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ON LAVINE'S FORMULA FOR TIME-DELAY

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Abstract.

Lavine's formula gives a connection between time-delay and potential in scattering theory. A time-dependent proof is given for potentials $V=V_1+V_2$, $V_1(x)=O(|x|^{-1-\varepsilon})$, $x\cdot\nabla V_1(x)=O(|x|^{-1-\varepsilon})$, $V_2(x)=O(|x|^{-2-\varepsilon})$, as $|x|\to\infty$.

I. Introduction.

The present note is devoted to an essentially time-dependent proof of Lavine's formula for time-delay. It can be viewed as a continuation of [1]. To state the results, let $H_0=-\Delta$ and $H=H_0+V$ denote the free and full Hamiltonian, respectively, in $\mathscr{H}=L^2(\mathbf{R}^n)$, with $V(x)=O(|x|^{-\beta})$, $\beta>1$, as $|x|\to\infty$. For such short range potentials existence and completeness of the wave operators $W_\pm$ is well known. Let $S=W_+^*W_-$ denote the scattering operator, and $S=\{S(\lambda)\}$ its decomposition into scattering matrices in the spectral representation for H_0. The Eisenbud–Wigner time-delay operator is defined in this spectral representation by

$$T=\{-iS(\lambda)^*(d/d\lambda)S(\lambda)\},$$

see [1]. Let $D=(2i)^{-1}(x\cdot\nabla+\nabla\cdot x)$ denote the generator of dilations. Lavine's expression for time-delay is the right hand side of the following formula, which is the main result obtained here:

$$\langle f,TH_0g\rangle=\int_{-\infty}^{\infty}\langle e^{-isH}W_-f,(H-i/2[H,D])e^{-isH}W_-g\rangle\,ds \tag{1.1}$$

for a dense set of vectors $f,g\in\mathscr{H}$. Formally $H-i/2[H,D]=V+\frac{1}{2}x\cdot\nabla V$, so (1.1) establishes a connection between the potential and time-delay.

Here (1.1) is proved for potentials satisfying $V=V_1+V_2$, $V_1\in C^1(\mathbf{R}^n)$,

$$|V_1(x)|+|x\cdot\nabla V_1(x)|\leqq c(1+|x|)^{-1-\varepsilon},\quad \varepsilon>0,$$

and

$$V_2(x)=O(|x|^{-2-\varepsilon})$$

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as $|x| \to \infty$; V_2 can have some local singularities. The proof given here follows essentially the formal proof given in [5], see also [3]. In this proof technical results on $[H, D]$ and $[e^{-itH}, D]$ first given by Mourre [4] play an important role. The proof given here also shows that the alternative definition of time-delay [5] can be made rigorous, and agrees with the usual one.

The result in Lemma 2.7 might be of independent interest. Here it is shown that the four operators $W_{\pm}\varphi(H_0)$, $W_{\pm}^{*}\varphi(H)$, map the domain of D into itself. φ is a smooth function with compact support in $(0, \infty) \setminus \sigma_p(H)$. $\sigma_P(H)$ denotes the point spectrum of H.

In [2] Lavine proved that the right hand side of (1.1) equals an expression involving sojourn times. The result was proved in $L^2(\mathbf{R}^1)$ for $V = V_1$, V_1 satisfying the condition given above. The connection with T was not given in [2]. Combining (1.1) with the results in [1] a connection with sojourn times has been established.

Recently Martin [3] has given an extensive review of time-delay and related topics. See also [3] for applications of (1.1).

This note is a revision of a preliminary version, in which stronger conditions were imposed on V. Partly based on this preliminary version Narnhofer [6] has recently discussed (1.1) and related results, using a somewhat different approach, for essentially the same class of potentials as defined above.

