2})]X - P[A \cap [\tfrac{1}{2},1)]Y \quad (A \in \mathscr{A}).$$

Then M is a set-valued measure having convex values which is P-continuous and of bounded variation.

On the one hand, since

$$\operatorname{cl} M([0,\tfrac{1}{2})) + \operatorname{cl} M([\tfrac{1}{2},1)) = \tfrac{1}{2}(X-Y) \neq \tfrac{1}{2}\operatorname{cl}(X-Y) = \operatorname{cl} M([0,1)),$$

then $\operatorname{cl} M$ fails to be a set-valued measure.

On the other hand, by Property 2.1 (1), $\operatorname{cl} M$ is however a $\boldsymbol{K}_c$-valued $\dot{\sum}$-measure. At the same time, the example shows that the assumption that each $M(A)$ is weakly compact in Property 2.1. (3) cannot be omitted.

EXAMPLE 4.2. Following [5], call $\langle X_n \rangle$ an approximate martingale, if the net $\langle \operatorname{cl} \int_0 X_\tau dP \rangle_{\tau \in T}$ is bounded. We note that there is a L^1-potential (hence, L^1-amart) of nonnegative real-valued functions which fails to be an approximate martingale.

Indeed, let $(\Omega, \mathscr{A}, P)$ be as in Example 4.1 and $n \in \mathrm{N}$. Define $X_{n,k}$: $\Omega \to [0,\infty)$ by

$$X_{n,k} = n\mathbf{1}_{[(k-1)2^{-n},k2^{-n})} \qquad k=1,2,\ldots,2^n$$

where $\mathbf{1}_A$ denotes the characteristic function of A. By $(n,k)>(n',k')$ we mean either $n>n'$ or $n=n'$ and $k>k'$. Let $\langle P_n \rangle$ be the resulting sequence renumbered according to the above order. It is easy to see that $\langle P_n \rangle$ is a L^1-potential but $\sup_T \int_\Omega P_\tau dP = \infty$.

EXAMPLE 4.3. There is a regular martingale in $L^1(\boldsymbol{K}_c(l_2), \mathscr{A})$ which is H_w-convergent but it is not H-convergent.

CONSTRUCTION. (See [3, Example 1].) Let $(\Omega, \mathscr{A}, P)$ be as in the previous examples and $\boldsymbol{B}=l_2$. Let X be the multi-function, constructed by Costé [3], then X has the following properties,

a) $X(\omega) \not\subseteq \boldsymbol{K}_{cc}$ for all $\omega \in \Omega$

b) $\int_A X\,dP \in \boldsymbol{K}_{cc}$ $(A \in \mathscr{A})$.

Let $\langle \mathscr{A}_n \rangle$ be any but fixed increasing sequence of finite sub σ-fields of $\mathscr{A}$ with $\mathscr{A}_n \uparrow \mathscr{A}$. Thus by (b), $M_n = \mathscr{E}(X, \mathscr{A}_n) \in L^1(\boldsymbol{K}_{cc}, \mathscr{A})$ $(n \geqq 1)$. Hence by (a) and the H-completeness of $L^1(\boldsymbol{K}_{cc}, \mathscr{A})$ $\langle M_n \rangle$ cannot be H-convergent. But by (b) and Theorem 3.6, $\langle M_n \rangle$ is H_w-convergent to X.

Note that the above example with Theorem 6.1 in [7] shows that $L_c^1(\Omega, \boldsymbol{B}) \nsubseteq H_w[\boldsymbol{K}_c, \langle \mathscr{A}_n \rangle]$ even in the case where $\boldsymbol{B}$ is a Hilbert space and $L_c^1(\Omega, \boldsymbol{B})$ is borrowed from [7]. This is an essential difference between the theory of vector-valued martingales and that of multi-valued ones.

Acknowledgement. The author is very grateful to the referee for his invaluable comments and suggestions.

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INSTITUTE OF MATHEMATICS
HANOI
208Đ ĐỘI CẤN
NGHIA DO-TU LIEM - VIETNAM