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PROJECTIONS AND REFLECTIONS OF GENERIC SURFACES IN R^3

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In this article we describe a technique for investigating the differential geometry of generic surfaces in Euclidean space R^3. More precisely we are concerned with the study of differential geometric properties of embedded surfaces which hold for a residual set of such embeddings. For a general reference on generic geometry the reader is recommended Wall's excellent survey article [10]. Indeed this technique is related to Theorem D of Wall's paper.

The method differs from the, by now, standard approaches described in [10], where each different geometrical study requires a different transversality theorem. Here we prove *one* transversality result, which hopefully covers every case for which the standard methods work. The price one pays for this is that each new situation requires some special computations. In a sense this makes our approach rather inelegant, but it does seem quite a useful method for investigating new geometric phenomena. (We have used it for considering the geometry of wavefront evolution and generic isotopies of curves, where the standard methods do not appear to work.) We hope its lack of sophistication is another attraction, with the methods employed easier to understand for the non expert than the standard ones.

In section 1 we describe the method and prove the transversality theorem. We illustrate its use in section 2 by considering the contact of surfaces with spheres, obtaining a result here which one cannot obtain using the standard methods. In section 3 we show how the technique can be used to obtain a classification of the local normal forms of projections of generic surfaces onto planes. The resulting list was first obtained by Gaffney and Ruas (unpublished) and later by Arnold [1] using entirely different methods. Our derivation is along the lines of Gaffney and Ruas original method. It has the attraction however of being, at least initially, elementary. Using our technique and explicit changes of co-ordinates we can produce local forms up to some order. We then need to use some deep results of du Plessis to deduce that these are in fact smooth normal forms. (For the fascinating relationship of these projection

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with classical differential geometry and a wealth of interesting results the reader is referred to Gaffney's article in [5], and a forthcoming paper by Gaffney and Ruas. We must stress that our purpose here is merely to give a fairly rapid derivation of a list of normal forms.)

Finally in section 4 we study the infinitesimal reflective symmetry of surface by considering a natural family of fold maps on R^3 (those obtained from the fold $(x, y, z) \rightarrow (x, y^2, z)$ by varying the reflecting plane $y=0$) and their restriction to surfaces in R^3. In this section again we content ourselves with obtaining a list of local normal forms. Pictures of the normal formas and a discussion of the associated geometry will follow in a later paper. A more extensive list of normal forms, containing ours, has been obtained by Mond in his Liverpool Ph. D. thesis, in connection with the problem of projecting surfaces in R^4 into R^3.

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1. Monge-Taylor Expansions.

In the study of the generic geometry of smooth submanifolds of Euclidean space the most difficult and most interesting ingredient is usually the local geometry. The geometry of a submanifold at some point will be determined, generically, by the infinitesimal information concerning the embedding of the manifold stored at that point. Moreover one will only need to consider the infinitesimal information up to some fixed order. (All of this will be made more precise later on.)

How then can be store this information? Let $M \subset R^3$ be a compact surface. At each point $p \in M$ we can choose mutually perpendicular co-ordinate axes (x, y, z) with the z-axis in the normal direction at p. Locally M can be written as the graph of a function $z = f(x, y)$, i.e. in Monge form. Infinitesimal information can then be deduced from the Taylor expansion of f (the Monge-Taylor expansion of M at p). Of course there is a problem here: the choice of co-ordinates is not unique. Moreover usually it is not possible to choose a set of co-ordinates for each point $p \in M$ which vary smoothly with p. These problems are easily circumvented, however. Our underlying idea is the following.

In considering the geometry of M the conditions for the point p to have some geometric property will impose conditions on the coefficients appearing in the Monge-Taylor expansion. We have a 2-parameter family of such expansions, so clearly we can expect any property involving $\leqq 2$ conditions to occur generically, while any property involving $\geqq 3$ conditions one would hope would not so occur. The rest of this section is devoted to proving this assertion.

Give $\mathbf{R}^3$ an orientation and let $M \subset \mathbf{R}^3$ be a smooth compact surface, with outward normal vector field n. Cover M with finitely may open sets U_i each possessing a smooth unit vector field v_i. On each U_i there is a third unit vector field w_i with the property that $v_i(p)$, $w_i(p)$, $n(p)$ form a positively oriented triple at each $p \in U_i$ (this determines w_i uniquely).

If V_k denotes the vector space of polynomials in x and y of degree $\leqq k$ and $\geqq 2$ we have smooth maps θ_i: $U_i \to V_k$ defined as follows. At each point $p \in U_i$ let the three co-ordinate axes x, y, z be chosen to coincide with the oriented lines determined by $v_i(p)$, $w_i(p)$, $n(p)$ respectively. Writing M near p as $z = f_p(x, y)$ we can associate to p the Taylor expansion of f_p truncated to degree k, which we write $j^k f_p \in V_k$. The non-uniqueness of the polynomial $j^k f_p$ is clearly due to our initial choice of vector field v_i, indeed $j^k f_p$ obviously also depends on the subscript i, we obtain different polynomials for points p in distinct U_i's. However the different choices of x and y axes at such a point p are related by a change of co-ordinates via an element of the special orthogonal group SO (2). Now this group SO (2) acts on the space of polynomials V_k via its variables. Suppose that $X \subset V_k$ is an SO (2) invariant submanifold. Clearly the condition that θ_i: $U_i \to V_k$ is transverse to X at p is independent of the particular U_i chosen; indeed it is independent of the covering chosen for M.