II. The results.

Let $\mathscr{H} = L^2(\mathbf{R}^n)$ denote configuration space and $\mathscr{F}$ the Fourier transform. $\mathscr{D}(T)$ denotes the domain of an operator T. $\mathscr{B}(\mathscr{X}, \mathscr{Y})$ denotes the bounded operators from $\mathscr{X}$ to $\mathscr{Y}$. Let $\mathscr{S}'(\mathbf{R}^n)$ denote the tempered distributions. The weighted Sobolev space $H^{m,s} = H^{m,s}(\mathbf{R}^n)$ is given by

$$H^{m,s} = \{ f \in \mathscr{S}'(\mathbf{R}^n) \mid \|f\|_{m,s} = \|(1+x^2)^{s/2}(1-\Delta)^{m/2} f\|_{L^2} < \infty \} .$$

The free Hamiltonian is $H_0 = -\Delta$ with $\mathscr{D}(H_0) = H^{2,0}$. Let $L^2(S^{n-1})$ denote the square integrable functions on the unit sphere S^{n-1} in $\mathbf{R}^n$. The spectral representation for H_0 is given by

$$F \colon \mathscr{H} \to \mathscr{H}_s = L^2((0,\infty); L^2(S^{m-1}))$$

defined by

$$(Ff)(\lambda)(\omega) = 2^{-1/2} \lambda^{(n-2)/4} (\mathscr{F} f)(\lambda^{1/2}\omega) ,$$

$\lambda > 0$, $\omega \in S^{n-1}$. See [1] for further details.

The following short range assumption is imposed on the potential.

ASSUMPTION 2.1. Let V be multiplication by a real-valued function $V(x)$. Let $V(x) = V_1(x) + V_2(x)$, where V_1 is continuously differentiable with

$$|V_1(x)|+|x\cdot\nabla V_1(x)| \leqq c(1+|x|)^{-1-\varepsilon}$$

for some $\varepsilon>0$, $c>0$, and V_2 satisfies

$$V_2 : H^{2,0} \to H^{0,\beta}$$

is compact for some $\beta>2$.

$H=H_0+V$ is the operator sum. Let $\sigma_p(H)$ denote the point spectrum for H, and E the spectral measure for H. E_0 denotes the spectral measure for H_0. Under the above assumption existence and completeness of the wave operators

$$W_\pm = s-\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$$

is well known, see e.g. [8]. The scattering operator $S=W_+^*W_-$ is decomposable in $\mathscr{H}_s$, viz.

$$(FSf)(\lambda) = S(\lambda)(Ff)(\lambda), \qquad \lambda\in(0,\infty)\setminus\sigma_p(H)\,.$$

Usually this is written $S=\{S(\lambda)\}$. $S(\lambda)$ is the scattering matrix. If $V=V_1$, or $V=V_2$, the Eisenbud–Wigner time-delay operator was defined in [1] by

$$T = \{-iS(\lambda)^*(d/d\lambda)S(\lambda)\}\,.$$

The generalization to $V=V_1+V_2$ follows from the proof of Theorem 3.6 in [1]. Theorem 3.8 in [1] remains valid for this larger class of potentials.

Let $D=(2i)^{-1}\,(x\cdot\nabla+\nabla\cdot x)$ denote the generator of dilations. Note that $i[H_0,D]=2H_0$. $[V,D]$ can be defined on $\mathscr{D}(D)\cap\mathscr{D}(H_0)$ as a quadratic form. Assumption 2.1 implies that $[V,D]$ extends to a bounded operator, denoted $[V,D]^a$, from $H^{2,0}$ to $H^{-2,0}$. $[H,D]^a$ is defined similarly, and one has

$$H-i/2[H,D]^a = V-i/2[V,D]^a$$

as bounded operators from $H^{2,0}$ to $H^{-2,0}$. Sometimes it is convenient to use the notation

$$\tilde{V} = V-i/2[V,D]^a\,.$$

The main result of this note is the following theorem.

THEOREM 2.2. *Let V satisfy Assumption 2.1. Let $[a,b]\subset(0,\infty)\setminus\sigma_p(H)$ be a finite interval, and let $f,g\in E_0([a,b])\mathscr{H}$. Then one has*

$$(2.2)\qquad \langle f,TH_0g\rangle = \int_{-\infty}^{\infty}\langle e^{-isH}W_-f,(H-i/2[H,D]^a)e^{-isH}W_-g\rangle\,ds\,.$$

The proof is based on the following Lemmas.

