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EXCEPTIONAL SETS, SUBORDINATION, AND FUNCTIONS OF UNIFORMLY BOUNDED CHARACTERISTIC

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1. Introduction.

Let f be a function meromorphic in $D=\{|z|<1\}$, $0<r<1$, $z=x+iy$, $f^{\#}=|f'|/(1+|f|^2)$, and let

$$T(r,f) = \pi^{-1}\int_0^r t^{-1}\,dt \iint_{|z|<t} f^{\#}(z)^2\,dxdy .$$

Set

$$T(f) = T(1,f) = \lim_{r\to 1} T(r,f) .$$

For $w \in D$, set

$$f_w = f\circ\varphi_w, \quad \varphi_w(z) = (z+w)/(1+\bar{w}z), \quad z\in D .$$

Then f is said to be of uniformly bounded characteristic in D, $f\in$ UBC, in notation, if $\|f\|_T<\infty$, where

$$\|f\|_T = \sup_{w\in D} T(f_w) .$$

Obviously, UBC $\subset$ BC, where BC is the family of f with $T(f)<\infty$.

A subset E of $\mathbf{C}^* = \{|z|\leqq\infty\}$ is said to be of positive elliptic capacity, $\mathrm{cap}^* E>0$, in notation, if E contains a closed subset of $\mathbf{C}^*$ of positive elliptic capacity in the sense of M. Tsuji [11, p. 90]. It is easy to observe that $\mathrm{cap}^* E>0$ if and only if $E-\{\infty\}$ contains a bounded closed set of positive (logarithmic) capacity; see [11, p. 55] for the definition of capacity.

Let f be meromorphic in D, let $n(a,f)$ for $a\in\mathbf{C}^*$, be the number of the roots of the equation $f(z)=a$ in D, where the roots are counted according to their multiplicities. Our first result is

THEOREM 1. *Let f be a function meromorphic in D, and let $k\geqq 0$ be an integer. If*

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$$\text{cap}^*\{a \in \mathbf{C}^* \ ; \ n(a, f) \leqq k\} > 0 , \tag{1.1}$$

then $f \in \text{UBC}$.

A weaker and familiar conclusion is that $f \in \text{BC}$.

Let H^p, $0<p<\infty$, denote the Hardy classes in D [2], and let their intersection be

$$H = \bigcap_{0<p<\infty} H^p .$$

Let N be the family of functions meromorphic and normal in D in the sense of O. Lehto and K. I. Virtanen [8].

THEOREM 2. *There exists* $f \in (H \cap N) - \text{UBC}$.

Since $H \subset \text{BC}$, it follows that the inclusion formula $\text{UBC} \subset \text{BC} \cap N$, observed in [13], is strict.

For the proof of Theorem 2, use is made of Theorem 3 described below.

Let B be the family of functions f holomorphic and bounded, $|f|<1$, in D. Let U be the family of $f \in B$ such that

$$\lim_{r \to 1-0} |f(re^{it})| = 1$$

for almost every t, $0 \leqq t < 2\pi$. For f meromorphic in D, and for $h \in B$, the composed function $f \circ h$ is considered as being "roughly" subordinate to f in the sense that the condition $h(0)=0$ is not necessarily assumed.

THEOREM 3. *The following two propositions are true for f meromomorphic in D.*

(I) *For* $f \in \text{UBC}$ *and* $h \in B$,

$$\|f \circ h\|_T \leqq \|f\|_T , \tag{1.2}$$

so that $f \circ h \in \text{UBC}$.

(II) *If* $h \in U$ *and* $f \circ h \in \text{UBC}$, *then*

$$\|f \circ h\|_T = \|f\|_T , \tag{1.3}$$

so that $f \in \text{UBC}$.

