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ON GAMES OF TOPSØE

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F. Topsøe [17] introduced several topological games, called strong games with clustering, and by means of the games SC^M and kSC^M he characterized sieve-complete spaces. Both these games and the game $P(X, Y)$ (in the author's notation) considered by E. Porada [10] are natural modifications of the strong game of G. Choquet [1].

In the present paper there are introduced two more games: the k-modification of the game $P(X, Y)$ denoted by $kP(X, Y)$ and game $H(X)$, called the Hurewicz game. The following relations are established ($\sim$ denotes the equivalence of games): $SC^M(X) \sim P(\beta X, X)$, $kSC^M(X) \sim kP(\beta X, X) \sim H(X^*)$, where $X^* = \beta X - X$ and βX is the Čech-Stone compactification of X. Moreover, some relations to the game $G(C, X)$ of the author [12, 13, 15] are derived. For another results concerning $P(X, Y)$ we refer to [14, 16].

All spaces considered here are assumed to be completely regular and all games involve perfect information for both players. N denotes the set of all positive integers. Following the notation of [5], $\mathrm{I} \uparrow G$ ($\mathrm{II} \uparrow G$) denotes that Player I (respectively Player II) has a winning strategy (w.s. for short) in the game G. $G_1 \sim G_2$ denotes that $\mathrm{I} \uparrow G_1 \Leftrightarrow \mathrm{I} \uparrow G_2$ and $\mathrm{II} \uparrow G_1 \Leftrightarrow \mathrm{II} \uparrow G_2$. For topological background the reader is referred to [2].

Recall the game $kSC^M(X)$, where X is a given space. Here, however, the players β and α are called Player I and Player II, respectively. Player I chooses a compact non-empty set C_1 and an open neighbourhood (nbhd. for short) U_1 of C_1. After that Player II chooses an open nbhd. V_1 of C_1 with $V_1 \subset U_1$. Now Player I chooses a compact non-empty set $C_2 \subset V_1$ and an open nbhd. U_2 of C_2 with $U_2 \subset V_1$. After that Player II chooses an open nbhd. V_2 of C_2 with $V_2 \subset U_2$, and so on. Player II wins the play $((C_1, U_1), V_1, (C_2, U_2), V_2, \ldots)$ of $kSC^M(X)$ if each filter base $\mathscr{F}$ such that for each $n \in \mathrm{N}$, there is a $F_n \in \mathscr{F}$ with $F_n \subset V_n$ *clusters* in X, that is $\bigcap \{\bar{F}\colon F \in \mathscr{F}\} \neq \emptyset$; otherwise Player I wins. The games SC^M and kSC^M differ just in the following: in SC^M the sets C_n are singletons.

Let Y be a subset of a space X. The game $kP(X, Y)$ is played as $kSC^M(X)$ except for the extra rule that C_n's must be subsets of Y. Player II wins the play

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of $kP(X, Y)$, if $\emptyset \neq \bigcap\{V_n: n \in \mathbb{N}\} \subset Y$; otherwise Player I wins. In $P(X, Y)$ it is further demanded that C_n's are singletons.

In the Hurewicz game $H(X)$, where X is a given space, Player I chooses an open cover $\mathscr{G}_1$ of X and after that Player II chooses a finite subfamily $\mathscr{H}_1$ of $\mathscr{G}_1$. Again Player I chooses an open cover $\mathscr{G}_2$ of X and Player II chooses a finite subfamily $\mathscr{H}_2$ of $\mathscr{G}_2$, and so on. Player II wins the play $(\mathscr{G}_1, \mathscr{H}_1, \mathscr{G}_2, \mathscr{H}_2, \ldots)$ of $H(X)$ if $\bigcup\{\bigcup \mathscr{H}_n: n \in \mathbb{N}\} = X$; otherwise Player I wins.

