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# ON THE GENERALIZED HARDY'S INEQUALITY OF McGEHEE, PIGNO AND SMITH AND THE PROBLEM OF INTERPOLATION BETWEEN BMO AND A BESOV SPACE

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## 0. Introduction.

Littlewood conjectured (1948) that

$$\int_0^\infty \left|\sum_1^N e^{in_k x}\, dx\right| \geqq C \log N$$

where $0 < n_1 < \ldots < n_N$. The truth of this conjecture was established only recently (1981) apparently quite independently on the one hand by Konjagin [19] and on the other hand by McGehee, Pigno and Smith [20], [21]. Indeed, the last mentioned authors, whose proof is remarkably simple, prove the stronger estimate

$$(*) \qquad \int_0^{2\pi} \left|\sum_1^N c_k e^{in_k x}\right| dx \geqq C \sum_1^N |c_k|/k\,,$$

which incidentally also entails Hardy's well-known inequality (1927), the special case $n_k = k$.

In section 1 of the present paper we present a possibly even simpler arrangement of the details of the proof of this generalized Hardy's inequality. (The proof has also been simplified by Rudin [37] and others; see in particular Fournier [12].) As in [20], [21] the idea is roughly speaking the following. It is clear that by duality (*) is essentially equivalent to the following interpolation problem for Fourier coefficients: *Given a sequence $\{a_k\}$ of complex numbers such that $a_k = O(1/k)$ to find a function $F$ in $L^\infty = L^\infty(\boldsymbol{T})$ ($\boldsymbol{T}$ denotes the unit circle) such that $\hat{F}(n_k) = a_k$.* An approximative solution of this problem is gotten by setting

$$F = \sum_j e^{-\varepsilon(h_j + h_{j+1} + \ldots)} f_j \qquad (\varepsilon > 0)\,,$$

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where $f_j = \sum_{\Delta_j} a_k e^{in_k x}$, $\{\Delta_j\}$ being a suitable partition of the positive integers, and $h_j = 2P_-|f_j| + |f_j|\hat{}(0)$, $P_-$ being the orthogonal projection in $L^2 = L^2(\boldsymbol{T})$ onto the subspace $H^2_-$ (=the orthogonal complement of the usual Hardy space $H^2$). Taking $\varepsilon$ sufficiently small one can reiterate this construction and the existence of an exact solution readily follows.

In the remainder of the paper we turn to a somewhat different topic. It is well-known (essentially Grisvard's thesis [13], see e.g. [2], [23]) that for Besov spaces (in fact, in any number of variables, both in the periodic case ($\boldsymbol{T}^n$) and the non-periodic case ($\mathbf{R}^n$)) one has the following interpolation theorem:

$$(B^{s_0 p_0}_{p_0}, B^{s_1 p_1}_{p_1})_{\theta p} = B^{sp}_p \quad \text{for } s = (1-\theta)s_0 + \theta s_1 \;,$$

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad p_0, p_1 \in (0,\infty],\ \theta \in (0,1)\,.$$

Here (section 2 for $n=1$ and section 4 for $n$ general) we shall show that in the extremal case $s_0 = 0$, $p_0 = \infty$ one can do considerably better: the space $B^{0\infty}_\infty$ can be replaced by BMO, at least if $s_1 \neq 0$. (If $p_1 \geqq 1$ we get by duality a similar result with $H^1$.) It is remarkable that the proof utilizes the *same* construction (the special case $n_k = k$) as for the generalized Hardy's inequality alluded to. The special case $s_1 = p_1 = 1$, $n=1$ of this result was previously obtained by Peller [28] (research announcement in [27]), as a byproduct of his work on Hankel operators of trace class. Here we shall go the other way round. Indeed, we shall show (section 3) that our interpolation theorem for BMO entails Peller's main result for Hankel operators, at least in the "intermediate" case $1 < p < \infty$. Finally, let us mention that in Grisvard's case [13] our result extends to Besov–Lorentz spaces as well.

Some of the results of this paper were announced by one of the authors at the meeting of the Swedish Mathematical Society in Umeå (January 82), and later on at the Conference on Approximation Theory in Edmonton (June 1982) [25].

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**1. The generalized Hardy's inequality.**

Now we carry out the details of the construction already described in the introductory section 0.

The main step is embodied in the following general

LEMMA. *Let* $\{\Delta_j\}_{j=1}^{\infty}$ *be any partition of the set* $\mathbf{Z}_+$ *of positive integers where the* $\Delta_j$ *are disjoint intervals taken in increasing order* (i.e. $\Delta_{j+1}$ *is to the right of* $\Delta_j$). *Then, given any increasing sequence of integers* $0 < n_1 < \ldots$, *any sequence of complex numbers* $a_1, a_2, \ldots$ *and any number* $\varepsilon > 0$, *one can find a function* $F \in L^\infty = L^\infty(\boldsymbol{T})$ *such that*

$$1^\circ \qquad \|F\|_\infty \leqq 1/\varepsilon$$

*and*

$$2^\circ \qquad |\hat{F}(n_k) - a_k| \leqq \sqrt{2}\,\varepsilon \sum_{\nu \geqq j} b_\nu(b_\nu + b_{\nu+1} + \ldots)$$

*whenever* $k \in \Delta_j$, *where we have put* $b_\nu = (\sum_{m\in\Delta_\nu} |a_m|^2)^{1/2}$.

PROOF. Put $f_j = \sum_{k\in\Delta_j} a_k e^{in_kx}$ $(j = 1, 2, \ldots)$. Then obviously $\hat{f}_j(n_k) = a_k$ for $k \in \Delta_j$ but the function $\sum_{j=1}^{\infty} f_j$ is in general not in $L^\infty$ so we have to modify it a little. Define therefore

$$F = \sum_j e^{-\varepsilon(h_j + h_{j+1} + \ldots)} f_j$$

with $h_j = 2P_-|f_j| + |f_j|\hat{\ }(0)$. Here, as in section 0, $P_-$ is the orthogonal projection of $L^2 = L^2(\boldsymbol{T})$ onto the span $H^2_-$ of the $\{e^{inx}\}_{n<0}$ (that is, $H^2_-$ is the orthogonal complement of the usual Hardy space $H^2$). Since $|f_j|$ is real valued, in our case this gives $\operatorname{Re} h_j = |f_j|$. Moreover holds $\|h_j\|_2 \leqq \sqrt{2} \cdot \|f_j\|_2$. Clearly

$$|F| \leqq \sum_j e^{-\varepsilon(|f_j| + |f_{j+1}| + \ldots)} |f_j| \qquad \text{a.e.}$$

Upon comparing suitable Riemann sums of the convergent integral $\int_0^\infty e^{-\varepsilon x}\,dx$ we see that $F \in L^\infty$ with $\|F\|_\infty \leqq 1/\varepsilon$. Indeed, consider the finite sums

$$S = \sum_{j=1}^{N} e^{-\varepsilon(c_j + c_{j+1} + \ldots + c_N)} c_j \qquad (c_j \geqq 0)\,.$$

Clearly the $j$th term has the majorant

$$\int_{c_{j+1} + \ldots + c_N}^{c_j + c_{j+1} + \ldots + c_N} e^{-\varepsilon x}\,dx$$

so we find $S \leqq \int_0^{c_1 + \ldots + c_N} e^{-\varepsilon x}\,dx \leqq 1/\varepsilon$. (For this simple argument see a recent paper by Jones [17].) On the other hand (by the construction of the $h_j$) we have the identity

$$\hat{F}(n_k) - a_k = \sum_{\nu \geqq j} [(e^{-\varepsilon(h_\nu + h_{\nu+1} + \dots)} - 1) f_\nu]\hat{}\,(n_k)$$

if $k \in \Delta_j$. (The essential thing here is that the factor in front of $f_\nu$ in the formula for $F$ is a function whose Fourier transform is supported by the set $\mathbf{Z}_-$; therefore the terms with $\nu < j$ drop out in the above sum. A more general theorem holds true cf. Remark 5, *infra*.) Using the elementary inequality $|e^{-z} - 1| \leqq |z|$ $(\operatorname{Re} z \geqq 0)$ along with Schwarz's inequality it follows that

$$|\hat{F}(n_k) - a_k| \leqq \sqrt{2} \cdot \varepsilon \sum_{\nu \geqq 1} \|f_\nu\|_2 (\|f_\nu\|_2 + \|f_{\nu+1}\|_2 + \dots) .$$

This obviously proves 2°.