2. Proof of Theorem 1.

The chordal distance of z and w in $\mathbf{C}^*$ is

$$X(z, w) = |z-w|/[(1+|z|^2)(1+|w|^2)]^{1/2} ,$$

with the usual device for $z=\infty$ or $w=\infty$. For a closed set $E \subset \mathbf{C}^*$ with $\text{cap}^* E > 0$, there exists the elliptic conductor potential of E,

$$P(w) = -\int_E \log X(w,a)\,d\mu(a), \qquad w \in \mathbf{C}^*,$$

where μ is the elliptic equilibrium distribution of E with the support on E and $\mu(E)=1$. Then

$$0 \leqq P(w) \leqq -\log(\operatorname{cap}^* E) \equiv V(E), \qquad w \in \mathbf{C}^*; \tag{2.1}$$

see [11, Theorem III.46, p. 90]. A few modifications of the proof of [11, Theorem III.7, p. 56] yield that if $e \subset E$ is a closed set and $\operatorname{cap}^* e = 0$, then $\mu(e) = 0$. In particular, $\mu(\{a\})=0$ for each point $a \in E$.

For g meromorphic in D and for $0<r<1$ we set

$$N(r,a,g) = \int_0^r t^{-1} n(t,a,g)\,dt, \qquad a \in \mathbf{C}^*,$$

where $n(r,a,g)$ is the number of the roots of $g(z)=a$ on $|z| \leqq r$, the orders of their multiplicites being again considered. We denote

$$N(1,a,g) = \lim_{r \to 1} N(r,a,g).$$

For later uses we set

$$I(r,g) = (4\pi)^{-1} \int_0^{2\pi} \log(1+|g(re^{it})|^2)\,dt, \qquad 0<r<1,$$

and further we remark that

$$T(r,f) = \int_{\mathbf{C}^*} N(r,a,g)\,d\sigma(a), \qquad 0<r\leqq 1, \tag{2.2}$$

where

$$d\sigma(a) = \pi^{-1}(1+|a|^2)^{-2}|a|\,d|a|\,d\arg a.$$

LEMMA 2.1. *Suppose that g is meromorphic in D and $g(0)=0$. Let E be a closed set on $\mathbf{C}^*$, $\operatorname{cap}^* E>0$, and let μ be the elliptic equilibrium distribution of E. Then, for each r, $0<r<1$, we have*

$$\left| \int_E N(r,a,g)\,d\mu(a) - T(r,g) \right| \leqq V(E). \tag{2.3}$$

PROOF. Since $g(0)=0$ it follows that

$$T(r,g) = I(r,g) + N(r,\infty,g), \qquad 0<r<1; \tag{2.4}$$

see [9, p. 180, the formula at the bottom].

Let $a \in \mathbf{C}^* - \{0,\infty\}$. Add then the quantity

$$-(4\pi)^{-1}\int_0^{2\pi}\log[(1+|g(re^{it})|^2)(1+|a|^2)]\,dt$$

to both sides of the Jensen formula [9, (2′), p. 164], applied to $g-a$:

$$\log|a| = (2\pi)^{-1}\int_0^{2\pi}\log|g(re^{it})-a|\,dt+N(r,\infty,g)-N(r,a,g)\,.$$

On integrating both sides of the resulting equality:

$$\log X(0,a)-I(r,g) = (2\pi)^{-1}\int_0^{2\pi}\log X(g(re^{it}),a)\,dt+N(r,\infty,g)-N(r,a,g)$$

with respect to $d\mu(a)$ on E, and on observing (2.4), we obtain, after the obvious arrangement, that

$$\int_E N(r,a,g)\,d\mu(a)-T(r,g) = P(0)-(2\pi)^{-1}\int_0^{2\pi}P(g(re^{it}))\,dt\,.$$

Combining this with (2.1) we have (2.3).

REMARK. A few verbal changes yield the same lemma for g defined in $|z|<R \leqq \infty$.