The Hurewicz game is related to Hurewicz spaces studied by A. Lelek [8], the author [11, 12] and others, but introduced in 1926 by W. Hurewicz [6]. A space X is said to be a Hurewicz space if for each sequence $(\mathscr{G}_1, \mathscr{G}_2, \ldots)$ of open covers of X, there is a cover $\mathscr{H}$ of X such that $\mathscr{H} = \bigcup\{\mathscr{H}_n: n \in \mathbb{N}\}$, where each $\mathscr{H}_n$ is a finite subfamily of $\mathscr{G}_n$. (Or, equivalently: for each sequence $(\mathscr{G}_1 \cdot \mathscr{G}_2, \ldots)$ of finitely additive open covers of X, there is a selector $(G_1, G_2, \ldots) \in \mathscr{G}_1 \times \mathscr{G}_2 \times \ldots$ such that $\bigcup\{G_n: n \in \mathbb{N}\} = X$.) It is easy to check that if X is not a Hurewicz space, then $\mathrm{I} \uparrow H(X)$; hence, if $\mathrm{II} \uparrow H(X)$, then X is a Hurewicz space.

Finally, recall the games $G(\boldsymbol{C}, X)$ and $G(\boldsymbol{F}, X)$ of [12]. In $G(\boldsymbol{C}, X)$ Player I chooses a compact set C_1 in X and after that Player II chooses a closed set $E_1 \subset X - C_1$. Player I chooses a compact set $C_2 \subset E_1$ and Player II chooses a closed set $E_2 \subset E_1 - C_2$, and so on. Player I wins the play $(C_1, E_1, C_2, E_2, \ldots)$ of $G(\boldsymbol{C}, X)$ if $\bigcap\{E_n: n \in \mathbb{N}\} = \emptyset$; otherwise Player II wins. The game $G(\boldsymbol{F}, X)$ is played as $G(\boldsymbol{C}, X)$, but with the restriction that C_n's are finite sets.

THEOREM 1. *Let X be a compactification of Y. Then $SC^M(Y) \sim P(X, Y)$, and $kSC^M(Y) \sim kP(X, Y)$.*

The proofs of both statements are similar and quite easy when indicating their essential points: without loss of generality we may assume that Player II always chooses the set V_n with $\bar{V}_n \subset U_n$; $x \in \bigcap\{V_n: n \in \mathbb{N}\}$, iff there is a filter base $\mathscr{F}$ of subsets of Y so that $\bigcap\{\bar{F}: F \in \mathscr{F}\} = \{x\}$ and for each $n \in \mathbb{N}$, there is a $F_n \in \mathscr{F}$ with $F_n \subset V_n$. Therefore the details of the proof are left to the reader.

THEOREM 2. *Let Y be a subset of a compact space X. Then $kP(X, Y) \sim H(X - Y)$.*

PROOF. For each compact set $C \subset Y$ and each open set U in X with $C \subset U$ denote by $\mathscr{V}(C, U)$ a base of nbhds. of C in X with $\bar{V} \subset U$ for each $V \in \mathscr{V}(C, U)$. For each open set G in $X - Y$ denote by G' an open set in X such that $G' \cap (X - Y) = G$.

Let s be a w.s. of Player I in $kP(X, Y)$. We define a w.s. t for the player in $H(X - Y)$. Put

$$(C_1, U_1) = s(\emptyset) \quad \text{and} \quad \mathscr{G}_1 = \{(X - Y) - \bar{V}: V \in \mathscr{V}(C_1, U_1)\}.$$

Then $\bigcup \mathscr{G}_1 = X - Y$. Put $t(\emptyset) = \mathscr{G}_1$. For each finite subfamily $\mathscr{H}_1$ of $\mathscr{G}_1$, there is a finite subfamily $\mathscr{V}_1$ of $\mathscr{V}(C_1, U_1)$ such that

$$\mathscr{H}_1' = \{(X-Y) - \bar{V}: V \in \mathscr{V}_1\} .$$

Put $V_1 = \bigcap \mathscr{V}_1$. Then $C_1 \subset V_1 \subset U_1$. Put

$$(C_2, U_2) = s(V_1) \quad \text{and} \quad \mathscr{G}_2 = \{(X-Y) - \bar{V}: V \in \mathscr{V}(C_2, U_2)\} .$$