We now specialize the sequence $\{a_k\}$ imposing the condition $|a_k| \leqq M/k$ $(k = 1, 2, \dots)$ and further take the $\Delta_j$ to be dyadic intervals: $\Delta_j = [2^{j-1}, 2^j)$ $(j = 1, 2, \dots)$. Then as is readily seen, $b_\nu = O(2^{-\nu/2})$ so with $\varepsilon$ sufficiently small the lemma gives us an $F \in L^\infty$ such that

$$|\hat{F}(n_k) - a_k| \leqq \eta \frac{M}{k} \qquad (k = 1, 2, \dots)$$

where $\eta$ is any fixed number with $0 < \eta < 1$. If we now reiterate this construction in a well-known fashion (in step 2 we apply it with $a_k$ replaced by $a_k - F\hat{}\,(n_k)$ and so forth) we end up with the following

THEOREM. *If $a_k = O(1/k)$ then for some $F \in L^\infty$ holds $\hat{F}(n_k) = a_k$ $(k = 1, 2, \dots)$.*

It is clear that this implies the inequality (*) of section 0.

Several comments are now in order.

REMARK 1. There is nothing peculiar about the function $e^{-z}$. Instead we could have used any function $\varphi(z)$ holomorphic in the halfplane $x = \operatorname{Re} z \geqq 0$ such that

1° $|\varphi(z)| \leqq \psi(x)$ where $\psi(x)$ is a decreasing function such that $\int_0^\infty \psi(x)\,dx < \infty$,

2° $|\varphi'(z)| \leqq C$

and

3° $\varphi(0) = 1$ .

(From 2° and 3° it follows, for instance, that $|\varphi(z)-1| \leqq C|z|$; cf. the previous inequality $|e^{-z}-1| \leqq |z|$.) E.g. one can take $\varphi(z)=(1+z)^{-2}$ and more generally $\varphi(z)=(1+z)^{-n}$, $n>1$. Notice that $e^{-z}=\lim_{n\to\infty}(1+z/n)^{-n}$.

REMARK 2. More generally the following result (cf. Remark 1, *infra*) holds true.

THEOREM. *Let $G$ be a locally compact Abelian group with dual $\Gamma$. Let $S \subset \Gamma$ be such that $S+S \subset S$, and let $f \in L^2(G)$ be such that* supp $\hat{f} \subset S$. *Let $E$ be the closed convex hull in $\mathbf{C}$ of $\{0\}$ and the range of $f$, and suppose that $\varphi$ is a function which is analytic on an open set containing $E$ and in addition satisfies $|\varphi'(z)| \leqq C$ on $E$. Then $\varphi \circ f - \varphi(0) \in L^2(G)$, $\|\varphi \circ f - \varphi(0)\|_2 \leqq C\|f\|_2$, and* supp $(\varphi \circ f - \varphi(0))\hat{} \subset S$.

PROOF. Let $\{V_j\}$ be neighborhoods of the origin in $G$ such that $\|f(\cdot - x) - f(\cdot)\|_2 < 1/j$ if $x \in V_j$, and define a sequence of functions $\{f_j\}$ by putting $\varrho_j = 1/|V_j|\chi_{V_j}$ and $f_j = \varrho_j * f$. Then $f_j \in A(G) \cap L^2(G)$ (since $L^2(G) * L^2(G) = A(G)$), $\|f - f_j\|_2 \to 0$, supp $\hat{f}_j \subset S$ and for each $f_j$ there is a compact convex set $F_j \subset E$ containing 0 and the range of $f_j$. Now if $p$ is a polynomial it is obvious that $p \circ f_j - p(0) \in A(G)$ and that supp $(p \circ f_j - p(0)) \subset S$ (since $S+S \subset S$). By Runge's theorem we can for each set $F_j$ find a sequence of complex polynomials $\{p_{jn}\}$ such that $p'_{jn}(z) \to \varphi'(z)$ $(n \to \infty)$ uniformly on $F_j$. By the mean value theorem we have

$$|p_{jn}(z) - p_{jn}(w)| \leqq \sup_{\zeta \in E} |p_{jn}(\zeta)||z-w|$$

if $z, w \in F_j$ and

$$|\varphi(z) - \varphi(w)| \leqq C|z-w|$$

if $z, w \in E$. It follows that $p_{jn} \circ f_j - p_{jn}(0)$ and $\varphi \circ f_j - \varphi(0)$ are in $L^2(G)$. Since $p'_{jn}(z) \to \varphi'(z)$ $(n \to \infty)$ uniformly on $F_j$ it is also easy to see that

$$p_{jn} \circ f_j - p_{jn}(0) \to \varphi \circ f_j - \varphi(0) \qquad (n \to \infty)$$

in $L^2(G)$. Therefore supp $(\varphi \circ f_j - \varphi(0))\hat{} \subset S$ since supp $(p_{jn} \circ f_j - p_{jn}(0))\hat{} \subset S$ for each $n$. Now applying once more $|\varphi(z) - \varphi(w)| \leqq C|z-w|$ if $z, w \in E$ we see that $\varphi \circ f - \varphi(0) \in L^2(G)$ and that $\varphi \circ f_j - \varphi(0) \to \varphi \circ f - \varphi(0)$ in $L^2(G)$. Hence supp $(\varphi \circ f - \varphi(0))\hat{} \subset S$ since supp $(\varphi \circ f_j - \varphi(0))\hat{} \subset S$ for each $j$. Also $\|\varphi \circ f - \varphi(0)\|_2 \leqq C\|f\|_2$.

REMARK 3. Literally the same proof works for any locally compact Abelian group $G$ with a *totally* ordered dual $\Gamma$. We now take $\{\Delta_j\}$ to be a partition of $\Gamma^+ = \{\xi \in \Gamma, \xi > 0\}$, each $\Delta_j$ being an interval, with $\Delta_{j+1}$ to the "right" of $\Delta_j$.