A meromorphic function g in D is normal, $g \in N$, if and only if $c(g)<\infty$, where

$$c(g) = \sup_{z\in D}(1-|z|^2)g^{\#}(z)\,.$$

If g assumes three distinct points of $\mathbf{C}^*$ only a finite number of times in D, then it follows from [8, p. 54 and Theorem 3] that $g \in N$, or, $c(g)<\infty$. This is the case if there exists an integer $k\geqq 0$ such that

$$\operatorname{cap}^*\{a\in\mathbf{C}^*\ ;\ n(a,g)\leqq k\} > 0\,.$$

With this in mind we propose

LEMMA 2.2. *Suppose that g is meromorphic in D and $g(0)=0$. Suppose that the set*

$$\{a\in\mathbf{C}^*\ ;\ n(a,g)\leqq k\}$$

contains a closed set $E\subset\mathbf{C}^*$ with $\operatorname{cap}^* E>0$, *where $k\geqq 0$ is an integer. Let μ be the elliptic equilibrium distribution of E. Then,*

$$\int_E N(1,a,g)\,d\mu(a) \leqq k[A(g)+V(E)]\,, \tag{2.5}$$

where

$$A(g) = \log[(e+1)(c(g)+2)/2] .$$

PROOF. We may suppose that g is nonconstant. First of all, $N(1,a,g)=0$ for $a \in \mathbf{C}^*$ with $n(a,g)=0$. We show that, for $a \in E-\{0\}$ with $n(a,g)>0$,

$$N(1,a,g) \leqq k[A(g)-\log X(0,a)] . \tag{2.6}$$

Then (2.5) is obtained by integration on $E-\{0\}$, together with $P(0) \leqq V(E)$. For the proof of (2.6), we choose $z_0 \neq 0$ with $g(z_0)=a$ such that the disk $|z|<|z_0|=r_0$ contains no root of the equation $g(z)=a$. For each r, $r_0<r<1$, we then obtain

$$N(r,a,g) = \int_{r_0}^{r} t^{-1} n(t,a,g)\,dt \leqq k\log(r/r_0) ,$$

whence

$$N(1,a,g) \leqq -k\log r_0 . \tag{2.7}$$

Since $c(g)<\infty$, and since

$$X(0,a) = X(g(0),g(z_0)) \leqq [c(g)/2]\log[(1+r_0)/(1-r_0)]$$

it follows that

$$r_0 > (e^\beta-1)/(e^\beta+1), \qquad \beta = 2X(0,a)/[c(g)+2] .$$

Since

$$e^\beta+1 < e+1 \quad \text{and} \quad e^\beta-1 > \beta ,$$

it follows that $1/r_0<(e+1)/\beta$. We now obtain (2.6) by (2.7).

PROOF OF THEOREM 1. For $w \in D$ let

$$Q_w(z) = (z-f(w))/(1+\overline{f(w)}z), \qquad z \in \mathbf{C}^* , \tag{2.8}$$

be the rotation of $\mathbf{C}^*$. Fix $w \in D$ arbitrarily and let

$$g = Q_w \circ f_w = Q_w \circ f \circ \varphi_w .$$

Since f takes three distinct points of $\mathbf{C}^*$ only a finite number of times in D, it follows that $c(f)<\infty$. Furthermore, since

$$(1-|z|^2)|(\varphi_w)'(z)| = 1-|\varphi_w(z)|^2, \qquad z \in D ,$$

it follows that $c(g)=c(f)$.

Let E be a closed set with $\text{cap}^* E>0$ contained in the set

$$\{a \in \mathbf{C}^* ;\ n(a,f) \leqq k\} .$$

Since for each $a \in \mathbf{C}^*$,

$$n(a,f) \leqq k \Leftrightarrow n(Q_w(a), g) \leqq k ,$$

it follows that the image set $E_w = Q_w(E)$ is contained in the set

$$\{b \in \mathbf{C}^* \; ; \; n(b,g) \leqq k\} .$$

Apparently, E_w is closed and $\mathrm{cap}^* E_w = \mathrm{cap}^* E$, whence $V(E_w) = V(E)$.