Then $\bigcup \mathscr{G}_2 = X - Y$. Put $t(\mathscr{H}_1) = \mathscr{G}_2$. Again, for each finite subfamily $\mathscr{H}_2$ of $\mathscr{G}_2$, there is a finite subfamily $\mathscr{V}_2$ of $\mathscr{V}(C_2, U_2)$ such that

$$\mathscr{H}_2 = \{(X-Y) - \bar{V}: V \in \mathscr{V}_2\} .$$

Put $V_2 = \bigcap \mathscr{V}_2$. Then $C_2 \subset V_2 \subset U_2$. Put $(C_3, U_3) = s(V_1, V_2)$, and so on. Since

$$(X-Y) \cap \bigcap \{V_n : n \in \mathbf{N}\} \neq \emptyset ,$$

t is easy to check that $\bigcup\{\bigcup \mathscr{H}_n : n \in \mathbf{N}\} \neq X - Y$.

Let s be a w.s. of Player I in $H(X-Y)$. We define a w.s. t for the player in $kP(X, Y)$. Put $\mathscr{G}_1 = s(\emptyset)$, $C_1 = X - \bigcup\{G': G \in \mathscr{G}_1\}$ and $U_1 = X$. Then $C_1 \subset Y$. Put $t(\emptyset) = (C_1, U_1)$. Since X is compact, for each open nbhd. V_1 of C_1, there is a finite subfamily $\mathscr{H}_1$ of $\mathscr{G}_1$ such that $X - \bigcup\{G': G \in \mathscr{H}_1\} \subset V_1$. Put

$$\mathscr{G}_2 = s(\mathscr{H}_1) \quad \text{and} \quad C_2 = X - \bigcup\{G' : G \in \mathscr{H}_1 \cup \mathscr{G}_2\} .$$

Then $C_2 \subset V_1 \cap Y$. Put $U_2 = V_1$ and $t(V_1) = (C_2, U_2)$. Since X is compact, for each open nbhd. V_2 of C_2 with $V_2 \subset U_2$, there is a finite subfamily $\mathscr{H}_2$ of $\mathscr{G}_2$ such that $X - \bigcup\{G' : G \in \mathscr{H}_1 \cup \mathscr{H}_2\} \subset V_2$. Put $\mathscr{G}_3 = s(\mathscr{H}_1, \mathscr{H}_2)$, and so on. Since $\bigcup\{\bigcup \mathscr{H}_n : n \in \mathbf{N}\} \neq X - Y$, it is easy to check that

$$(X-Y) \cap \bigcap \{V_n : n \in \mathbf{N}\} \neq \emptyset .$$

Let s be a w.s. of Player II in $kP(X, Y)$. We define a w.s. t for the player in $H(X-Y)$. For each open (in $X-Y$) cover $\mathscr{G}_1$ of $X-Y$ we put $C_1 = X - \bigcup\{G': G \in \mathscr{G}_1\}$, $U_1 = X$ and $V_1 = s(C_1, U_1)$. Since $C_1 \subset V_1$, there is a finite subfamily $\mathscr{H}_1$ of $\mathscr{G}_1$ such that $X - \bigcup\{G': G \in \mathscr{H}_1\} \subset V_1$. Put $t(\mathscr{G}_1) = \mathscr{H}_1$. For each open (in $X-Y$) cover $\mathscr{G}_2$ of $X-Y$ we put

$$C_2 = X - \bigcup\{G' : G \in \mathscr{H}_1 \cup \mathscr{G}_2\}, \quad U_2 = V_1, \quad \text{and} \quad V_2 = s(C_1, U_1, C_2, U_2) .$$

Since $C_2 \subset V_2$, there is a finite subfamily $\mathscr{H}_2$ of $\mathscr{G}_2$ such that $X - \bigcup\{G': G \in \mathscr{H}_1 \cup \mathscr{H}_2\} \subset V_2$. Put $t(\mathscr{G}_1, \mathscr{G}_2) = \mathscr{H}_2$, and so on. Since $\bigcap\{V_n: n \in \mathbf{N}\} \subset Y$, it is easy to check that $\bigcup\{\bigcup \mathscr{H}_n: n \in \mathbf{N}\} = X - Y$.