However, if $G$ is not compact we must also allow *doubly infinite* sequences (i.e. $j$ runs through all of $\mathbf{Z}$, not just $\mathbf{Z}^+$). The "interpolation data" are a measurable subset $E$ of $\Gamma^+$ and a measurable function $a$ defined on $E$. We are primarily interested in finding a function $F$ in $L^\infty(G)$ such that $\hat{F}(\xi)=a(\xi)$ if $\xi \in E$. As before we first construct an approximative solution. The (second) estimate of the lemma now takes the form

$$|\hat{F}(\xi)-a(\xi)| \leqq \sqrt{2}\cdot\varepsilon \sum_{\nu\geqq j} b_\nu(b_\nu+b_{\nu+1}+\ldots)$$

for $\xi \in \Delta_j \cap E$, with $b_\nu=(\int_{\Delta_j\cap E}|a(\xi)|^2\,d\xi)^{1/2}$. Let $\delta(\xi)$ be the (Haar) measure of the set $(0,\xi)\cap E$. Then the relevant condition on $a$ in the interpolation theorem is $|a(\xi)|\leqq M/\delta(\xi)$ for $\xi \in E$. In particular we will thus end up with a "Hardy's inequality" of the form

$$\|f\|_1 \geqq C\int_{\Gamma^+}|\hat{f}(\xi)|/\delta(\xi)\,d\xi \qquad (C>0)$$

valid for all $E\subset\Gamma^+$ and for all $f\in L^1(G)$ with $\operatorname{supp}\hat{f}\subset E$. As already told, the previous proof goes over after merely formal changes; in particular, the projection is given by the formula $(P_-g)\hat{}(\xi)=\hat{g}(\xi)$ if $\xi<0$, 0 else, and thus

$$h_j = 2P_-|f_j|+\varkappa|f_j|\hat{}(0)$$

($\varkappa$ = measure of the set $\{0\}$). One can also consider the case of *partially* ordered duals. One instance of such a situation will be considered in section 4.

REMARK 4. At least in the compact case (i.e. both $G$ and $E$ (relatively) compact) it is also easy to get a *continuous* interpolating function. For simplicity let us consider just the case $G=T$. Then we just have to replace the original interpolation function $F$ (i.e., the one provided by the theorem) by $\chi * F$ where $\chi$ is a suitable function in $L^1$ such that $\chi\equiv 1$ on $E$ (take a kernel of de la Vallée-Poussin type).

More generally, we can always get a continuous interpolating function, *provided* we assume that $a_k=o(1/k)$ not just $O(1/k)$. Indeed (this argument we owe to Janson [15]), let $1=k_0<k_1<k_2<\ldots$ be integers such that $k|a_k|<2^{-j}$ for $k\geqq k_j$ $(j=1,2,\ldots)$. Pick functions $F_j\in L^\infty$ such that $\|F_j\|_\infty\leqq C2^{-j}$ $(j=0,1,\ldots)$ and

$$\hat{F}_j(n_k) = \begin{cases} a_k & \text{if } k_j\leqq k<k_{j+1} \ (j=0,1,\ldots) \\ 0 & \text{else} \end{cases}$$

(where $E=\{n_1,n_2,\ldots\}$). By convoluting with suitable functions $\chi_j$ we get continuous functions $\Phi_j$ with $\|\Phi_j\|_\infty\leqq 3C2^{-j}$ such that

$$\hat{\Phi}_j(n_k) = \begin{cases} a_k & \text{if } k_j\leqq k<k_{j+1} \\ 0 & \text{else} \end{cases}.$$

It is clear that we can take the interpolating function $F=\sum \Phi_j$.

With $O(1/k)$ the above result is *not* true. Here is a counter-example in part due to Janson [15]. Let $f(x)=\operatorname{sign} x$, $-\pi<x\leqq\pi$, so that

$$\hat{f}(k) = \begin{cases} \dfrac{2}{\pi}\cdot\dfrac{1}{ik} & \text{if } k=\pm 1, \pm 3, \dots \\ 0 & \text{else}. \end{cases}$$

Assume that there exists a continuous function $F$ such that $\hat{F}(k)=\hat{f}(k)$ for $k=0,1,2,\dots$. Then $f-F\in\overline{H^\infty}$, that is $f\in\overline{H^\infty+C}$. But $f$ is real valued, whence $f\in(H^\infty+C)\cap(\overline{H^\infty+C})$. By a theorem of Sarason's [38] (cf. [39]), to the effect that $(H^\infty+C)\cap(\overline{H^\infty+C})=\mathrm{VMO}\cap L^\infty$ (this is the famous class QC of *quasi-continuous* functions) it follows that $f\in\mathrm{VMO}$. But VMO does not contain any functions with jumps. Contradiction.

Alternatively we could in this connection (this remark) have used the theory of Hankel operators, viz. the theorems of Nehari and Hartmann (cf. section 3, *infra*).

REMARK 5. It is likewise easy to extend the generalized Hardy's inequality to the classes $H^p$, $0<p<1$. Indeed, let $F\in L^\infty$ be the function constructed in the theorem ($G=\boldsymbol{T}$) with a suitable sequence $\{a_k\}$. Consider the function $\Psi$ whose Fourier coefficients are $\hat{\Psi}(n)=\hat{F}(n)/n^\alpha$ where $\alpha>0$. Then $\Psi$ belongs to the Lipschitz class $\Lambda_\alpha$. Invoke now the celebrated duality theorem of Duren–Romberg–Shields [9] stating that $(H^p)'\approx\Lambda_\alpha$, $\alpha=1/p-1$. Then we may conclude that $\sum|c_k|/kn_k^\alpha<\infty$ for any function $g$ in $H^p$ ($L^p$) of the form $g=\sum c_k e^{in_k x}$ (that is, $\hat{g}(n)=0$ if $n\neq n_k$). Again the case $n_k=k$ is classical: $\sum|c_k|/k^{1/p}<\infty$ in that situation.

## 2. Interpolation between BMO(A) and a (Bergman-)Besov space.

In this section we take for convenience $G=\mathbf{R}$. The necessary changes for the case $G=\boldsymbol{T}$ will be indicated at the end (see Remark 9). The extension to $n$ variables will be briefly treated in section 4.

First let us consider the relevant definitions (cf. [2], [23]). We consider sequences $\{\omega_j\}_{j\in\mathbf{Z}}$ of test functions on the real line $\mathbf{R}$ (i.e. each $\omega_j$ belongs to the Schwartz class $\mathscr{S}$) such that

$$\operatorname{supp}\hat{\omega}_j \subset [2^{j-i},2^{j+1}]\cup -[2^{j-1},2^{j+1}],$$

$$|\xi|^l\omega_j^{(l)}(\xi)| \leqq C_1 \qquad (l=0,1,2,\dots).$$

Then a tempered distribution $f$ on $\mathbf{R}$ ($f\in\mathscr{S}'$) is said to be in the Besov space

$B_p^{sq}$, where $s$ is arbitrary real (and finite) and $p,q \in (0,\infty]$, if for each such sequence holds

$$\left[\sum_{j\in\mathbf{Z}} (2^{js}\|\omega_j * f\|_p)^q\right]^{1/q} < \infty .$$

Indeed one can do with only *one* such sequence if one imposes a suitable Tauberian condition; in particular, the latter will be fulfilled if the $\omega_j$ form a "partition of unity", i.e. $\sum_{j\in\mathbf{Z}} \omega_j = \delta$. If we in this definition replace $L^p$ by the Lorentz space $L^{pr}$ we get the spaces $B_{pr}^{sq}$ ("Besov-Lorentz space"). If $f$ is in $B_p^{sq}$ and $\operatorname{supp} \hat{f} \subset [0,\infty)$ we say that $f$ is in the class $A_p^{sq}$ ("Bergman-Besov space"); that is, $A_p^{sq} = PB_p^{sq}$, where $P$ is the projection complementary to our previous $P_-$, that is $P = I - P_-$. The spaces $A_{pr}^{sq}$ are defined in a similar manner; cf. *infra* Remark 10. In what follows we are mostly concerned with the case $q = r$. Finally we also require the spaces BMOA and BMO; we can define them by the formulae $\text{BMOA} = PL^\infty$ and $\text{BMO} = L^\infty + PL^\infty$ (linear hull) respectively.