We now apply Lemma 2.2 to g and E_w to obtain the estimate

$$\int_{E_w} N(1,a,g)\, d\mu(a) \leqq k[A(g) + V(E)] .$$

Since $T(r, f_w) = T(r, g)$ for $0 < r < 1$, it follows from Lemma 2.1, applied to g and E_w, that

$$T(r, f_w) \leqq kA(g) + (k+1)V(E) .$$

Letting $r \to 1$, observing that

$$A(g) = \log[(e+1)(c(f)+2)/2]$$

is independent of $w \in D$, and letting w run over D, we finally conclude that $\|f\|_T < \infty$.

3. Proof of Theorem 3.

For f meromorphic in D, M. Heins' theorems [5, Theorems 11.1 and 11.2, p. 440] in the specified case read:

$$f \in \mathrm{BC} \text{ and } h \in B \Rightarrow f \circ h \in \mathrm{BC};$$

$$h \in U \text{ and } f \circ h \in \mathrm{BC} \Rightarrow f \in \mathrm{BC} .$$

For the proof of Theorem 3 we rapidly review the proofs of them in terms of $T(r, f)$.

Let B_0 be the family of $f \in B$ with $f(0) = 0$. We shall later use (III) and the "if" part of (V) in

LEMMA 3.1. *For f nonconstant and meromorphic in D, the following three propositions hold.*

(III) *For each $h \in B_0$ and each r, $0 < r \leqq 1$,*

$$T(r, f \circ h) \leqq T(r, f) .$$

(IV) *Let $h \in B_0$. Then*

$$T(r, f \circ h) = T(r, f)$$

holds for an r, $0 < r < 1$, if and only if h is a rotation, $h(z) = cz$, $|c| = 1$.

(V) *Let* $h \in B_0$. *Then*

$$T(f\circ h) = T(f) < \infty$$

holds if and only if $T(f\circ h)<\infty$ *and* $h \in U$.

Lehto [6, p. 9] essentially proved that

$$N(r,a,f\circ h) \leqq N(r,a,f)$$

for $0<r<1$, $a \in \mathbf{C}^*$ and $h \in B_0$; the role of φ, there, is played by our h. The proof of (III) is then obvious in view of (2.2).

Supposing for the moment that (V) is true, we prove (IV). We must prove the non-obvious part, the "only if" part.

For g meromorphic in D, the identity holds:

$$T(r,g) = \pi^{-1}\iint_{|z|<r} g^{\#}(z)^2 \log|r/z|\,dx\,dy$$

for $0<r\leqq 1$ [13, (2.5)]. For $0<r<1$, let

$$g_{(r)}(z) = g(rz), \quad z \in D\,.$$

Then,

$$T(r,g) = T(g_{(r)})$$

because

$$(g_{(r)})^{\#}(z) = rg^{\#}(rz), \quad z \in D\,.$$

For $0<r<1$ and for our $h \in B_0$ we set

$$h_0 = r^{-1}h_{(r)}\,.$$

Schwarz's lemma teaches then that $h_0 \in B_0$. Furthermore,

$$(f\circ h)_{(r)} = f_{(r)}\circ h_0\,.$$

Therefore, if $T(r,f\circ h)=T(r,f)$ for an r, $0<r<1$, then

$$T(f_{(r)}\circ h_0) = T(r,f\circ h) = T(r,f) = T(f_{(r)})\,.$$

By (V), $h_0 \in U$, and hence $|h(z)|=|z|$ on $|z|=r$. The Schwarz lemma asserts then that h is a rotation.

For the proof of (V) we shall make use of

LEMMA 3.2. *Let* $h \in B_0$ *and* $b \in D-\{0\}$. *Then*

$$N(1,b,h) \leqq \log|1/b|\,. \tag{3.1}$$

If the equality in (3.1) *holds for a certain* $b \in D - \{0\}$, *then* $h \in U$. *Conversely, if* $h \in U$, *then there exists a set* $A \subset D$ *of capacity zero such that the equality in* (3.1) *holds for each* $b \in D - A$.

Note that

$$N(1, b, h) = \sum \log |1/b_n| ,$$

where b_n $(n \geqq 1)$ are the roots of the equation $h(z) = b$ in D, multiple roots appearing as their multiplicities.

