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ON INTEGER-VALUED FOURIER-STIELTJES TRANSFORMS ON A LCA GROUP WITH A CERTAIN DIRECTION

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1. Introduction.

For a locally compact Hausdorff space Ω, $M(\Omega)$ denotes the Banach space of complex-valued bounded regular measures on Ω with the total variation norm. Let G be a LCA group with the dual group $\hat{G}$. G_d is the group G with the discrete topology and m_G is the Haar measure of G. Let $M(G)$ be the measure algebra and "$\hat{\ }$" denotes the Fourier–Stieltjes transform. We denote by $B(\hat{G})$ the set $\{\hat{\mu};\ \mu \in M(G)\}$. For $\hat{\mu} \in B(\hat{G})$, we signify $\|\hat{\mu}\|$ by $\|\hat{\mu}\| = \|\mu\|$. According to the notation of [9], let $F(G)$ be the set of measures in $M(G)$ whose Fourier–Stieltjes transforms are integer-valued.

In their proof of the Cohen Idempotent Theorem, Ito and Amemiya proved the following theoerem.

THEOREM 1.1 (See [5].) *Let μ be a measure in $F(G)$. Then μ can be represented as follows:*

$$\mu = \sum_{i=1}^{n} \sum_{j=1}^{l_i} n_{ij}\gamma_{ij}m_{H_i},$$

where $n_{ij} \in \mathbf{Z}$, $\gamma_i \in \hat{G}$ and H_i are compact subgroups of G such that $\{\sum_{j=1}^{l_i} n_{ij}\gamma_{ij}m_{H_i}\}$ are mutually singular.

In his Semi-Idempotent Theorem, Kessler proved the following theorem.

THEOREM 1.2 (See [6].) *Let G be a compact abelian group such that $\hat{G}$ is ordered. Suppose $\hat{\mu} \in B(\hat{G})$ is integer-valued on $\{\gamma \in \hat{G};\ \gamma > 0\}$. Then there exists a measure $\nu \in F(G)$ such that $\hat{\nu}(\gamma) = \hat{\mu}(\gamma)$ for $\gamma > 0$.*

On the other hand, Svensson has recently obtained the following theorem.

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THEOREM 1.3 ([10, Theorem 3.3.1]). *Let Γ be a* LCA *group and Ω a bounded open convex set in* $\mathbf{R}^n$. *Suppose $\hat{\mu} \in B(\mathbf{R}_d^n \times \Gamma)$ is integer-valued on $\Omega \times \Gamma$. Then there exists an integer-valued $\hat{\nu} \in B(\mathbf{R}_d^n \times \Gamma)$ such that $\hat{\nu}(\gamma)=\hat{\mu}(\gamma)$ on $\Omega \times \Gamma$.*

In this paper we prove that Svensson's theorem is satisfied for a LCA group G such that there exists a continuous homomorphism from $\hat{G}$ into $\mathbf{R}^n$. We first state our result in this paper.

MAIN THEOREM. *Let G be a* LCA *group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into* $\mathbf{R}^n$. *Let Ω be a bounded open convex set in* $\mathbf{R}^n$. *Suppose $\hat{\mu} \in B(\hat{G})$ is integer-valued on $\psi^{-1}(\Omega)$. Then there exists a measure $\nu \in F(G)$ such that $\hat{\nu}(\gamma)=\hat{\mu}(\gamma)$ on $\psi^{-1}(\Omega)$.*

LEMMA 1.4. *Let G be a* LCA *group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into* $\mathbf{R}^n$. *Let τ be an automorphism on $\hat{G}$, and let β be an automorphism on* $\mathbf{R}^n$. *We put $\psi_{\#}=\beta\circ\psi\circ\tau$. Then the following are equivalent:*