LEMMA 2.3. *Let $[a,b] \subset (0,\infty) \setminus \sigma_p(H)$ be a finite interval. There exists $c>0$, depending only on a and b, such that*

$$\int_{-\infty}^{\infty} |\langle e^{-itH}E([a,b])f, Ve^{-itH}E([a,b])g\rangle|\, dt \leqq c\|f\|\,\|g\|$$

for all $f,g \in \mathscr{H}$. The same result is true with V replaced by $\tilde{V}$.

PROOF. Assumption 2.1 implies $V \in \mathscr{B}(H^{2,-\delta}, H^{0,\delta})$ and $\tilde{V} \in \mathscr{B}(H^{2,-\delta}, H^{-2,\delta})$ for some $\delta > 1/2$. The result now follows from well known local smoothness results due to Kato and Lavine, see e.g. [9].

LEMMA 2.4. (i) *$[D, e^{-itH}]$ extends to a bounded operator from $H^{2,0}$ to $H^{-2,0}$, which satisfies*

$$\|[D, e^{-itH}]^a\|_{\mathscr{B}(H^{2,0},H^{-2,0})} \leqq c(1+|t|)$$

for all $t \in \mathbf{R}$.

(ii) *Let $\varphi \in C_0^\infty(\mathbf{R}^n)$. Then $[D, \varphi(H)]$ extends to a bounded operator from $H^{-1,0}$ to $H^{1,0}$.*

PROOF. See [7; Lemma 7.4]. These results extend slightly results due to Mourre [4]. Note that the extension is needed here.

LEMMA 2.5. *Let $\varphi \in C_0^\infty((0,\infty) \setminus \sigma_p(H))$. Then one has for all $t \in \mathbf{R}$*

$$\|(D+i)^{-1}e^{-itH}\varphi(H)(D+i)^{-1}\|_{\mathscr{B}(\mathscr{H})} \leqq c(1+|t|)^{-1}\,. \tag{2.3}$$

PROOF. The following commutator is computed on $\mathscr{D}(D) \cap \mathscr{D}(H)$ as a quadratic form:

$$\begin{aligned}[D, e^{-itH}] &= e^{-itH}(e^{itH}De^{-itH} - D)\\ &= e^{-itH}\int_0^t e^{isH}i[H,D]e^{-isH}\, ds\\ &= e^{-itH}2tH + e^{-itH}\int_0^t e^{isH}(i[V,D] - 2V)e^{-isH}\, ds\,.\end{aligned}$$

This result now extends as an equality between bounded operators from $H^{2,0}$ to $H^{-2,0}$. Let $\varphi \in C_0^\infty((0,\infty) \setminus \sigma_p(H))$ be given, and let $\chi \in C_0^\infty((0,\infty) \setminus \sigma_p(H))$ be identically one on the support of φ. Let $\psi(\lambda) = \lambda^{-1}\chi(\lambda)$. Then $\psi \in C_0^\infty$, $\varphi(H) = H\psi(H)\varphi(H)$, and

$$
\begin{aligned}
e^{-itH}\varphi(H) &= \frac{1}{2t}\psi(H)e^{-itH}2tH\varphi(H) \\
&= \frac{1}{2t}\psi(H)[D,e^{-itH}]\varphi(H)+\frac{1}{t}\psi(H)\int_0^t e^{isH}\tilde{V}e^{-isH}\,ds\,\varphi(H)\,.
\end{aligned}
$$

Lemma 2.3 implies

$$
\left\|\psi(H)\int_0^t e^{isH}\tilde{V}e^{-isH}\,ds\,\varphi(H)\right\| \leqq c
$$

for all $t \in \mathbf{R}$. The result now follows using Lemma 2.4 (ii).

REMARK 2.6. (2.3) and related results were proved in [1] for $V=0$. The idea used in handling $[D,e^{-itH}]$ above is due to Mourre [4].