We can now announce the main results of this section (in its final form due to Janson [15]; cf. *infra* Remark 8).

THEOREM A. *Assume that* $p_1, q_1, r_1 \in (0,\infty)$, $s_1 \neq 0$ *and let* $1/p = \theta/p_1$, $1/q = \theta/q_1$, $s = \theta s_1$ *with* $\theta \in (0,1)$. *Then*:

$$(\text{BMOA}, A_{p_1 r_1}^{s_1 q_1})_{\theta q} = A_{pq}^{sq} .$$

THEOREM B. *Similar result with the space* BMO, $B_{pr}^{sq}$.

In view of the above relation between the two types of spaces we need only to prove Theorem A.

First we give, however, some comments on the theorems (Remark 1–6, *infra*). For matter of convenience we take below $p = r$ (and $p_1 = r_1$).

REMARK 1. As already noted in the Introduction (section 0), in the special case $p_1$ $(= r_1) = s_1 = 1$ both theorems were obtained by Peller [27], [28] in a quite different way, namely as a by-product of his work on Hankel operators ("the trace ideal criterion"; see section 3).

REMARK 2. Certain limiting cases of our theorems are at least implicit in the literature. Consider the case $p = p_1 = \infty$. For instance (alternatively one could imbed the spaces $\Lambda_\theta$ in the scale of Besov spaces $B_\infty^{sq}$ and use the reiteration theorem [2]) from Lemma 5 and Lemma 6 in Janson [14] follows that $(\text{BMO}, \Lambda)_{\theta\infty} = \Lambda_\theta$ (the usual Lipschitz space; $\Lambda = \Lambda_1$). Indeed, one has in this special case also a corresponding result for the more general interpolation spaces $(.,.)_{\varrho\infty;K}$; $\varrho$ a concave function. (The spaces $(.,.)_{\theta\infty}$ correspond to $\varrho(t) = t^{\theta}$.) In the light of this observation one gains a new insight into the proof of the second half of p. 269 in [14]. Notice that in this limiting case one obtains a necessary decomposition of the function using *linear* operators (indeed, the simplest type of *mollifiers* will do), whereas our construction (*infra*), extracted from the Lemma in section 1, is highly non-linear. Obvious question: Are the pairs $(\mathrm{BMO}, B_{pr}^{sq})$ "quasi-linearizable" (in the sense of [24]) or not? Obvious guess: Not!

REMARK 3. Another extremal case is $s=s_1=0$. Now by a theorem by Fefferman and Stein [11] $(\mathrm{BMO}, L^{p_1})_{\theta p} = L^p$, as above, with $1/p = \theta/p_1$ $(0<\theta<1)$. Now $B_2^{02} = L^2$ so if Theorem B were true in this case we would get $B_p^{0p} = L^p$ if $2<p<\infty$, which is known to be false. This counter-example clearly shows that the restriction $s_1 \neq 0$ is in fact necessary.

REMARK 4. We consider here only *real* interpolation. In fact, there are counter-examples showing that the corresponding result with *complex* interpolation is not true; one such is due to Peller himself [30] (Sub-remark. Rochberg [36], who essentially gets Peller's main results for Hankel operators, nevertheless uses complex interpolation (Riesz-Thorin), and so does Peller [29] in his recent extension of the trace ideal criterion to the vector valued case; it is an obvious challenge to extend the present approach to the trace ideal criterion (section 3) to cover this case too.) After the first version of this paper was completed (December 1981) Jöran Bergh kindly turned the authors' attention to Rychener's paper [35], where the problem of complex interpolation between BMO and a potential (Sobolev) space is treated. Per Nilsson (unpublished) has gotten further results in that direction.

REMARK 5. We have only considered the case when the "second" interpolation parameter $q$ is adjusted to the situation at hand; as in the case of Grisvard's theorem (see section 0) we could of course have considered the general space $(.,.)_{\theta q}$ but the description of the resulting spaces is then less explicit: they are not any longer Besov spaces (cf. the discussion in [25]).

REMARK 6. It is likewise of some interest to point out that the obvious analogue of Theorem B with BMO replaced by $L^{\infty}$ is *not* true. Indeed, it is not even true, for instance, that

$$(1) \qquad B_2^{\frac{1}{2},\infty} \subset (L^{\infty}, B_1^{1,\infty})_{\frac{1}{2},\infty} \,.$$

For consider the function $f = \sum a_k e^{ikx}$ (for simplicity's sake we revert temporarily to the case of $\boldsymbol{T}$) where $a_k$ is a *twosided* sequence with $|a_k| \leqq /|k|$.

If (1) were true we could for any $\varepsilon>0$ find an $F \in L^\infty$ such that

1° $\|F\|_\infty \leqq C\varepsilon^{-1}$

and

2° $\|\omega_j * (F-f)\|_1 \leqq C2^{-j}\varepsilon$ .

But 2° implies that $|\hat{F}(h)-a_k| \leqq \eta/|k|$ with $0<\eta<1$, provided we take $\varepsilon$ sufficiently small. If we apply the same reiteration as in the proof of the Theorem in section 1 we get a contradiction — the result obtained would imply a bilateral Hardy's inequality, which is known to be false (it is easy to produce a function $f$ in $L^1$ such that $\sum_{k\neq 0}|\hat{f}(k)|/|k|=\infty$).

Proof of Theorem A (after Svante Janson). One inclusion ($\subset$) is as usual trivial. (Alternatively we could have used Grisvard's result [13] quoted in the Introduction (section 0).) So our chief concern is the inclusion $\supset$.

In view of Wolff's theorem [46] (and the (usual) reiteration theorem [2]) we may take $p_1>1$, $r_1\geqq 1$. We consider first the case $s_1$ (and $s$) $>0$.

Let thus $f \in A^{sq}_{pr}$, where $s>0$, $1<p<\infty$, $1\leqq q$, $r\leqq\infty$, be given. (So in particular $\operatorname{supp}\hat{f}\subset[0,\infty]$.) Put $f_j=\omega_j * f$ ($j \in \mathbf{Z}$), where we now assume that $\sum_{j\in\mathbf{Z}}\omega_j=\delta$, and let $h_j$ be defined as in section 1, that is $h_j=2P_-|f_j|$. Write also $H_j=h_j+h_{j+1}+\ldots$. Set $F=\sum_{j\in\mathbf{Z}}\varphi(\varepsilon H_j)f_j$ where $\varphi$ is as in Remark 1 of the same section assuming, however, that furthermore

$$\varphi'(0) = \varphi''(0) = \ldots = \varphi^{(m-1)}(0) = 0; \qquad |\varphi^{(m)}(z)| \leqq C\,,$$