THEOREM 1. *Let $X \subset V_k$ be an* SO (2) *invariant submanifold. For a dense set of embeddings of M in $\mathbf{R}^3$ the mappings θ_i: $U_i \to V_k$ are transverse to X.*

PROOF. Consider the space of all polynomial mappings $\mathbf{R}^3 \to \mathbf{R}^3$ of degree d, P_d. The identity mapping I is in P_d and given an open bounded region containing M we may choose a neighbourhood U of I consisting of mappings which embed this region, and hence M. Since the U_i are embedded by mappings in U the vector fields v_i are mapped to non zero vector fields which can be normalized and used, together with the new outward normal vector field, to determine new sets of co-ordinate axes. Thus we have a smooth family of maps

$$\tilde{\theta}_i: U_i \times U \to V_k$$

parametrized by the polynomial maps $\psi \in U$, defined by taking the truncated Monge-Taylor expansion of $\psi(M)$ at the points $p \in \psi(U_i)$ with respect to the given co-ordinate axes. We claim that if $d \geqq k$ then the maps $\tilde{\theta}_i$ are submersions. First consider points of the form $(p, 0) \in U_i \times U$. At p write M locally as above $z = f(x, y)$. The deformation of M induced by the polynomial mappings $\Phi_s(x, y, z) = (x, y, z + s\varphi(x, y))$, where s is small and $2 \leqq \deg \varphi \leqq k$ clearly, differentiating with respect to s, gives the tangent vector $\varphi(x,y)$ at the point $j^k f_p \in V_k$ and the result follows. (Using the compactness of M it is now clear that by possibly shrinking the U_i, while still retaining a covering of M, and shrinking U the assertion then follows.) The general case, where we consider points of the form (p, ψ) with ψ non zero follows in the same way. Indeed if for our deformation we chose $\Phi_s \circ \psi$ exactly the same argument gives the result. (Unfortunately $\Phi_s \circ \psi$ need not give a path in P^d; instead we consider $j^d(\Phi_s \circ \psi)$, one easily checks this gives the same vector $\varphi(x,y)$ as before.)

Now given any submanifold X as above by Thom's basic transversality lemma ([8, p. 53]) for almost all P in U (in the sense of Lebesgue measure) $P(M)$ has the required property of yielding maps transverse to X.

We shall in practice always use the theorem above in the case when codim $X > 2$, so that the maps θ_i miss X. (This then only uses, as its crucial technical ingredient, the trivial version of Sard's theorem). In fact the sets X arising in practice are algebraic, so that we need to stratify X into smooth manifolds X_α. Note that if X is closed and made up of smooth manifolds of codimension >2 the set of embeddings of M yielding mappings θ_i: $U_i \to V_k$ transverse to X is clearly open. (This needs the compactness of M.) More generally the same is true, with no codimension restriction on the X_α, if X is a closed set with its constituent manifolds yielding a *Whitney* (A) *regular stratification.* (See [10]. Such technicalities are rarely needed in this paper.) One rather technical point we shall need however: often the subset $X \subset V_k$ are most easily obtained in some product space $V_k \times \mathbf{R}^N$ via projection onto V_k. So although the subsets of $V_k \times \mathbf{R}^N$ will be algebraic we can only deduce that the resulting sets in V_k are semi-algebraic. The dimension of a semi-algebraic set can only drop under such a projection so in general we shall be interested in showing that certain "bad" sets in $V_k \times \mathbf{R}^N$ have codimension $> N+2$. We also note that we can take the closure of a semi-algebraic set without affecting its dimension. Thus the generic embeddings of M, consisting of those whose maps θ_i miss some subset X of codimension >2, will always be open as well as dense. (For a discussion of semi-algebraic sets and proofs of the assertions made above see [7, part I].)

2. Generic Umbilics.

In this section we prove one of the results stated by Porteous in [9] concerning generic umbilics on surfaces in $\mathbf{R}^3$. The method used is applicable in many situations where we wish to study the geometry of various subsets of the surface itself, a weak point of the traditional approach.

The one preliminary result we shall need computes the image of the derivative of the Monge-Taylor expansion.

PROPOSITION 2. *Let $\theta_i\colon U_i \to V^k$ be as in Theorem 1 and let $p \in U_i$, with M written locally as $z=f(x,y)$ with respect to the distinguished co-ordinate axes at p. The image of $d\theta_i(p)$ is spanned by*

$$j^k(-f_{xx}(0,0)x-f_{xy}(0,0)y+f_x(x,y)-f_x(x,y)f(x,y)f_{xx}(0,0) \\ -f_y(x,y)f(x,y)f_{xy}(0,0))$$

and

$$j^k(-f_{xy}(0,0)x-f_{yy}(0,0)y+f_y(x,y)-f_x(x,y)f_x(x,y)f(x,y)f_{xy}(0,0) \\ -f_y(x,y)f(x,y)f_{yy}(0,0))\,.$$

(We are using the standard notation here: $f_x=\dfrac{\partial f}{\partial x}$, $f_{xy}=\dfrac{\partial^2 f}{\partial x \partial y}$ etc.)