LEMMA 2.7. *Let $\varphi \in C_0^\infty((0,\infty)\setminus\sigma_p(H))$. The operators $[D,W_-\varphi(H_0)]$, $[D,W_+\varphi(H_0)]$, $[D,W_-^*\varphi(H)]$, and $[D,W_+^*\varphi(H)]$, defined as quadratic forms on $\mathscr{D}(D)\times\mathscr{D}(D)$, extend to bounded operators on $\mathscr{H}$. In particular, all four operators $W_\pm\varphi(H_0)$, $W_\pm^*\varphi(H)$ leave $\mathscr{D}(D)$ invariant.*

PROOF. Consider first $[D,W_+\varphi(H_0)]$. Given $\varphi \in C_0^\infty((0,\infty)\setminus\sigma_p(H))$ let $\psi \in C_0^\infty((0,\infty)\setminus\sigma_p(H))$ be identically one on the support of φ. The following computation is justified using the mollified generator of dilation, $D(\lambda)= i\lambda D(D+i\lambda)^{-1}$, see [4, 7]. This step is omitted here and in the sequel. One finds as bounded operators from $H^{2,0}$ to $H^{-2,0}$:

$$
\begin{aligned}
&[D,\psi(H)e^{itH}\varphi(H)e^{-itH_0}\psi(H_0)] \\
&\quad= \psi(H)[D,e^{itH}\varphi(H)e^{-itH_0}]\psi(H_0)+ \\
&\quad\quad+[D,\psi(H)]e^{itH}\varphi(H)e^{-itH_0}\psi(H_0)+ \\
&\quad\quad+\psi(H)e^{itH}\varphi(H)e^{-itH_0}[D,\psi(H_0)]\,.
\end{aligned}
\tag{2.4}
$$

The last two terms above are bounded operators on $\mathscr{H}$ (Lemma 2.4 (ii)), uniformly bounded in t.

In the following computation one uses

$$
\begin{aligned}
[-iH_0,D]^a &= -2H_0 \\
[-iH,D]^a &= -2H_0-i[V,D]^a = -2H+2V-i[V,D]^a
\end{aligned}
$$

valid as bounded operators from $H^{2,0}$ to $H^{-2,0}$.

$$\psi(H)[D, e^{itH}\varphi(H)e^{-itH_0}]\psi(H_0)$$

$$= \psi(H)e^{itH}\left\{\int_0^t e^{-isH}[-iH, D]e^{isH}\,ds\varphi(H) + [D, \varphi(H)] - \right.$$

$$\left. -\varphi(H)\int_0^t e^{-isH_0}[-iH_0, D]e^{isH_0}\,ds\right\}e^{-itH_0}\psi(H_0)$$

$$(2.5) \qquad = \psi(H)e^{itH}\left\{\int_0^t e^{-isH}(-2H+2V-i[V,D])e^{isH}\,ds\varphi(H) + \right.$$

$$\left. + [D, \varphi(H)] + 2t\varphi(H)H_0\right\}e^{-itH_0}\psi(H_0)$$

$$= -2t\varphi(H)e^{itH}Ve^{-itH_0}\psi(H_0) +$$

$$+ \psi(H)e^{itH}[D, \varphi(H)]e^{-itH_0}\psi(H_0) +$$

$$+ \psi(H)e^{itH}2\int_0^t e^{-isH}\tilde{V}e^{isH}\,ds\varphi(H)e^{-itH_0}\psi(H_0)\,.$$

The last two terms define bounded operators on $\mathscr{H}$, with norm uniformly bounded in t. The first term is treated as follows. Let $f, g \in \mathscr{D}(D)$ be given. Using the local H- and H_0-smoothness properties of V (cf. the proof of Lemma 2.3) one can find a sequence $t_n \to \infty$ as $n \to \infty$, such that

$$\lim_{n\to\infty} t_n\langle f, \varphi(H)e^{it_nH}Ve^{-it_nH_0}\psi(H_0)g\rangle = 0\,.$$

