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A NEW TYPE OF AFFINE BOREL FUNCTION

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Abstract.

We construct a separable Banach space E which has the Schur property and such that there is $x \in E^{**} \setminus E$ which is Borel and strongly affine on (E^*, weak^*). If K denotes the unit ball of (E^*, weak^*), x is Borel, affine and strongly affine on K. However, x cannot be obtained from affine continuous functions on K by taking pointwise limits and repeating this operation any number of times.

1. Introduction.

Let K be a metrisable convex compact of a locally convex vector space. We denote by $A(K)$ the space of continuous affine functions on K. By induction over the ordinal α, we define the class $A_\alpha(K)$ of functions of Baire affine class α. We set $A_0(K) = A(K)$, and we take for $A_\alpha(K)$ the set of pointwise limits of sequences in $\bigcup_{\beta < \alpha} A_\beta(K)$.

Given a Borel probability measure μ on K, we define its barycenter b_μ as the unique point of K such that for each $f \in A(K)$, we have

$$f(b_\mu) = \int_K f(x)\, d\mu(x) . \tag{1}$$

A function f on K is called *strongly affine* if for each Borel probability μ it is μ-measurable and (1) holds. A strongly affine function f is affine.

By Lebesgue's theorem, and induction over α, it follows that for each α, each $f \in A_\alpha(K)$ is strongly affine.

The following natural question has been open for some time: given a strongly affine function f on K, which is Borel, does f belong to some $A_\alpha(K)$? The main result of this paper is to provide a negative answer.

For the convenience of the reader who is not familiar with this question, we discuss now some related results.

If f is an affine function on K, which is of first Baire class, then f is strongly affine, by a result of Choquet ([5, p. 100]). Moreover, f belongs to $A_1(K)$, by a

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result of Mokobodzki (unpublished). In other words, knowing that f is affine and a pointwise limit of a sequence of continuous functions, it is a pointwise limit of affine continuous functions.

If f is an affine function on K, which is of second Baire class, then f need not be strongly affine, as an easy example shows ([5, p. 104]). Our example will be of second Baire class, strongly affine, but is not any $A_\alpha(K)$.

A lot is known under some special assumptions on K. If K is a Choquet simplex, M. Capon has shown that an affine function on K which is Borel of Baire class α belongs to $A_{\alpha+1}(K)$. If K is the unit ball of $(L^\infty, \text{weak*})$, a remarkable result of J. P. R. Christensen [2] shows that each Borel affine function on K is automatically continuous. This result has been extended to the unit ball fo the dual of a Banach lattice which does not contain c_0 by G. Godefroy [4].

A closely connected problem is the structure of convex Borel sets. A convex set A is called strongly convex if for each compact set $L \subset A$, A contains the closed convex hull of L. Given a metrizable convex compact K we define the Borel convex classes $C_\alpha(K)$, in the following way. $C_0(K)$ is the class of convex closed sets. For α even, $C_{\alpha+1}(K)$ consists of the increasing unions of sequences in $C_\alpha(K)$. For α odd, $C_{\alpha+1}(K)$ consists of the countable intersections of elements of $C_\alpha(K)$. Finally, if α is a limit, then $C_\alpha(K)$ is the union of $C_\gamma(K)$ for $\gamma < \alpha$. A remarkable result of D. Preiss shows that a set belongs to some $C_\alpha(K)$ if it is Borel and strongly convex. We find it very surprising that the corresponding problem for linear functionals has a negative answer.

2. The Result.

We say that a Banach space E has the Schur property if each sequence (x_n) of E which goes to zero weakly also goes to zero in norm. It then follows that E is weakly sequentially complete. Denote by K the unit ball of $(E^*, \text{weak*})$. Then, E identifies with the subset $A^0(K)$ of functions which take values zero at zero. To say that E is weakly sequentially complete is equivalent to saying that each pointwise limit of functions in $A^0(K)$ still belongs to $A^0(K)$. It is then easily seen that $A_1(K) = A(K)$, and so $A_\alpha(K) = A(K)$ for each α, so for each α,

$$A_\alpha(K) \cap E^{**} = \{x \in A_\alpha(K) : x(0) = 0\} = E .$$

THEOREM. *There exists a separable Banach space E which has the Schur property, and $x \in E^{**} \setminus E$ such that x is strongly affine and of second Baire class on the unit ball K of $(E^*, \text{weak*})$.*

3. Construction of E.

Let $I=\{(n,p) \in \mathbf{N}\times\mathbf{N}: p\leqq n\}$. For $(n,p)\in I$, we denote by $e_{n,p}$ the element of R^I such that $e_{n,p}(i)=1$ for $i=(n,p)$ and zero otherwise. We denote by F the linear span of the $(e_{n,p})$. We denote by $e^*_{n,p}$ the orthogonal system to $(e_{n,p})$.