Let s be a w.s. of Player II in $H(X-Y)$. We define a w.s. t for the player in $kP(X, Y)$. For each compact subset C_1 of Y and each open nbhd. U_1 of C_1 we put

$$\mathscr{G}_1 = \{(X-Y)-\bar{V}: V \in \mathscr{V}(C_1, U_1)\} \quad \text{and} \quad \mathscr{H}_1 = s(\mathscr{G}_1).$$

Since $\mathscr{H}_1$ is finite, there is a finite subfamily $\mathscr{V}_1$ of $\mathscr{V}(C_1, U_1)$ such that $\mathscr{H}_1 = \{(X-Y)-\bar{V}: V \in \mathscr{V}_1\}$. Put $V_1 = \bigcap \mathscr{V}_1$ and $t(C_1, U_1) = V_1$. For each compact subset C_2 of $V_1 \cap Y$ and each open nbhd. U_2 of C_2 with $U_2 \subset V_1$ we put

$$\mathscr{G}_2 = \{(X-Y)-\bar{V}: V \in \mathscr{V}(C_2, U_2) \quad \text{and} \quad \mathscr{H}_2 = s(\mathscr{G}_1, \mathscr{G}_2).$$

Since $\mathscr{H}_2$ is finite, there is a finite subfamily $\mathscr{V}_2$ of $\mathscr{V}(C_2, U_2)$ such that $\mathscr{H}_2 = \{(X-Y)-\bar{V}: V \in \mathscr{V}_2\}$. Put $V_2 = \bigcap \mathscr{V}_2$, $t(C_1, U_1, C_2, U_2) = V_2$, and so on. Since

$$\bigcup\{\bigcup \mathscr{H}_n : n \in \mathbb{N}\} = X - Y \quad \text{and} \quad V_n = \bigcap \mathscr{V}_n \subset \bigcap\{\bar{V}: V \in \mathscr{V}_n\} \subset U_n,$$

it is easy to check that $\emptyset \neq \bigcap\{V_n : n \in \mathbb{N}\} \subset Y$.

From Theorems 1 and 2 we immediately get, denoting by X^* the remainder $\beta X \setminus X$:

COROLLARY 1. $\mathrm{I} \uparrow kSC^M(X) \Leftrightarrow \mathrm{I} \uparrow kP(\beta X, X) \Leftrightarrow \mathrm{I} \uparrow H(X^*)$.

Moreover, we get

COROLLARY 2. *The following conditions are equivalent*:

2.1. $\mathrm{II} \uparrow SC^M(X)$.
2.2. $\mathrm{II} \uparrow P(\beta X, X)$.
2.3. $\mathrm{II} \uparrow kSC^M(X)$.
2.4. $\mathrm{II} \uparrow kP(\beta X, X)$.
2.5. $\mathrm{II} \uparrow H(X^*)$.
2.6. $\mathrm{I} \uparrow G(C, X^*)$, that is, X^* is compact-like (cf. [12, p. 195]).
2.7. X is sieve-complete (cf. [9, 15, 17]).

PROOF. The equivalence of 2.1, 2.3, and 2.7 was proved by F. Topsøe [17], the equivalence of 2.6 and 2.7 was proved by the author in [15], and the remaining equivalences follow from the above theorems.

The list of conditions in Corollary 2 can be considerably extended. F. Topsøe showed in [17, p. 620], that $SC^M \sim S^*C^M$ and $kSC^M \sim kS^*C^M$, where the games S^*C^M and kS^*C^M are defined precisely as SC^M and kSC^M respectively, except for the extra rules that $U_1 = X$, $U_2 = V_1$, $U_3 = V_2, \ldots$ Other conditions can be derived from recent results of F. Galvin and the author [4]; it turns out that in 2.1–2.4 and 2.6 one can equivalently state: the player has a stationary w.s. in the corresponding game (the same also holds for $\mathrm{II}\uparrow S^*C^M(X)$ and $\mathrm{II}\uparrow kS^*C^M(X)$). For further conditions we refer to [15].

Since each space X is the remainder of some space Y for a compactification Z of Y (e.g., for $Y=([0,1)\times\beta X)\cup(\{1\}\times X^*)$ and $Z=[0,1]\times\beta X$ we get $Z-Y=\{1\}\times X$), from Corollary 2 we get

COROLLARY 3. $\mathrm{II}\uparrow H(X)\Leftrightarrow \mathrm{I}\uparrow G(C,X)$.