Since the remaining terms in (2.4) and (2.5) are bounded operators on $\mathscr{H}$, uniformly bounded in t, one finds, using the intertwining relation and $\varphi(H_0) = \psi(H_0)\varphi(H_0)\psi(H_0)$

$$|\langle Df, W_+\varphi(H_0)g\rangle - \langle f, W_+\varphi(H_0)Dg\rangle|$$

$$= \left|\lim_{n\to\infty}\langle f, [D, \psi(H)e^{it_nH}\varphi(H)e^{-it_nH_0}\psi(H_0)]g\rangle\right|$$

$$\leqq C\|f\|\cdot\|g\|$$

with $c > 0$ independent of f and g. This proves the result for $W_+\varphi(H_0)$. A similar proof holds for $W_-\varphi(H_0)$. Since the wave operators are asymptotically complete, $W_\pm^*\varphi(H) = s-\lim_{t\to\pm\infty} e^{itH_0}e^{-itH}\varphi(H)$, and an analogous proof can be given.

PROOF OF THEOREM 2.2. It suffices to prove (2.2) for a dense subset of $E_0([a,b])\mathscr{H}$, since both sides in (2.2) define bounded quadratic forms on this space. Let $f, g \in E_0([a,b])\mathscr{H}$ with Ff, Fg smooth with compact support in

$(0,\infty)\setminus\sigma_p(H)$, and in particular $f,g\in\mathscr{D}(D)$. As noted above [1; Theorem 3.8] remains valid under Assumption 2.1. Thus one has

$$\begin{aligned}\langle f, TH_0 g\rangle &= -\tfrac{1}{2}\langle f, S^*[D,S]g\rangle \\ &= -\tfrac{1}{2}(\langle Sf, DSg\rangle - \langle f, Dg\rangle)\,.\end{aligned}$$

A computation as quadratic form on $\mathscr{D}(D)\cap\mathscr{D}(H_0)$ yields

$$\begin{aligned}&\frac{d}{dt}(e^{itH}e^{-itH_0}De^{itH_0}e^{-itH})\\ &= -2e^{itH}\tilde{V}e^{-itH}+2\frac{d}{dt}(te^{itH}Ve^{-itH})\,.\end{aligned} \tag{2.6}$$

Write $W(t)=e^{itH}e^{-itH_0}$. Let $u=\varphi(H)\tilde{u}$, $v=\varphi(H)\tilde{v}$, $\tilde{u},\tilde{v}\in\mathscr{D}(D)$, $\varphi\in C_0^\infty((0,\infty)\setminus\sigma_p(H))$. Integrating (2.6) gives

$$\begin{aligned}\langle W(t)^*u, DW(t)^*v\rangle &= \langle u, Dv\rangle - 2\int_0^t \langle u, e^{isH}\tilde{V}e^{-isH}v\rangle\, ds + \\ &\quad + 2t\langle u, e^{itH}Ve^{-itH}v\rangle\,.\end{aligned}$$

The local H-smoothness of V implies the existence of a sequence $t_n\to\infty$ as $n\to\infty$ such that

$$\lim_{n\to\infty} t_n\langle u, e^{it_nH}Ve^{-it_nH}v\rangle = 0 \tag{2.7}$$

(see also Remark 2.8 (i)).

Lemma 2.3 now implies

$$\lim_{n\to\infty}\langle W(t_n)^*u, DW(t_n)^*v\rangle = \langle u, Dv\rangle - 2\int_0^\infty \langle u, e^{isH}\tilde{V}e^{-isH}v\rangle\, ds\,.$$

To conclude that the left hand side equals $\langle W_+^*u, DW_+^*v\rangle$ it suffices to show that $\|DW(t)^*v\|\leqq c$ for all $t\in\mathbf{R}$. To prove this estimate let $\psi\in C_0^\infty((0,\infty)\setminus\sigma_p(H))$ be identically one on the support of φ. Write $v=\varphi(H)(D+i)^{-1}v_1$.

