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A NOTE ON COMPACT METRIC SPACES AS REMAINDERS

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1. Introduction.

Throughout this paper X denotes a non-compact, locally compact, Hausdorff space. If αX is any Hausdorff compactification of X, then $\alpha X - X$ is a remainder of X. The question of determining when all members of a certain class of compact spaces can serve as remainders of each space in another class of spaces has been a major problem in the theory of compactifications (cf. [2]). For example, Rogers [9] has determined conditions to insure that all Peano continua are remainders and Chandler [1] has provided sufficiency conditions for when all weak Peano continua are remainders. In [4] and [5] the condition that all compact metric spaces are remainders of X has been characterized. In this paper we provide an additional sufficiency condition for when all compact metric spaces are remainders of X. Related results and examples are also included.

2. Sufficiency conditions.

Notation concerning rings of continuous functions and the Stone-Čech compactification βX of X will follow that of [3]. Let $C(X)$ be the ring of continuous real-valued functions on X and, for $f \in C(X)$, let f^{β} be the continuous extension of f mapping βX into $\beta \mathsf{R}$, where R denotes the real numbers. Let $F(X)$ be the subring of $C(X)$ consisting of all members f of $C(X)$ for which f^{β} is constant on components of $\beta X - X$. Denote by f^* the extension of $f \in C(X)$ which maps βX into $\mathsf{R}^* = \mathsf{R} \cup \{\infty\}$, the one-point compactification of R. N denotes the natural numbers.

THEOREM 2.1. *If, for each $p \in \beta X - X$, there exists $f \in F(X)$ such that $f^*(p) = \infty$, then all compact metric spaces are remainders of X.*

PROOF. Let δX be the compactification of X obtained by identifying components of $\beta X - X$ to points and let t be the canonical mapping of βX

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onto δX which is the identity on X and which carries $\beta X - X$ onto $\delta X - X$ (cf. 6.12 of [3]).

If δ is the proximity relation on X associated with δX, let $P(X)$ be the collection of real-valued proximity functions on X, where R is equipped with the proximity δ_{R} determined by the usual metric. Let $\delta\mathsf{R}$ be the (Smirnov) compactification of R associated with δ_{R}. For $p \in \delta X - X$, let $\mathscr{C}_p$ be the component of $\beta X - X$ which satisfies $t[\mathscr{C}_p] = p$.

Let j be the continuous mapping of $\beta\mathsf{R}$ onto $\delta\mathsf{R}$ whose restriction to R is the identity. Now, for each $f \in F(X)$, define a mapping f^δ of δX into $\delta\mathsf{R}$ by taking

$$f^\delta(p) = j \circ f^\beta[\mathscr{C}_p], \quad \text{for } p \in \delta X - X ,$$

and

$$f^\delta(x) = f(x), \quad \text{for } x \in X .$$

Since $f^\delta \circ t = j \circ f^\beta$ and t is a projection, it follows that f^δ is continuous. Hence $f \in P(X)$ and f^δ is the Smirnov extension of f which carries δX into $\delta\mathsf{R}$.

Suppose p is a point of $\delta X - X$. Take $z \in \mathscr{C}_p$. Then there exists $g \in F(X)$ such that $g^*(z) = \infty$. Now $g \in P(X)$ and it follows that g^δ carries p onto a point of $\delta\mathsf{R} - \mathsf{R}$. Thus $p \notin \upsilon_\delta X$, where $\upsilon_\delta X$ is the (minimal) real-completion of (X, δ) (cf. [8]).

Now if p is an isolated point of $\delta X - X$, there is a set U, open in δX, such that $U \cap (\delta X - X) = \{p\}$. Set $H_1 = U$ and, for $n \geqq 2$, let H_n be the pre-image under g^δ of the set $\delta\mathsf{R} - [-n, n]$. Evidently, $p \in H_n$, for all n, and if $x \in X$, there exists $n \in \mathsf{N}$ such that $x \notin H_n$. Thus, $\{p\} = \bigcap \{H_n \mid n \in \mathsf{N}\}$ so that $\{p\}$ is a G_δ-point of δX. But since $p \notin \upsilon_\delta X$, p is not a G_δ-point (cf. Corollary 3.8 of [7]), which is a contradiction. Hence $\delta X - X$ contains no isolated points. Now $\delta X - X$ is totally disconnected, compact and non-scattered and it follows (cf. Theorem 8.5.4 of [10]) that there is a continuous mapping of $\delta X - X$ onto the Cantor set $\mathscr{C}$. Since all compact metric spaces are continuous images of $\mathscr{C}$, Magill's Theorem [6] implies that all compact metric spaces are remainders of X.

This completes the proof.

3. Further results and examples.

The following is immediate.

COROLLARY 3.1. *If $\beta X - X$ is totally disconnected and X is realcompact, then all compact metric spaces are remainders of X.*

Without realcompactness, Corollary 3.1 is false. Let W denote the space of all countable ordinals and let W^* be the one-point compactification of W (see 5.12 of [3]). If X is any space for which $\beta X - X = W^*$, then W^* is totally disconnected but any metric space which is a continuous image of W^* must be countable or finite. Hence not all compact metric spaces are remainders of X. Clearly, X is non-realcompact since X is pseudocompact.

Next, suppose $X = \beta \mathsf{R} - (\beta \mathsf{N} - \mathsf{N})$. Then $\beta X - X \approx \beta \mathsf{N} - \mathsf{N}$, so $\beta X - X$ is totally disconnected and non-scattered since $\beta \mathsf{N} - \mathsf{N}$ contains no discrete points. Thus, the Cantor set is a continuous image of $\beta X - X$ and all compact metric spaces are remainders of X. But X is pseudocompact so that every $f \in F(X) = C(X)$ satisfies $f^*(p) \neq \infty$, for all $p \in \beta X - X$. Thus, the converse of Theorem 2.1 is false.

Finally, the following result is immediate from the proof of Theorem 2.1.

COROLLARY 3.2. *If X admits a proximity δ for which (X, δ) is realcomplete and $\delta X - X$ is totally disconnected, then all compact metric spaces are remainders of X.*

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