(I) *For any bounded open convex set Ω' in* $\mathbf{R}^n$ *and $\hat{\mu}' \in B(\hat{G})$ such that $\hat{\mu}'$ is integer-valued on $\psi_{\#}^{-1}(\Omega')$, there exists an integer-valued $\hat{\nu}' \in B(\hat{G})$ such that $\hat{\nu}'(\gamma) =\hat{\mu}'(\gamma)$ on $\psi_{\#}^{-1}(\Omega')$;*

(II) *For any bounded open convex set Ω in* $\mathbf{R}^n$ *and $\hat{\mu} \in B(\hat{G})$ such that $\hat{\mu}$ is integer-valued on $\psi^{-1}(\Omega)$, there exists an integer-valued $\hat{\nu} \in B(\hat{G})$ such that $\hat{\nu}(\gamma) =\hat{\mu}(\gamma)$ on $\psi^{-1}(\Omega)$.*

PROOF. (I) $\Rightarrow$ (II): We note that $\psi^{-1}(\Omega)=(\beta\circ\psi)^{-1}(\beta(\Omega))$ and $\beta(\Omega)$ is a bounded open convex set in $\mathbf{R}^n$. We define $\hat{\mu}' \in B(\hat{G})$ by $\hat{\mu}'=\hat{\mu}\circ\tau$. Then $\hat{\mu}'$ is integer-valued on $(\beta\circ\psi\circ\tau)^{-1}(\beta(\Omega))=\tau^{-1}(\psi^{-1}(\Omega))$. Hence, by the hypothesis, there exists an integer-valued $\hat{\nu}' \in B(G)$ such that

$$\hat{\nu}'(\gamma) = \hat{\mu}'(\gamma) \quad \text{on } (\beta\circ\psi\circ\tau)^{-1}(\beta(\Omega)) .$$

We define $\hat{\nu} \in B(\hat{G})$ by $\hat{\nu}=\hat{\nu}'\circ\tau^{-1}$. Then $\hat{\nu}$ is integer-valued, and for $\gamma \in \psi^{-1}(\Omega)$, we have

$$\begin{aligned} \hat{\nu}(\gamma) &= \hat{\nu}'(\tau^{-1}(\gamma)) \\ &= \hat{\mu}'(\tau^{-1}(\gamma)) \qquad (\tau^{-1}(\gamma) \in (\beta\circ\psi\circ\tau)^{-1}(\beta(\Omega))) \\ &= \hat{\mu}(\gamma) . \end{aligned}$$

Thus (I) $\Rightarrow$ (II) is proved. Since $\psi=\beta^{-1}\circ\psi_{\#}\circ\tau^{-1}$, (II) $\Rightarrow$ (I) can be also proved as same as above. This completes the proof.

REMARK 1.5. Svensson proved in ([10, 4.2, p. 132]) the following:

Let Ω be a non-empty bounded open subset of $\mathbf{R}^n$ whose closure $\bar{\Omega}$ in $\mathbf{R}^n$ is non-convex. Then there exists $\hat{\mu} \in B(\mathbf{R}^n_d)$, integer-valued on Ω and such that $\hat{\mu}|_\Omega \neq \hat{v}|_\Omega$ for all integer-valued $\hat{v} \in B(\mathbf{R}^n_d)$.

2. Several lemmas.

Let e_k be an unit vector in $\mathbf{R}^n$ such that $e_k = (\overbrace{0,\ldots,1}^{k},\ldots,0)$.

DEFINITION 2.1. Let G be a LCA group and ψ a continuous homomorphism from $\hat{G}$ into $\mathbf{R}^n$ such that $e_k \in \psi(\hat{G})$ $(1 \leqq k \leqq n)$. Let χ_k be an element in $\hat{G}$ such that $\psi(\chi_k) = e_k$. We define a discrete subgroup Λ of $\hat{G}$ by

$$\Lambda = \{m_1\chi_1 + \ldots + m_n\chi_n;\ (m_1,\ldots,m_n) \in \mathbf{Z}^n\},$$

and we put $K = \Lambda^\perp$ (the annihilator of Λ).