$m$ a sufficiently large integer ($m\geqq p/p_1-1$ will do!). As in section 1 it is clear (see again Remark 1) that $F \in L^\infty$, $\|F\|_\infty=O(1/\varepsilon)$, and moreover holds

$$\omega_j * (\tilde{F}-f) = \sum_{\nu\geqq j-1} \tilde{\omega}_\nu * (\varphi(\varepsilon H_\nu)-1)f_\nu$$

with $\tilde{F}\overset{\text{def}}{=}PF$, $\tilde{\omega}_j\overset{\text{def}}{=}P\omega_j$; notice that by definition $\tilde{F} \in \text{BMOA}$, $\|\tilde{F}\|_{\text{BMOA}}=O(1/\varepsilon)$. Using Minkowski's inequality and M. Riesz's conjugate function theorem (for Lorentz space!) it is easy to see that

$$\|\{\|H_j\|_{pr}\}\|_{l^q_s} \leqq C\|f\|_{A^{sq}_{pr}}\,.$$

($l^q_s$ is $l^q$ with respect to the weight $\{2^{js}\}$: $\{x_j\} \in l^q_s$ iff $\{2^{js}x_j\} \in l^q$.) On the other hand, since $|\varphi(z)-1|\leqq C|z|^\alpha$ if $0\leqq\alpha\leqq m$ we get (Hölder's inequality for Lorentz space!) if $\alpha=p/p_1-1=(1-\theta)/\theta$ and $r_1=\theta r$:

$$\begin{aligned}
\|\omega_j * (\tilde{F}-f)\|_{p_1 r_1} &\leqq C \sum_{\nu=j-1}^{\infty} \| |\varepsilon H_\nu|^\alpha |f_\nu| \|_{p_1 r_1} \\
&\leqq C\varepsilon^\alpha \sum_{\nu=j-1}^{\infty} \| |H_\nu|^\alpha |f_\nu| \|_{p_1 r_1} \\
&\leqq C\varepsilon^\alpha \sum_{\nu=j-1}^{\infty} \|H_\nu\|_{pr}^\alpha \|f_\nu\|_{pr} .
\end{aligned}$$

Another application of Minkovski's inequality now yields

$$\begin{aligned}
\|\tilde{F}-f\|_{A^{s_1 q_1}_{p_1 r_1}} &= \|\{\|\omega_j * (\tilde{F}-f)\|_{p_1 r_1}\}\|_{l^{q_1}_{s_1}} \\
&\leqq C\varepsilon^\alpha \sum_{i=-1}^{\infty} \|\{\|H_{j+i}\|_{pr}^\alpha \|f_{j+i}\|_{pr}\}\|_{l^{q_1}_{s_1}} \\
&= C\varepsilon^\alpha \sum_{i=-1}^{\infty} 2^{-is_1} \cdot \|\{\|H_j\|_{pr}^\alpha \|f_j\|_{pr}\}\|_{l^{q_1}_{s_1}} \\
&\leqq C\varepsilon^\alpha \|\{\|H_j\|_{pr}\}\|_{l^q_s}^\alpha \cdot \|\{\|f_j\|_{pr}\}\|_{l^q_s} \\
&\leqq \varepsilon^\alpha \|f\|_{A^{sq}_{pr}}^{1+\alpha} .
\end{aligned}$$

Introducing the $K$-functional (see [2]) and choosing $\varepsilon = t^{-\theta}$ (recall that $\theta = 1/(1+\alpha)$) we now see that

$$K(t, f; \mathrm{BMOA}, A^{s_1 q_1}_{p_1 r_1}) \leqq Ct^\theta \|f\|_{A^{sq}_{pr}} ,$$

which apparently gives

$$A^{sq}_{pr} \subset (\mathrm{BMOA}, A^{s_1 q_1}_{p_1 r_1})_{\theta\infty} .$$

The reverse inclusion

$$(\mathrm{BMOA}, A^{s_1 q_1}_{p_1 r_1})_{\theta 1} \subset A^{sq}_{pr} ,$$

indeed, even with BMOA replaced by $A^{s\infty}_{\infty\infty}$, is well-known (see [2] or [23]). If we now use reiteration [2] (notice that

$$(A^{s'q'}_{p'r'}, A^{s''q''}_{p''r''})_{\eta q} = A^{sq}_{pq}$$

with suitable relations for $s$, $p$ and $q$, but *irrespective* of the choice of $r'$ and $r''$!) this clearly yields the desired result.

The case $s_1 < 0$ can be handeled in a similar way. Take $h_j = 2P_+|f_j|$ with now $f_j = \omega_{-j} * f$. We do not insist upon the details.

REMARK 7. The most important case of our theorems is when $p = r$ (i.e., we are dealing with genuine Besov spaces, not Besov-Lorentz spaces). In that case one can obtain for instance Theorem B by duality from the following result:

$$(*) \qquad (H_r^1, B_{p_1}^{s_1 p_1})_{\theta p} = B_p^{sp},$$

valid for $p_1 \in (1, \infty]$, $s_1 \neq 0$ and $1/p = 1 - \theta + \theta/p_1$ with $\theta \in (0,1)$. (Here $H_r^1$ stands for the "real form" of the Hardy class $H^1$.) (*), or rather its "A-version" (and in the "diagonal" case $s = -1/p$), was first discovered by Semmes [42] using a direct *ad hoc* approach and then one of the present authors (see [25]) remarked that it really is a very special case of a result of Triebel's for "Triebel-Lizorkin spaces" (see [45, p. 185]). We do not know if the analogue of (*) holds in the general case $p \neq r$. If Theorem B were true with BMO replaced by CMO (*continuous* mean oscillation) this would again follow by duality; apply Fefferman's duality theorem [11] to the effect that $(\mathrm{CMO})' \approx H_r^1$. After the first version of this paper was completed (December 1981) Per Nilsson kindly turned out attention to Bui's paper [4] where a proof of (*) is found, indeed, even in the more general case of *weighted* spaces. The method of the latter, which amounts to representing the couple in question as a *retract*, lends itself to a determination of the interpolation spaces also in the complex case.

REMARK 8. In an earlier version of this paper we considered only the special cases $q = p$ and $q = \infty$ (with $p_1 \geqq 1$). In the former case our proof depended on the use of the $E$-functional, along with the $E$-spaces or "approximation spaces" ([26]; see also [2, Chapter 7]). More specifically, we proved for the pair $(\dot{\mathrm{B}}\mathrm{MOA}, B_1^{s_1 1})$ the estimate

$$E(C/\varepsilon, f) \leqq C \sum_j 2^{js_1} \sum_{\nu \geqq j-1} \|\min(1, \varepsilon^m |H_\nu|^m f_\nu)\|_1;$$

we mention this because this might perhaps be of some interest in a more general context.

REMARK 9. Finally we take up for discussion the case of the torus $T$. We claim that with practically no changes of the proof, Theorem A and B go over to that case. It suffices only to remark that we can employ exactly the same definition of Besov spaces as on the line R. Namely, we can consider a function (or distribution) on $T$ as a periodic tempered distribution on the line and then convolution with a testfunction makes sense. (The Fourier transform of a periodic distribution is a series of the form $\sum a_k \delta_k$, where the $\delta_k$ are translates of the unit mass (delta functiom) at the origin and the $a_k$ can be identified with the Fourier coefficients of the corresponding object on $T$; this is essentially Laurent Schwartz's approach to Fourier series, see [40, p. 108–109].)

REMARK 10. In comparison with the Besov spaces $B_p^{sq}$ the Bergman-Besov spaces $A_p^{sq}$ have been little studied. We mention e.g. the paper by Stegenga [44], which is concerned with (pointwise) multipliers (the case $p = q = 2$). In higher dimensions they find their most natural generalization in the context of *several complex* variables; see notably [6], [7].

## 3. Application to Hankel operators.

For convenience we consider now the case $G=T$ only but similar results hold for $G=\mathbf{R}$. (Indeed, the whole set-up is essentially conformally invariant, provided we let our functions $f$ transform as forms of type $(0,\frac{1}{2})$.) Let $H^2$ be the usual Hardy class (i.e. $f\in H^2$ iff $f\in L^2$, $\operatorname{supp}\hat{f}\in[0,\infty)$) and let $H^2_-$ be its orthogonal complement in $L^2$, so that $H^2=PL^2$, $H^2_-=P_-L^2$ $(P=I-P_-)$.