The proof of Lemma 3.2 is familiar, see, for example, [7, p. 110], or, [4, p. 446ff.], under far general settings.

To prove (V) we first assume that $T(f \circ h) = T(f) < \infty$ for an $h \in B_0$. By (2.2), then,

$$N(1, a, f) < \infty \qquad \text{for each } a \in \mathbf{C}^* - E ,$$

where E is a certain set with $\sigma(E) = 0$. Let $\{z_n\}$ be all the roots of $f(z) = a$ in D, $a \in \mathbf{C}^* - E$. Then, it follows from (2.2), together with

$$(3.2) \qquad N(1, a, f \circ h) = \sum N(1, z_n, h) \leqq \sum \log |1/z_n| = N(1, a, f) ,$$

that the equality

$$N(1, z_n, h) = \log |1/z_n|$$

holds for all z_n for at least one a. Therefore, $h \in U$ by Lemma 3.2.

Suppose now that $T(f \circ h) < \infty$ and $h \in U$. It follows from Lemma 2.1 that the equality in (3.2) holds:

$$N(1, a, f \circ h) = N(1, a, f)$$

for all $a \in \mathbf{C}^* - E_1$, where $E_1 \supset f(A)$ and $\sigma(E_1) = 0$. The integration of both sides yields that $T(f) = T(f \circ h)$.

For the proof of Theorem 3 we further needs

LEMMA 3.3. *If* $f \in$ BC, *then* $T(f_w)$ *is a* C^∞ *function of real variables* u *and* v *with* $w = u + iv \in D$.

PROOF. Suppose that f is nonconstant and let $f = g/h$, where g and h are holomorphic and bounded function in D having no common zero in D [9, p. 189]. Then,

$$F = (1/2) \log (|g|^2 + |h|^2)$$

is a finite-valued subharmonic function which is C^∞ and bounded from above in D. Consequently, the known result [13, Lemma 5.1] admits

$$(3.3)\qquad (F\circ\varphi_w)\hat{} = F\hat{}\circ\varphi_w\,,$$

where $\hat{}$ denote the least harmonic majorants of the functions considered in D. Our aim is now to show that

$$(3.4)\qquad T(f_w) = F\hat{}(w) - F(w), \qquad w\in D\,,$$

from which Lemma 3.3 follows.

Suppose first that $f(0)\neq\infty$, so that $h(0)\neq 0$. Since a pole of f is a zero of h and vice versa, it follows from the Jensen formula [9, p. 164] that

$$(3.5)\qquad \log|h(0)| = (2\pi)^{-1}\int_0^{2\pi}\log|h(re^{it})|\,dt - N(r,\infty,f)$$

for $0<r<1$. On the other hand, for $0<r<1$,

$$T(r,f) = I(r,f) - (1/2)\log(1+|f(0)|^2) + N(r,\infty,f)\,,$$

[9, p. 180], which, together with (3.5), shows that

$$T(r,f) = (2\pi)^{-1}\int_0^{2\pi} F(re^{it})\,dt - F(0)\,.$$

Letting $r\to 1$ we have

$$(3.6)\qquad T(f) = F\hat{}(0) - F(0)\,.$$

Suppose next that $f(0)=\infty$. Then $g(0)\neq 0=h(0)$. By the same reasoning, applied to $1/f=h/g$ this time, and by the identity $T(f)=T(1/f)$, we again obtain (3.6).

Combining (3.3) and (3.6), we have (3.4).

REMARK. It follows from (3.4) that

$$\Delta T(f_w) = -2f^{\#}(w)^2, \qquad w\in D\,.$$

This is reasonable because [13, (2.6)]

$$T(f_w) = \pi^{-1}\iint_D f^{\#}(\zeta)^2\log|1/\varphi_{-w}(\zeta)|\,d\xi\,d\eta$$

is a Green potential in D.