For $y \in F$, we set

$$\|y\|_1 = \sum_n \sup_{p\leqq n} |e^*_{n,p}(y)| .$$

We denote by Σ the set of increasing sequences of integers, that is $\sigma \in \Sigma$ if $\sigma(n) \leqq \sigma(m)$ for $n\leqq m$. For $\sigma \in \Sigma$, $\sigma=(\sigma(n))$, we define the linear functional g_σ on F by $g_\sigma(e_{n,p})=1$ if $p\geqq\sigma(n)$ and $g_\sigma(e_{n,p})=0$ otherwise. For $y\in F$, let

$$\|y\|_2 = \sup\{|g_\sigma(y)| : \sigma\in\Sigma\} .$$

Finally, let $\|y\|=\sup(\|y\|_1, \|y_2\|)$.

We denote by E_1 (respectively E_2, E) the completion of F for $\|\cdot\|_1$ (respectively $\|\cdot\|_2, \|\cdot\|$). The map $y \to (y,y)$ extends into an isometry of E as a subspace of $E_1\times E_2$. We shall identify E with its image under this map. We denote by $\|\cdot\|$ the norm of $E_1\times E_2$.

4. Construction of x.

For simplicity, we set $G=E_1\times E_2$. For $p\leqq n$, let $f_{n,p}=(0,e_{n,p})\in G$.

For $\sigma\in\Sigma$, we have $\lim_{n\to\infty} g_\sigma(e_{n,p})=1$ if $\lim_n \sigma(n)<p$ and $=0$ otherwise. It follows that $\lim_p \lim_n g_\sigma(e_{n,p})$ exists. Since the set $(g_\sigma, \sigma\in\Sigma)$ is weak* compact, it follows from Krein–Millman's theorem and Lebesgue's theorem that $\lim_p \lim_n z(e_{n,p})$ exists for $z\in E_2^*$. It follows that $x=\lim_p\lim_n f_{n,p}$ exists in (G^{**}, weak^*).

We show now that $x\in E^{**}$. For a sequence of real numbers a_n with $a_n\to a$, we have

$$a = \lim_k k^{-1} \sum_{i\leqq k} a_i .$$

It follows that in (G^{**}, weak^*), we have

$$x = \lim_k \lim_n k^{-1} \sum_{p\leqq k} f_{n,p} .$$

Let $g_{n,p}=(e_{n,p}, e_{n,p})\in E$. Let us fix k and $n\geqq k$. Let

$$\bar{f}_{n,k} = k^{-1}\sum_{p\leqq k} f_{n,p} \quad\text{and}\quad \bar{g}_{n,k} = k^{-1}\sum_{p\leqq k} g_{n,p} .$$

For $z\in E_2^*$, we have $z(\bar{f}_{n,k})=z(\bar{g}_{n,k})$. For $z\in E_1^*$, we have $z(\bar{f}_{n,k})=0$, while

$$|z(\bar{g}_{n,k})| \leqq \left|k^{-1} z\left(\sum_{p\leqq k} e_{n,p}\right)\right| \leqq k^{-1}\|z\|_1 .$$

It follows that $\|\bar{f}_{n,k}-\bar{g}_{n,k}\| \leqq k^{-1}$. Now, we notice the $\bar{g}_{n,k}$ belongs to the unit ball of E, and that the unit ball E_1^{**} of E^{**} and G_1^{**} of G^{**} are weak*-compact. It follows that

$$\lim_n k^{-1}\bar{f}_{n,k} \in E_1^{**}+k^{-1}G_1^{**}$$

and hence that $x^* \in E_1^{**}$.