The following lemma is due to [11].

LEMMA 2.2 ([11, Proposition 2.7]). *$\{(\psi(\gamma), \gamma|_K);\ \gamma \in \hat{G}\}$ is a closed subgroup of $\mathbf{R}^n \oplus \hat{K}$ and topologically isomorphic to $\hat{G}$.*

From above lemma, the following lemma is obtained.

LEMMA 2.3. *If* $\ker(\psi)$ *is open* $\{(\psi(\gamma), \gamma|_K);\ \gamma \in \hat{G}\}$ *is an open subgroup of $\mathbf{R}^n_d \oplus \hat{K}$. In particular, $\hat{G}$ is topologically isomorphic to an open subgroup $\{(\psi(\gamma), \gamma|_K);\ \gamma \in \hat{G}\}$ of $\mathbf{R}^n_d \oplus \hat{K}$.*

PROOF. Since $\ker(\psi)$ is open, $\{\gamma|_K;\ \gamma \in \ker(\psi)\}$ is an open subgroup of $\hat{K}$. Hence $\{(\psi(\gamma), \gamma|_K);\ \gamma \in \hat{G}\}$ is an open subgroup of $\mathbf{R}^n_d \oplus \hat{K}$. This completes the proof.

The following proposition is a special case of Main Theorem.

PROPOSITION 2.4. *Let G be a* LCA *group and ψ a continuous homomorphism from $\hat{G}$ into $\mathbf{R}^n$ such that $\psi(\hat{G})$ contains e_k $(1 \leqq k \leqq n)$. We assume that* $\ker(\psi)$ *is open. Let Ω be a bounded open convex set in $\mathbf{R}^n$. Suppose $\hat{\mu} \in B(\hat{G})$ is integer-valued on $\psi^{-1}(\Omega)$. Then there exists a measure $v \in F(G)$ such that $\hat{v}(\gamma) = \hat{\mu}(\gamma)$ on $\psi^{-1}(\Omega)$.*

PROOF. Let $\Gamma = \{(\psi(\gamma), \gamma|_K);\ \gamma \in \hat{G}\}$. Then, by Lemma 2.3, $\hat{\mu}$ can be regarded as a function in $B(\Gamma)$. Since Γ is an open subgroup of $\mathbf{R}^n_d \oplus \hat{K}$, there exists $\hat{\xi}_\mu \in B(\mathbf{R}^n_d \oplus \hat{K})$ such that

$$\hat{\xi}_\mu(\gamma) = \begin{cases} \hat{\mu}(\gamma) & \text{for } y \in \Gamma \\ 0 & \text{for } \gamma \notin \Gamma . \end{cases}$$

Evidently $\hat{\xi}_\mu$ is integer-valued on $\Omega \times \hat{K}$. Hence, by Theorem 1.3, there exists an integer-valued $\hat{v}_0 \in B(\mathrm{R}^n_d \oplus \hat{K})$ such that $\hat{v}_0(\sigma) = \hat{\xi}_\mu(\sigma)$ for $\sigma \in \Omega \times \hat{K}$. We define $\hat{v} \in B(\hat{G})$ by $\hat{v}(\gamma) = \hat{v}_0(\psi(\gamma), \gamma|_K)$. Then $\hat{v}$ is integer-valued and $\hat{v}(\gamma) = \hat{\mu}(\gamma)$ on $\psi^{-1}(\Omega)$. This completes the proof.

We need the following two lemmas in order to prove Main Theorem.

LEMMA 2.5. *Let Γ_i be proper subgroups of R^m $(i = 1, 2, \ldots, l)$. Then, for any $x_i \in \mathrm{R}^m$, $n_i \in \mathrm{Z}$ and open set V in R^m, we have*

$$\sum_{i=1}^{l} n_i \chi_{(x_i + \Gamma_i)} \neq M \quad \text{on } V$$

for any M, where $\chi_{(x_i + \Gamma_i)}$ is the characteristic function of $x_i + \Gamma_i$.