PROOF OF THEOREM 3. We may suppose that f is nonconstant. To begin with, for each $h\in B$ and $w\in D$,

$$(3.7)\qquad (f\circ h)_w = f_{h(w)}\circ h^*, \qquad \text{where } h^* = \varphi_{-h(w)}\circ h\circ\varphi_w \in B_0\,.$$

Therefore, (1.2) follows from (III) of Lemma 3.1, because

$$T((f\circ h)_w) \leqq T(f_{h(w)}) \leqq \|f\|_T$$

for each $w \in D$.

To prove (1.3) we observe the "if" part of (V) and (3.7) with $h^* \in U$. It then follows that

$$T(f_{h(w)}) = T((f\circ h)_w) \leqq \|f\circ h\|_T < \infty$$

for each $w \in D$. Since $D - h(D)$ is of capacity zero (see [3, Theorem, p. 111]), $h(D)$ is dense in D. Since $T(f_\zeta)$ is continuous for $\zeta \in D$, it follows that $\|f\|_T \leqq \|f\circ h\|_T$, which together with (1.2) in (I), proves (1.3).

4. Proof of Theorem 2.

Let us regard $\mathbf{C}^*$ as the sphere of center $Z = (0, 0, 1/2)$ in the Euclidean space, which touches the complex plane at the origin. We can then find a finite number of distinct points

$$\alpha_1, \ldots, \alpha_n, \quad \text{on } \mathbf{C}^*,$$

which we fix once and for all, such that, for each $\alpha \in \mathbf{C}^*$, there exists at least one α_j which lies in the $\pi/4$ "neighborhood" of α. More precisely, the smaller, nonnegative angle between the radial vectors $Z\alpha$ and $Z\alpha_j$ is less than $\pi/4$.

For g nonconstant and meromorphic in D we let u_g be a conformal homeomorphism from D onto the universal covering surface of the subdomain $D - g^{-1}(\{\alpha_1, \ldots, \alpha_n\})$ of D. Regarding u_g as a holomorphic function with the image in D we then consider the composed function $\Phi_g = g\circ u_g$.

Theorem 2 is an immediate consequence on setting $f = \Phi_g$ in

PROPOSITION. *If $g \in H - N$, then $\Phi_g \in (H \cap N) - \mathrm{UBC}$.*

It is familiar that $H - N$ is non-empty; see, for example, [1] and [12, p. 296].

PROOF OF PROPOSITION. It follows from C. Pommerenke's theorem [10, Theorem 1] that $\Phi_g \in N$. It is easy to observe that $\Phi_g \in H$ because $g \in H$. Therefore $\Phi_g \in H \cap N$.

Since $g^{-1}(\{\alpha_1, \ldots, \alpha_n\})$ is a relatively closed subset of D whose capacity is zero, it follows from O. Frostman's result [3, p. 113] that $u_g \in U$.

Suppose that $\Phi_g \in \mathrm{UBC}$. Since $u_g \in U$, it follows from Theorem 3, (II), that $g \in \mathrm{UBC}$, so that $g \in N$ by [13, Theorem 3.1]. This contradiction shows that $\Phi_g \notin \mathrm{UBC}$.

5. A remark on UBC functions.

A characterization of a function of UBC in terms of the Nevanlinna characteristic [9, (6), p. 168] is appended.

Let f be meromorphic in D and let $T_N(r, f)$ be the Nevanlinna characteristic function of f, $0<r<1$. Let

$$T_N(f) = \lim_{r\to 1} T_N(r, f) .$$

Then, *f is a member of* UBC *if and only if*

$$\sup_{w\in D} T_N(Q_w \circ f_w) < \infty ,$$

where Q_w is defined in (2.8). We let $g = Q_w \circ f_w$. Then $T(g) = T(f_w)$. On the other hand, an obvious estimate yields that

$$|T(g) - T_N(g)| \leqq (1/2)\log 2$$

because $g(0)=0$, which completes the proof.

ADDED IN PROOF. In Section 4 we have only to choose three distinct point $\alpha_1, \alpha_2, \alpha_3$ on C^* without any further condition. Then the resulting function Φ_g, omitting $\alpha_1, \alpha_2, \alpha_3$ in D, must be normal in D without appealing to Pommerenke's theorem.

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