A *Hankel operator* is a (bounded) linear map $U$ from $H^2$ into $H^2_-$ such that $U\chi=P_-\chi U$. Here $\chi$ stands for (multiplication by) the principal character of $T$, that is $\chi(x)=e^{ix}$, $x\in T$. It follows from this definition that (Nehari's theorem) every Hankel operator $U$ can be expressed in the form $U=H_f\overset{\text{def}}{=}P_-f$ where $f$ (the symbol of $U$) is a function in $L^\infty$ determined up to an element of $H^\infty$. Thus the space of (bounded) Hankel operators is essentially, i.e., up to complex conjugation, isomorphic to BMOA. Similarly the space of compact Hankel operators is isomorphic to VMOA (Hartman's theorem). A recent survey of Hankel operators (matrices) is the article by Power [33], containing in particular a discussion of the above theorems by Nehari and Hartman. See also his book [34] and further the books Nikol'skiĭ [22, chapter 8] and especially App. 4 of the forthcoming English translation, and Sarason [38, Chapter 9].

Peller [27], [28] (see also Peller-Hruščev [32] for further work along this line) solved the problem which Hankel operators are of trace class. (Regarding the trace classes $\mathfrak{S}_p$ see [43] and also [2, Chapter 7].) His result ([28, Theorems 1 and 2]) says that $H_f$ is in $\mathfrak{S}_p$, $1\leq p<\infty$ iff $\overline{P_-f}\in A_p^{1/p,p}$ (complex conjugate). As we have already mentioned (Introduction and Remark 1 in section 2) Peller gets from this as an application a special case of our Theorems A and B (section 2). Here we shall go in the opposite direction: we show that Theorem A entails Peller's result at least in the intermediate case $1<p<\infty$, the extremal case $p=1$ requiring a special proof ($p=\infty$ is of course just Nehari (or Hartman)).

Before entering into the details let us mention that Peller in [28] gives yet many other, most striking applications, of which we here wish to emphasize the quite surprising result (Theorem 7 and 7′ in [28]) on the rate of approximation by rational functions in the BMOA or the BMO metric. (See also [27].) Is it possible to prove such a result directly without invoking Hankel operators? (This would call for an elimination of the deep theorem of Adamjan–Arov–Kreĭn [1] used by Peller.) Let us also point out (cf. Remark 4 of section 2) that Rochberg [36] has obtained Peller's main results by yet another route, via a certain type of molecular decomposition in the spaces $A_p^{sp}$. After the above lines where written (December 1981) the trace ideal criterion has been extended to the case $0<p<1$ by Peller [31] himself, and independently and simultaneously by Semmes [41], the latter using an approach in Rochberg's spirit [36] ("the St. Louis spirit"). This implies in particular that the above mentioned result on rational approximation extends to all values of the degree. For a "vector" generalization, that is, with "block" Hankel operators (matrices), see further [29]. There he uses complex interpolation à la Rochberg [36] (the Riesz–Thorin method). Is there any hope to extend our result to a vector-valued situation so as to cover that result too?

Perhaps it is also appropriate here to quote the work of Janson and Wolff [16] where the problem of trace class commutators of Calderón–Zygmund operators is solved. At least in *one* variable the two problems are intimately connected (see [36]). There is further a notion of Hankel (and Toeplitz) operators in the context of several *complex* variables (see e.g. [7]).

Let us now give the proof promised of Peller's result, that is, we want to prove the

THEOREM (Peller [28, Theorem 2]). *Let* $1<p<\infty$. *Then* $H_f \in \mathfrak{S}_p \Leftrightarrow \overline{P_- f} \in A_p^{1/p,p}$.

PROOF. Let us denote by $\alpha$ the mapping $\overline{P_- f} \to H_f$. Then by Nehari's theorem, and by [28, Theorem 1]

$$\begin{cases} \alpha : \mathrm{BMOA} \to \mathfrak{S}_\infty . \\ \alpha : A_1^{1,1} \to \mathfrak{S}_1 . \end{cases}$$

Thus by interpolation

$$\alpha : (\mathrm{BMOA}, A_1^{1,1})_{\theta p} \to (\mathfrak{S}_\infty, \mathfrak{S}_1)_{\theta p} \qquad (\theta \in (0,1)) .$$

If $\theta = 1/p$ we have $(\mathrm{BMOA}, A_1^{1,1})_{\theta p} = A_p^{1/p,p}$ (our Theorem A for the case of $\boldsymbol{T}$) and $(\mathfrak{S}_\infty, \mathfrak{S}_1)_{\theta p} = \mathfrak{S}_p$ (well-known, see e.g. [2, Chapter 7]). This gives

$$\alpha : A_p^{1/p,p} \to \mathfrak{S}_p \qquad (1<p<\infty)$$

and proves half of the theorem. The remaining half will be obtained simply by duality. Indeed taking the transpose we get

$$\alpha^t : S_p \to A_p^{1/p-1,p} \qquad (1<p<\infty)$$

(we use here the obvious generalization to the A case of the well-known fact that $(B_p^{sq})' \approx B_{p'}^{-sq'}$, see [2] or [23], and the corresponding result for trace classes: $\mathfrak{S}'_p \approx \mathfrak{S}_{p'}$, see [43]). There remains to "explicate" the map $\alpha^t$. We claim that $\alpha^t \circ \alpha$, which thus is a map that to a function ("symbol") assigns another function, is essentially derivation, $\alpha^t \circ \alpha = D \overset{\mathrm{def}}{=} 1/i \cdot d/dx$. This is most simply seen by introducing the matrix of the Hankel operator (a Hankel matrix). Then $\alpha$ is a map which to a vector $(a_1, a_2, \ldots)$ assigns the matrix

$$(a_{i+k}) = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots \\ a_2 & a_3 & a_4 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

It is clear that the transpose $\alpha^t$ (for the natural dualities) now is the operator which to a given matrix $(b_{ik})$ assigns the vector whose entries are the sums $\sum_{i+k=n} b_{ik}$. (Taking $b_{ik} = a_{i+k}$ we get back the $n$th component of the original vector $(a_1, a_2, \ldots)$ multiplied by $n$; it is the factor $n$ that accounts for $D$.) Introducing formally the map $\beta = D^{-1}\alpha^t$ we have indeed $\beta \circ \alpha = \mathrm{id}$, whence $\alpha$: $A_p^{1/p,p} \to \mathfrak{S}_p$ and $\beta$: $\mathfrak{S}_p \to A_p^{1/p,p}$. We have thereby represented the spaces of symbols $A_p^{1/p,p}$ as retracts of the corresponding trace classes $\mathfrak{S}_p$. A well-known elementary argument (see [23] for many instances of it) now gives the desired conclusion: $H_f \in \mathfrak{S}_p \Rightarrow \overline{P_- f} \in A_p^{1/p,p}$.

Remark 1. The above duality is of course also present in Peller's treatment [28] but we believe that in this way the state of affairs becomes much more transparent.