This corollary provides a new characterization of compact-like spaces. Comparing to Corollary 5 in [15], this one involves families of open sets only.

By 6.4 of [12] from Corollary 3 we get

COROLLARY 4. *Let X be a metrizable space. Then* $\mathrm{II}\uparrow H(X)\Leftrightarrow$ *X is σ-compact.*

Let X_δ denote the topological space obtained from X by taking its G_δ-sets as basic open sets. Since

$$\mathrm{I}\uparrow G(F,X)\Leftrightarrow \mathrm{I}\uparrow G(F,X_\delta)\Leftrightarrow \mathrm{I}\uparrow G(C,X_\delta)$$

(cf. [13, 5.3]), by Corollary 3 we get

COROLLARY 5. $\mathrm{II}\uparrow H(X_\delta)\Leftrightarrow \mathrm{I}\uparrow G(F,X)$.

This corollary provides a new characterization of finite-like spaces (cf. [12, p. 195], [3, p. 445], and [15, Corollary 8]).

It is easy to verify the following.

PROPOSITION 1. $\mathrm{I}\uparrow H(X)\Rightarrow \mathrm{II}\uparrow G(C,X)$.

It is, however, an usettled question whether or not the converse implication holds.

PROPOSITION 2. $\mathrm{I}\uparrow H(X)$, *iff either X is not a Lindelöf space or there is a continuous map from X onto a separable metric space Y such that* $\mathrm{I}\uparrow H(Y)$.

PROOF. Let s be a w.s. of Player I in $H(X)$, where X is a Lindelöf space. Then we may assume that each cover determined by s is countable, and moreover, that it consists of cozero sets in X. Therefore the strategy s is associated with a countable family Φ of real-valued continuous functions. Then the topology $\mathcal{T}$ generated by Φ is separable, pseudometrizable and coarser than the original topology of X. Let Y be the quotient space obtained from X by identifying points which are not distinguished by Φ and endowed with the quotient topology determined by $\mathscr{T}$. Then Y is separable, metrizable and the natural map from X onto Y is continuous. It is easy to check that $\mathrm{I}\uparrow H(Y)$. Conversely, if X is not a Lindelöf space, then any open cover with no countable subcover provides a w.s. for Player I. If X admits a continuous map f onto a separable metric space Y with $\mathrm{I}\uparrow H(Y)$, then f^{-1} transfers w.s. from $H(Y)$ into $H(X)$, and therefore $\mathrm{I}\uparrow H(X)$.

We conclude with two examples of certain singular spaces.

EXAMPLE 1. Let X be any Bernstein set contained in the closed unit interval J. Then the game $P(J,X)$ is not determined (cf. [10, p. 353]), however, $\mathrm{II}\uparrow G(C,X)$ (cf. [12, 5.11]). Moreover, $\mathrm{I}\uparrow kP(J,X)$. For, Player I in $kP(J,X)$ can choose finite sets and small nbhds. of these sets so that $\bigcap\{V_n\colon n\in\mathbf{N}\}$ will be a copy of the Cantor discontinuum; therefore he can win each play. By Theorem 2 we infer that $\mathrm{I}\uparrow H(X)$, and moreover, X is not a Hurewicz space, because it is not totally paracompact (cf. [7, 8]).

EXAMPLE 2. Let X be a Lusin set in J so that $G(F,X)$ is undetermined (cf. [3, p. 448]; the continuum hypothesis is assumed). Since X is not σ-compact, the game $G(C,X)$ is also undetermined. As in Example 1, the game $P(J,X)$ is undetermined and $\mathrm{I}\uparrow kP(J,X)$. By Proposition 1, Player I has no w.s. in $H(X)$. Since X is not σ-compact, Player II has no w.s. in $H(X)$. Therefore $H(X)$ is undetermined. By Theorem 2, the game $kP(J,J-X)$ also is undetermined. However, $\mathrm{II}\uparrow G(C,J-X)$ (cf. [12, 5.11]). By Theorem 2, $\mathrm{I}\uparrow H(J-X)$, because $\mathrm{I}\uparrow kP(J,X)$. The space X has the property C'' and therefore it is a Hurewicz space (cf. [8, p. 210]).

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