We denote by K the unit ball of E^* and L the unit ball of G^*. We denote by φ the canonical map from L to K. It is continuous. Let us consider x as an element of E^{**}, that is a function on K. Then, $x\circ\varphi$ is a function on L, and identifies with x considered as an element of G^{**}. The definition of x shows that $x\circ\varphi \in A_2(L)$. In particular, $x\circ\varphi$ is of second Baire class on L. A deep result of J. Saint-Raymond [6] shows that x is of second Baire class on K (hence also on E^*, weak*). We do not know an easy way to write x as a limit of a limit of continuous functions on K. It is however possible to check that, for $z \in E^*$,

$$x(z) = \lim_k \limsup_n p^{-1} \sum_{p\leqq k} z(e_{n,p})$$

but this formula shows only that x is of third Baire class.

Since $x\circ\varphi \in A_2(L)$, $x\circ\varphi$ is strongly affine. If μ is a probability measure on K, there is a probability ν on L with $\varphi(\nu)=\mu$, and since φ is affine, we have $\varphi(b_\nu) = b_\mu$. Now,

$$\int x\,d\mu = \int x\circ\varphi\,d\nu = x\circ\varphi(b_\nu) = x(b_\mu)$$

which shows that x is strongly affine.

5. Proof that E has the Schur Property.

It is enough to prove the following stronger fact: each sequence (y_n) of E, of norm one, and such that $e_i^*(y_n) \to 0$ for each $i \in I$, contains a subsequence which spans a complemented copy of l^1.

A standard pertubation argument reduces the problem to the case where there is a sequence (k_n) such that y_n belongs to the linear span of the vectors $e_{p,q}$ for $k_n \leqq p < k_{n+1}$.

Assume first that there is a subsequence of (y_n), still called y_n, such that $\|y_n\|_1 > \alpha > 0$ for each n. Then, y_n is equivalent to the unit vector basis of l^1. Denote by z_n a sequence of E_1^* with $z_n(y_n) > \alpha$, $\|z_n\|_1 = 1$, and $z_n(e_{p,q})=0$ for $p<k_n$ or $p \geqq k_{n+1}$. The map

$$\varphi : x \to \sum (z_n(x)/z_n(y))y_n$$

is a projection of E on the span of (y_n).

Assume now that $\|y_n\|_1 \to 0$. Since $\|y_n\| = 1$, there exists a sequence $\sigma^n \in \Sigma$ such that $g_{\sigma^n}(y_n) > \frac{1}{2}$. For each q, we have

$$\lim_{n\to\infty} \sum_{\substack{r \leqq q \\ p \geqq r}} |e^*_{p,r}(y_n)| = 0 .$$

By extracting a subsequence, we can assume that

$$\sum_{\substack{r \leqq k_n \\ p \geqq r}} |e^*_{p,r}(y_n)| < \tfrac{1}{4} .$$

For a subset P of N, let us denote by σ_P the following sequence.

For $k_n \leqq p < k_{n+1}$,

$$\sigma_P(p) = p+1 \qquad \text{if } n \notin P$$

$$\sigma_P(p) = \sup(\sigma_n(p), k_n) \qquad \text{if } n \in P .$$

It is easy to check that σ_P is increasing. For $n \notin P$, we have $g_{\sigma_P}(y_n) = 0$. For $n \in P$, we have

$$g_{\sigma_P}(y_n) = g_{\sigma^n}(y_n) - \sum e^*_{p,n}(y_n)$$

where the summation is taken for $k_n \leqq p \leqq k_{n+1}$ and $\sigma_n(p) \leqq q < k_n$ and hence

$$g_{\sigma_P}(y_n) \geqq \tfrac{1}{4} .$$

Using a lemma of Rosenthal ([7, Proposition 4]), this shows that y_n is equivalent to the l^1-basis.

For each n, let $z_n \in E^*$ be given by $z_n = \sum e_{p,q}$, where the summation is taken for $k_n \leqq p < k_{n+1}$ and $\sup(\sigma_n(p), k_n) < q \leqq p$. It is routine to check that the map

$$y \to \sum (z_n(y) z_n(y_n)^{-1}) y_n$$

is a projection of E onto the span of the y_n.

The theorem is proved.

It should also be noticed that E also shows that the converse of Theorem 13 of [3] does not hold. Indeed, the result of this paragraph shows that each infinite-dimensional subspace of E contains a copy of l^1 which is complemented in the whole space. However, for each bounded operator T: $E \to l^1$, $T^{**}(x)$ is Borel on $(l^\infty, \text{weak}^*)$, so it belongs to l^1.

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