Remark 2. We conclude this section with the following observation. Consider any "pure" *Toeplitz operator*, i.e. an operator of the type $V = T_f \overset{\text{def}}{=} Pf$ where $f$ is a bounded *anti-holomorphic* function (i.e. $\bar{f} \in H^\infty$). Then $V$ maps every Bergman-Besov space $A_p^{sq}$, where $s > 0$, $0 < p$, $q \leqq \infty$, into itself. This generalizes a statement of Peller's [28, p. 563, 1.9–10], the special case $s = 1/p$, $q = p$, $1 \leqq p < \infty$. It suffices to prove the result for $q = \infty$ and then use interpolation (with $p$ fixed!). Let thus $u \in A_p^{s\infty}$ where $s > 0$ and, for simplicity's sake, $1 \leqq p < \infty$, too. (Cf. also Kahane [18] who gives practically the same argument. An alternative approach uses duality and the explicit representation of $A_p^{sq}$ $(s < 0)$ as a "true" Bergman space.) Then $Vu = \sum P(fu_j)$ with $u_j = \omega_j * u$ and $\omega_\nu$ (and below $\tilde{\omega}_j$) as in section 2. It follows again that (cf. the corresponding argument in section 2!)

$$\omega_j * Vu = \sum_{\nu \geqq j-1} \tilde{\omega}_\nu * fu_\nu ,$$

whence

$$2^{js} \|\omega_j * Vu\|_p \leqq \sum_{\nu \geqq j-1} 2^{(j-\nu)s} 2^{\nu s} \|u_\nu\|_p \|f\|_\infty .$$

Since by assumption (definition of $A_p^{s\infty}$!) we get $\|\omega_j * Vu\|_p = O(2^{-js})$, too, which proves $Vu \in A_p^{s\infty}$. Notice that this observation sheds new light on the construction in section 2 (and section 1): The function $F$ is obtained by the intermediation of special (pure) Toeplitz operators. The argument in [28], attributed to N. K. Nikol'skiĭ, is given for any "approximation space" and uses again the theorem of Adamjan–Arov–Kreĭn [1]. However, a direct proof, without invoking Hankel operators, can be obtained by "interpolation", simply by remarking that $V$ obviously preserves BMO and maps rational functions onto rational functions without increasing the degree.

## 4. Generaliztion to $\mathbf{R}^n$.

We now give a generalization of the previous considerations to the case of $\mathrm{R}^n$. Since no genuinely new ideas are needed, we allow ourselves to be somewhat sketchy. But in a way it is this section which is really the *raison d'être* of the entire paper.

Our notation is more or less standard: elements of $\mathrm{R}^n$ are denoted by $x = (x_1,\ldots,x_n)$ and elements of the "dual" $\mathrm{R}^n$ by $\xi = (\xi_1,\ldots,\xi_n)$, the two types of vectors being connected by the duality $\langle x,\xi\rangle = x_1\xi_1 + \ldots + x_n\xi_n$. In one case we have the Haar measure $dx = dx_1 \ldots dx_n$, in the other case $(2\pi)^{-n}d\xi = (2\pi)^{-n}d\xi_1 \ldots d\xi_n$.

Let $E$ be an (open) subset of $\mathrm{R}^n$ contained in an open halfspace. Without loss of generality we may assume that it is the half-space $\mathrm{R}^n_+ = \{\xi_1 > 0\}$: $E \subset \mathrm{R}^n_+$. If $a$ is a given locally integrable function on $E$ we are interested in finding a function $f$ in $L^\infty = L^\infty(\mathrm{R}^n)$ such that $\hat{f}(\xi) = a(\xi)$ for $\xi \in E$. To this end we try to imitate the procedure of section 1.

Let thus $\{E_j\}$ be a suitable partition of $E$. Set

$$f_j(x) = (2\pi)^{-n} \int_{E_j} a(\xi) e^{i\langle x,\xi\rangle} d\xi$$

so that obviously $\hat{f}_j(\xi) = a(\xi)$ for $\xi \in E_j$ and define

$$F = \sum e^{-\varepsilon(h_j + h_{j+1} + \ldots)} f_j$$

with $h_j = 2P_-|f_j|$. Here $P_-$ is defined in the obvious way:

$$(P_-g)\hat{}(\xi) = \hat{g}(\xi) \qquad \text{if } \xi_1 < 0,\ 0 \text{ else},$$

for (say) $g \in L^2$. As in section 1 we see that $F \in L^\infty$ and $\|F\|_\infty \leqq 1/\varepsilon$. Furthermore we have the basic identity

$$\hat{F}(\xi) - a(\xi) = \sum_{\nu \in N_j} [(e^{-\varepsilon(h_\nu + h_{\nu+1} + \ldots)} - 1) f_\nu]\hat{}(\xi)$$

for $\xi \in E_j$, where $N_j$ denotes the set of indices $\nu$ such that the sum of $E_\nu$ and the complementary halfspace $\mathrm{R}^n_- = \{\xi_1 < 0\}$ has non-empty intersection with $E_j$ (the family $\{N_j\}$ thus depends on the "geometry" of the situation). We therefore get as a generalization of the estimate of the Lemma in section 1 the following inequality

$$(1) \qquad |\hat{F}(\xi)-a(\xi)| \leqq \sqrt{2\varepsilon} \sum_{\nu \in N_j} b_\nu(b_\nu+b_{\nu+1}+\dots)$$

with $b_\nu = ((2\pi)^{-n}\int_{E_\nu} |a(\xi)|^2 \, d\xi)^{\frac{1}{2}}$.

Next we seek conditions such that the reiterative procedure can be applied.

Let $\delta_j$ be the (Haar) measure of $E_j$. Assume that the function $a$ satisfies the condition $|a(\xi)| \leqq M/\delta_j$ for $\xi \in E_j$. Then $b_\nu = O(\delta_\nu^{-\frac{1}{2}})$ so that (1) gives

$$|\hat{F}(\xi)-a(\xi)| \leqq \text{const}\cdot M^2\sqrt{2\varepsilon} \sum_{\nu \in N_j} \delta_\nu^{-\frac{1}{2}}(\delta_\nu^{-\frac{1}{2}}+\delta_{\nu+1}^{-\frac{1}{2}}+\dots)\,.$$

Next if we impose the extra (hypothesis that (say)

$$(2) \qquad \sum_{\nu \in N_j} \delta_\nu^{-\frac{1}{2}}(\delta_\nu^{-\frac{1}{2}}+\delta_{\nu+1}^{-\frac{1}{2}}+\dots) \leqq C\delta_j^{-1}\,,$$

we get by choosing $\varepsilon$ sufficiently small

$$|\hat{F}(\xi)-a(\xi)| \leqq \eta M/\delta_j \qquad (\xi \in E_j)$$

where we can take $0<\eta<1$ (along with the estimate $\|F\|_\infty \leqq CM$). Thus we can effectively set the reiteration at work and finally solve our interpolation problem.

Using the interpolation result the following inequality of Hardy type can easily be proved

$$(3) \qquad \|f\|_1 \geqq C\int |\hat{f}(\xi)|\, d\xi/\delta(\xi_1) \qquad (C>0)\,.$$

Here $f$ is any function in $L^1$ with $\operatorname{supp}\hat{f} \subset \mathbf{R}^n$ and $\delta(t)$ denotes the measure of the set $\operatorname{supp}\hat{f} \cap \{0 \leqq \xi_1 \leqq t\}$. The inequality is non-trivial only for those functions for which $\delta(t)<\infty$ for some $t>0$ and in these cases the inequality follows from the above if we put $E = \operatorname{supp}\hat{f}$ and make a "dyadic" partition of $\operatorname{supp}\hat{f}$ in slices orthogonal to the $\xi_1$-axis. If $\operatorname{supp}\hat{f}$ is contained in a proper cone (in turn contained in $\mathbf{R}^n_+$) then of course $\delta(\xi_1) \leqq \text{const}\cdot|\xi|^n$ and therefore

$$(3') \qquad \|f\|_1 \geqq C\int |\hat{f}(\xi)|\, d\xi/|\xi|^n \qquad (C>0)\,.$$

REMARK 1. From this it is easy to get the corresponding result for the real Hardy class $H^1_r = H^1_r(\mathbf{R}^n)$ (cf. *infra*). This $n$-dimensional form of the classical Hardy's inequality seems to have been known in 1974 to a number of authors (see [3], [10]). The proof is essentially based on the use of "atoms".