PROOF. Suppose $\sum_{i=1}^{l} n_i \chi_{(x_i + \Gamma_i)} = M$ on V for some $M \in \mathrm{Z}$. Let H_i be the annihilator of Γ_i in $\bar{\mathrm{R}}^m$, and we put $I = \{1 \leqq i \leqq l;\ \Gamma_i \text{ is dense in } \mathrm{R}^m\}$. Then, by ([10, Theorem 3.1.2, p. 126]), we have

$$\sum_{i \in I} n_i x_i m_{H_i} = M \delta_0 . \tag{1}$$

However, since Γ_i are proper subgroups of R^m, m_{H_i} are continuous measures (cf. [4, (24.23) Theorem, p. 384]). Hence (1) yields a contradiction, and the proof is complete.

LEMMA 2.6. *Let F be a LCA group. Let E be an open coset in $\mathrm{R}^k_d \oplus R^{m-k} \oplus F$ such that E is not open in $\mathrm{R}^k \oplus R^{m-k} \oplus F$, where R^k_d is the group R^k with discrete topology. Then for each $u \in F$,*

$$E_u = \{x \in \mathrm{R}^k \oplus \mathrm{R}^{m-k};\ (x, u) \in E\}$$

is not open in $\mathrm{R}^k \oplus \mathrm{R}^{m-k}$ if $E_u \neq \emptyset$.

PROOF. Suppose there is $u \in F$ with $E_u \neq \emptyset$ such that E_u is open. Let x_0 be an element in $\mathrm{R}^k \oplus \mathrm{R}^{m-k}$ such that $(x_0, u) \in E$. Then, since E is open in $\mathrm{R}^k_d \oplus \mathrm{R}^{m-k} \oplus F$, there is an open neighborhood U of u in F such that $\{x_0\} \times U \subset E$. Then $\{x_0\} \times U + E_u \times \{u\} - E_u \times \{u\}$ is an open set in $\mathrm{R}^k \oplus \mathrm{R}^{m-k} \oplus F$, and it is contained in E because E is a coset. This contradicts the hypothesis that E is not open in $\mathrm{R}^k \oplus \mathrm{R}^{m-k} \oplus F$, and the proof is complete.

3. Proof of Main Theorem.

THEOREM 3.1. *Let G be a* LCA *group and ψ a continuous homomorphism from $\hat{G}$ into $\mathbf{R}^n$ such that $[\psi(\hat{G})]$ coincides with $\mathbf{R}^n$, where $[\psi(\hat{G})]$ is the subspace of $\mathbf{R}^n$ generated by $\psi(\hat{G})$. Let Ω be a bounded open convex set in $\mathbf{R}^n$. Suppose $\hat{\mu} \in B(\hat{G})$ is integer-valued on $\psi^{-1}(\Omega)$. Then there exists an integer-valued $\hat{\nu} \in B(\hat{G})$ such that $\hat{\nu}(\gamma)=\hat{\mu}(\gamma)$ on $\psi^{-1}(\Omega)$.*

PROOF. We prove the theorem by dividing two cases that ker (ψ) is open or not.

CASE 1. Suppose ker (ψ) is open. By the hypothesis that $[\psi(\hat{G})]$ coincides with $\mathbf{R}^n$, there exists an automorphism β on $\mathbf{R}^n$ such that $\beta\circ\psi(\hat{G})$ contains e_k $(1\leqq k\leqq n)$. Hence, by Lemma 1.4 and Proposition 2.4, the theorems is obtained.