$$\begin{aligned}\|DW(t)^*v\| &= \|De^{itH_0}\psi(H)e^{-itH}\varphi(H)(D+i)^{-1}v_1\|\\ &\leqq \|D\varphi(H)(D+i)^{-1}v_1\| + \|[D, e^{itH_0}\psi(H)e^{-itH}]\varphi(H)(D+i)^{-1}v_1\|\,.\end{aligned}$$

As in (2.5) above one finds in $\mathscr{B}(\mathscr{H})$

$$[D, e^{itH_0}\psi(H)e^{-itH}]\varphi(H)$$

$$= e^{itH_0}\{[D,\psi(H)] + 2tV\psi(H) - \tag{2.8}$$

$$-2\psi(H)\int_0^t e^{-isH}\tilde{V}e^{isH}\,ds\}\varphi(H)e^{-itH}.$$

Lemma 2.5 and Assumption 2.1 imply

$$\|Ve^{-itH}\varphi(H)(D+i)^{-1}v_1\| \leqq c(1+|t|)^{-1}.$$

The estimate $\|DW(t)^*v\| \leqq c$ now follows from Lemma 2.3.

Thus one has

$$\langle W_+^*u, DW_+^*v\rangle = \langle u, Dv\rangle - 2\int_0^\infty \langle u, e^{isH}\tilde{V}e^{-isH}v\rangle\,ds$$

for $u=\varphi(H)\tilde{u}$, $v=\varphi(H)\tilde{v}$, $\tilde{u},\tilde{v}\in\mathscr{D}(D)$. Similarly, one finds

$$\langle W_-^*u, DW_-^*v\rangle = \langle u, Dv\rangle + 2\int_{-\infty}^0 \langle u, e^{isH}\tilde{V}e^{-isH}v\rangle\,ds$$

and thus

$$\langle W_+^*u, DW_+^*v\rangle - \langle W_-^*u, DW_-^*v\rangle$$

$$= -2\int_{-\infty}^\infty \langle u, e^{isH}\tilde{V}e^{-isH}v\rangle\,ds.$$

Take now $u=W_-\varphi(H_0)f$, $v=W_-\varphi(H_0)g$, $f,g\in\mathscr{D}(D)$. Then $W_-^*u=\varphi(H_0)f$, $W_+^*u=S\varphi(H_0)f$, etc. and the equation (2.2) has been proved for the dense set

$$\{\varphi(H_0)f \mid f\in\mathscr{D}(D),\ \varphi\in C_0^\infty((a,b))\}.$$

REMARK 2.8 (i) Note that (2.7) can be improved, since only a dense set of u, v is considered. Lemma 2.5 and interpolation imply

$$\|(1+x^2)^{-\delta/2}e^{-itH}\varphi(H)(D+i)^{-1}\| \leqq c(1+|t|)^{-\delta}, \quad 0\leqq\delta\leqq 1.$$

Under assumption 2.1, $V(x)=O(|x|^{-1-\varepsilon})$ as $|x|\to\infty$, so one has

$$|\langle u, e^{itH}Ve^{-itH}v\rangle|$$

$$\leqq \|(1+x^2)^{-\varepsilon/2}e^{-itH}u\|\cdot\|(1+x^2)^{\varepsilon/2}Ve^{-itH}v\|$$

$$\leqq c(1+|t|)^{-1-\varepsilon}$$

for $u=\varphi(H)\tilde{u}$, $v=\varphi(H)\tilde{v}$, $\tilde{u},\tilde{v}\in\mathscr{D}(D)$.

(ii) The computation (2.8) gives a simpler proof of the fact that $W_+^*\varphi(H)$ leaves $\mathscr{D}(D)$ invariant, but the result in Lemma 2.7 is stronger. Note that one has $\|DW(t)^*\varphi(H)(D+i)^{-1}\| \leqq c$ for all $t \in \mathbf{R}$, but only $\|D\psi(H)W(t)\varphi(H_0)(D+i)^{-1}\| \leqq c$ for all $t \in \mathbf{R}$, cf. (2.5).

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