REMARK 2. It is also of interest to notice that (3′) is still valid if we only assume that supp $\hat{f}$ is contained in a half-space, say, $\mathbf{R}^n_+ = \{\xi_1 > 0\}$. A proof can be readily obtained by "transference" (cf. [8]) using the classical (one dimensional) result. Indeed, we have the following more general result: if $f \in L^1$ with supp $\hat{f} \subset \{\xi_1 > 0\}$ and $k$ is any positive, homogeneous of degree $-n$ function whose restriction to the unit sphere $S^{n-1}$ in $\mathbf{R}^n$ is an integrable function, $\int_{S^{n-1}} k(\xi)\,dA < \infty$ ($dA$ = area element), we have

$$\|f\|_1 \geqq C \int k(\xi)|\hat{f}(\xi)|\,d\xi \qquad (C>0)\,.$$

PROOF. Setting $\omega = \xi/|\xi|$ ($\omega \in S^{n-1}$) and $y = x_1 + \omega_2/\omega_1 \cdot x_2 + \ldots + \omega_n/\omega_1 \cdot x_n$ $(= \langle x, \omega\rangle/\omega_1)$ we may write

$$\hat{f}(\xi) = \hat{f}\left(\xi_1, \xi_1 \cdot \frac{\omega_2}{\omega_1}, \ldots, \xi_1 \cdot \frac{\omega_n}{\omega_1}\right)$$

$$= \int e^{i\xi_1 y}\left(\int f\left(y - \frac{\omega_2}{\omega_1}x_2 - \ldots - \frac{\omega_n}{\omega_1}x_n, x_2, \ldots, x_n\right)dx_2 \ldots dx_n\right)dy$$

$$= \int e^{i\xi_1 y} G(\omega, y)\,dy$$

where $G(\omega, y) = F(\omega, y\omega_1)\cdot\omega_1$ and $F(\omega, y)$ stands for the *Radon transform* of $f$,

$$F(\omega, y) = \frac{d}{dy}\int_{\langle x,\omega\rangle \leqq y} f(x)\,dx = \int_{\langle x,\omega\rangle = y} f(x)\frac{dx}{d\langle x,\omega\rangle}\,.$$

Using the 1-dimensional Hardy's inequality (keeping $\omega$ fixed so that $\hat{f}(\xi) = \hat{f}(\xi_1, \xi_1\cdot\omega_2/\omega_1, \ldots, \xi_1\cdot\omega_n/\omega_1)$ may be regarded as the Fourier transform of $G(\omega, y)$ we see that

$$\int |\hat{f}(\xi)|\,d\xi_1/\xi_1 \leqq C\int |F(\omega, y)|\,dy \leqq C\|f\|_1\,.$$

Multiply this inequality by $k(\omega)$ and integrate over $S^{n-1}$, noticing that $d\xi_1 dA/\xi_1 = d\xi_1 \ldots d\xi_n/|\xi|^n$. Indeed by $\xi_k = \xi_1 . \omega_k/\omega_1$ we have

$$d\xi_k = d\xi_k = d\xi_1\cdot\omega_k/\omega_1 + \xi_1 d(\omega_k/\omega_1) \qquad (k = 2, 3, \ldots)\,.$$

This gives

$$d\xi_1 \ldots d\xi_n = \xi_1^{n-1} d\xi_1 d(\omega_2/\omega_1) \ldots d(\omega_n/\omega_1)\,.$$

Dividing by $|\xi|^n$ the desired relation follows. Now the above inequality results. For an alternative proof, due to Janson, see [25].

Now we leave the (pointwise) interpolation problem and, taking the word interpolation in a different sense, turn to interpolation spaces.

We claim that the obvious analogous of at leat Theorem B of section 2 holds true in $\mathbf{R}^n$, viz.

$$(\mathrm{BMO}, B^{s_1 q_1}_{p_1 r_1})_{\theta q} = B^{sq}_{pq},$$

in the same hypothesis as in that section.

First we must define the spaces involved. As for the Lorentz-Besov spaces $B^{sq}_{pr}$ this is obvious (cf. [2], [21]); we use as before sequences of test functions $\{\omega_j\}_{j\in\mathbf{Z}}$, now of course in $\mathbf{R}^n$, requiring from the supports that $\operatorname{supp}\hat{\omega}_j \subset \{2^{j-1} \leqq |\xi| \leqq 2^{j+1}\}$. As for BMO we use a "microlocal" definition inspired by a paper by Carleson [5]: $f \in \mathrm{BMO} = \mathrm{BMO}(\mathbf{R}^n)$ iff for every direction $\xi$ $(\neq 0)$ there exists a Calderón–Zygmund (CZ) operator $K$, which is *non-characteristic* in that direction, and a function $g$ in $L^\infty$ such that $Kf = Kg$. A few words of explanation: A CZ operator is a convolution with a smooth (in $\mathbf{R}^n \setminus \{0\}$) function $k$, which is homogeneous of degree $-n$ and has zero mean value over the unit sphere $S^{n-1}$, $\int_{S^{n-1}} k(x)\,dA = 0$; it is non-characteristic in the $\xi$ direction if $\hat{k}(\xi) \neq 0$. If $n=1$ there are essentially only two CZ transforms, viz. the (Riesz) projections $P$ and $P_-$, so it is clear that this is essentially the previous definition. If $n>1$ one needs a formal proof. Using a finite partition of unity one sees that an $f$ which is in BMO according to our definition can be written in the form $f = \sum_{i=1}^N K_i g_i$, where the $K_i$ are suitable CZ operators and each $g_i$ is in $L^\infty$, so that $f$ is certainly in BMO in the usual sense (CZ operators act on BMO). The other direction follows by duality from Carleson's result ([5, Theorem 3]).

With this preliminaries settled it is now obvious how to carry over the arguments of section 2. Let thus $f$ in $B^{sq}_{pr}$ be given. Without loss of generality we may assume that $\operatorname{supp}\hat{f}$ is contained in a "narrow" cone $E$ inside the halfspace $\mathbf{R}^n_+$. Set again $f_j = \omega_j * f$ and

$$F = \sum_{j\in\mathbf{Z}} \varphi(\varepsilon(h_j + h_{j+1} + \ldots))f_j$$

with $\varphi$ as in section 2 and $h_j = 2P_-|f_j|$ as earlier in the present section. Again it is plain that $\|F\|_\infty = O(1/\varepsilon)$. Define $\tilde{F} = KF$, $\tilde{\omega}_j = K\omega_j$, where $K$ is a CZ transform such that $\hat{k}$ has its support in a slightly larger cone $E_1 \subset E$ still inside $\mathbf{R}^n_+$ and equal to 1 on $E$ (so that $Kf = f$). Then clearly

$$\omega_j * (\tilde{F} - f) = \sum_{\nu \geqq j - j_0} \tilde{\omega}_j * (\varphi(\varepsilon(h_\nu + h_{\nu+1} + \ldots)) - 1)f_\nu$$

where $j_0$ is a constant depending solely on the "geometry" (the choice of $E$ and $E_1$). Also by definition $\|\tilde{F}\|_{\mathrm{BMO}} = O(1/\varepsilon)$. It is now obvious how to complete the proof (following the pattern laid down in section 2) and we are thus lead to a conclusion of the type $f \in (\text{BMO}, B^{s_1 q_1}_{p_1 r_1})_{\theta q}$.

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