CASE 2. Suppose ker (ψ) is not open. By the structure theorem of LCA groups, we have $\hat{G}\cong\mathbf{R}^m\oplus F$, where m is a nonnegative integer and F is a LCA group which contains a compact open subgroup F_0. Since ker (ψ) is not open, $\psi(\mathbf{R}^m)\neq\{0\}$. We put $k=\dim\psi(\mathbf{R}^m)$. Then $1\leqq k\leqq\min(m,n)$, and there exist an automorphism τ on $\hat{G}$ $(\cong\mathbf{R}^m\oplus F)$ and an automorphism β on $\mathbf{R}^n$ such that

$$\beta\circ\psi\circ\tau(x_1,\dots,x_k,\dots,x_m) = (x_1,\dots,x_k,0,\dots,0) \quad \text{for } (x_1,\dots,x_k,\dots,x_m)\in\mathbf{R}^m .$$

Hence, by Lemma 1.4, we may assume that ψ satisfies

$$\psi(x_1,\dots,x_k,\dots,x_m) = (x_1,\dots,x_k,0,\dots,0) \quad \text{for } (x_1,\dots,x_k,\dots,x_m)\in\mathbf{R}^m .$$

Let $\hat{G}_\tau=\mathbf{R}^k_d\oplus\mathbf{R}^{m-k}\oplus F$, and let G_τ be the dual group of $\hat{G}_\tau$. Then ψ: $\hat{G}_\tau\mapsto\mathbf{R}^n$ is a continuous homomorphism and ker (ψ) is open in $\hat{G}_\tau$. Hence by Case 1 there is an integer-valued $\hat{\nu}\in B(\hat{G}_\tau)$ such that $\hat{\nu}=\hat{\mu}$ on $\psi^{-1}(\Omega)$. Then ν can be written as follows:

$$\nu = \sum_{i=1}^{l}\sum_{j=1}^{l_i} n_{ij}\gamma_{ij}m_{H_i} \qquad (n_{ij}\in\mathbf{Z},\ \gamma_{ij}\in\hat{G}_\tau)\,,$$

where H_i are compact subgroups of G_τ. Let $H_i^\perp$ be the annihilator of H_i in $\hat{G}_\tau$, let I_1 be the set of those i such that $H_i^\perp$ is open in $\hat{G}$ and let I_2 be the set of those i such that $H_i^\perp$ is not open in $\hat{G}$.

CLAIM. $(\sum_{i\in I_2}\sum_{j=1}^{l_i} n_{ij}\gamma_{ij}m_{H_i})\hat{} = 0$ *on* $\psi^{-1}(\Omega)$.

Suppose there exist a nonzero integer M and $\gamma_0=(x_0,u_0)\in\psi^{-1}(\Omega)$ ($x_0\in\mathbf{R}^k\oplus\mathbf{R}^{m-k}$, $u_0\in F$) such that $(\sum_{i\in I_2}\sum_{j=1}^{l_i}n_{ij}\gamma_{ij}m_{H_i})\hat{}(\gamma_0)=M$. Since $(\sum_{i\in I_2}\sum_{j=1}^{l_i}n_{ij}\gamma_{ij}m_{H_i})\hat{}$ is integer-valued and continuous on the open set $\psi^{-1}(\Omega)$, there is an open neighborhood V of x_0 in $\mathbf{R}^k\oplus\mathbf{R}^{m-k}$ such that

$$(1)\qquad \left(\sum_{i\in I_2}\sum_{j=1}^{l_i}n_{ij}\gamma_{ij}m_{H_i}\right)^{\wedge}(x,u_0) = M \quad \text{for } x\in V.$$

We put $E_{ij}=\{x\in\mathbf{R}^k\oplus\mathbf{R}^{m-k};\ (x,u_0)\in\gamma_{ij}+H_i^{\perp}\}$ and $I_2'=\{i\in I_2;\ E_{ij}\neq\varnothing\}$. Then E_{ij} are cosets in $\mathbf{R}^k\oplus\mathbf{R}^{m-k}$, and it follows from (1) that

$$(2)\qquad \sum_{i\in I_2'}\sum_{j=1}^{l_i}n_{ij}\chi_{E_{ij}}(x) = M \quad \text{for } x\in V.$$

By Lemma 2.6, if $i\in I_2'$, E_{ij} are not open in $\mathbf{R}^k\oplus\mathbf{R}^{m-k}$, hence proper cosets in $\mathbf{R}^k\oplus\mathbf{R}^{m-k}$. Hence by Lemma 2.5 and (2) we have a contradiction. Thus Claim is obtained.

We put $\sigma=\sum_{i\in I_1}\sum_{j=1}^{l_i}n_{ij}\gamma_{ij}m_{H_i}$. Then $\sigma\in M(G)$ and $\hat{\sigma}$ is integer-valued. Moreover, by Claim, we get $\hat{\sigma}=\hat{\mu}$ on $\psi^{-1}(\Omega)$. This completes the proof.

COROLLARY 3.2 (Main Theorem). *Let G be a* LCA *group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into $\mathbf{R}^n$. Let Ω be a bounded open convex set in $\mathbf{R}^n$. Suppose $\hat{\mu}\in B(\hat{G})$ is integer-valued on $\psi^{-1}(\Omega)$. Then there exists an integer-valued $\hat{\nu}\in B(\hat{G})$ such that $\hat{\nu}=\hat{\mu}$ on $\psi^{-1}(\Omega)$.*

PROOF. Let $[\psi(\hat{G})]$ be the linear subspace of $\mathbf{R}^n$ generated by $\psi(\hat{G})$. We put $k=\dim[\psi(\hat{G})]$. Then there exist independent vectors $u_1,\dots,u_k$ in $\psi(\hat{G})$. Hence there exists an automorphism β on $\mathbf{R}^n$ such that $\beta(u_i)=e_i$ $(1\leqq i\leqq k)$. Therefore, by Lemma 1.4, we may assume that ψ satisfies the following:

(1) $e_i\in\psi(\hat{G})\quad(1\leqq i\leqq k)$;
(2) $\psi(\hat{G})\subset\mathbf{R}^k=\{(x_1,\dots,x_k,0,\dots,0)\in\mathbf{R}^n;\ x_i\in\mathbf{R}\ (1\leqq i\leqq k)\}$.

By (1) and (2), we can regard ψ as a nontrivial continuous homomorphism from $\hat{G}$ into $\mathbf{R}^k$ such that $[\psi(\hat{G})]=\mathbf{R}^k$. We put $\Omega_k=\mathbf{R}^k\cap\Omega$. Then Ω_k is a bounded open convex set in $\mathbf{R}^k$ and $\psi^{-1}(\Omega_k)=\psi^{-1}(\Omega)$. Hence, by Theorem 3.1, there exists an integer-valued $\hat{\nu}\in B(\hat{G})$ such that $\hat{\nu}=\hat{\mu}$ on $\psi^{-1}(\Omega)$. This completes the proof.

COROLLARY 3.3. *Under the assumption of Main Theorem, we suppose that $\hat{\mu}(\gamma)=1$ or 0 on $\psi^{-1}(\Omega)$. Then there exists an idempotent measure $\nu\in M(G)$ such that $\hat{\nu}(\gamma)=\hat{\mu}(\gamma)$ on $\psi^{-1}(\Omega)$.*

PROOF. By Main Theorem, there exists an integer-valued $\hat{\nu}_0 \in B(\hat{G})$ such that $\hat{\nu}_0(\gamma)=\hat{\mu}(\gamma)$ on $\psi^{-1}(\Omega)$. By Theorem 1.1, $F=\{\gamma \in \hat{G};\ \nu_0(\gamma)=1\}$ belongs to the open coset ring of $\hat{G}$. Hence there exists an idempotent measure $\nu \in M(G)$ such that $\hat{\nu}(\gamma)=\chi_F(\gamma)$, where χ_F is the characteristic function of F. Since $F\supset\{\gamma \in \psi^{-1}(\Omega);\ \hat{\mu}(\gamma)=1\}$, ν is the dsired one. This completes the proof.

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