PROOF. As one might guess from the appearance of the statement, this is the result of a straightforward but rather brutal computation which we omit.

We shall assume that the reader is familiar with the results concerning the distance squared functions given in [10]. (These can also be obtained using our methods.) We also assume, in this section, that the reader knows a little about stratifications, in particular about the use of transversals when working with a stratification invariant under a group action. (See [16] and [7, part I]).

We now set about proving the required result concerning the umbilics. We will *not* give full details since the result has been in print for some time, if not formally proved. Indeed the method is perhaps of greater interest than the result itself.

Write M in Monge form at p, $z=f(x,y)$ with

$$f(x,y) \;=\; f_2(x,y)+f_3(x,y)+g(x,y)\,,$$

where f_j is a homogeneous polynomial of degree j and g vanishes at 0 to order 3. The distance squared function F_a from $a \in \mathbb{R}^3$ will have a singular point at p if and only if a is on the normal to M at p. Thus if $a=(0,0,\varrho)$ the function

$$F_a(x,y) \;=\; \|(x,y,f(x,y))-a\|^2 \;=\; \varrho^2+x^2+y^2-2\varrho f_2(x,y)-2\varrho f_3(x,y)+G(x,y)$$

where G vanishes at 0 to order 3. Thus F_a has a corank 2 singularity at p if and only if $f_2(x,y)=(2\varrho)^{-1}(x^2+y^2)$, so $F_a(x,y)=\varrho^2-2\varrho f_3+G$. This singularity is of type D_4 if in addition the cubic form f_3 has 3 distinct (real or complex) roots. We have an elliptic umbilic (denoted D_4^-) if f_3 has 3 real roots and a hyperbolic umbilic (denoted by D_4^+) if f_3 has 1 real root. We want to describe the rib set (that is the set of points on the surface where the distance squared function has an A_3 singularity) near the umbilics.

PROPOSITION 3. *For an open dense set of the space of smooth embeddings the rib lines through the umbilic p have local models* $xy(x-y)=0$ *(for* D_4^-*) and* $x=0$ *(for* D_4^+*). (Here by local model we mean that there is a local diffeomorphism* $\mathbf{R}^2, 0 \to M, p$ *which takes the given lines to the ribs.)*

PROOF. In the space V_3 we want to consider the way the D_4 and the A_3 sets fit together. As previously described there is an action of SO (2) on V_3 via the corresponding linear action on the variables. There is also an action by the group of positive reals under multiplication which comes from dilatations of the ambient space $\mathbf{R}^3$ (that is $t.(x,y,z)=(tx,ty,tz)$). Clearly this action will leave all of the relevant subsets of V_3 invariant. The effect on elements of V_3 is clearly that given by $t(f_2, f_3)=(tf_2, t^2 f_3)$. We consider the case of the elliptic umbilic first. Using the group actions above we need only consider points of V_3 of the form $(\sigma(x^2+y^2), x(x+\alpha y)(x+\beta y))$ with $\alpha\beta(\alpha-\beta)\sigma \neq 0$. A transversal to the orbit of this point is given by

$$\big(\sigma(x^2+y^2)+Ax^2+2Bxy+Cy^2, x(x+(\alpha+a)y)(x+(\beta+b)y)\big) = (F_2, F_3)$$

If the point given by (F_2, F_3) is to be of type $A_{\geqq 3}$, then for some ϱ we must have $(x^2+y^2)-2\varrho F_2$ a repeated square $\pm L^2$, with L a factor of F_3. In the case when L corresponds to the first factor of F_3 we want

$$x^2(1-2\varrho(A+\sigma))+xy(-4\varrho B)+y^2(1-2\varrho(C+\sigma))$$

to be a multiple of x^2, so clearly $B=0$ (and $\varrho=(2(C+\sigma))^{-1}$). Similarly for the other two factors one obtains the conditions

$$B(1-(\alpha+a)^2)-(\alpha+a)(A-C) = 0$$
$$B(1-(\beta+b)^2)-(\beta+b)(A-C) = 0 .$$

Thus the A_3 set consists of 3 smooth manifolds intersecting in the D_4 set (which is given by $B=0$, $A=C$). We now seek conditions under which these components of the A_3 set intersect pair-wise transversally. These are easily found to be $((\alpha+a)(\beta+b)+1)(\alpha+a)(\beta+b) \neq 0$ which in invariant form means that the roots of the cubic F_3 are *not* orthogonal. The umbilics in V_3 with orthogonal roots have co-dimension $\geqq 3$ and so can generically be avoided. For the remainder, we know that the Monge-Taylor map vill be transverse to the D_4 stratum generically, and a transversal will meet the A_3 stratum in three smooth curves through the umbilic which have distinct tangent at the umbilic, whence the result. The hyperbolic umbilic is of course much the same but considerably easier. On easily checks that the closure of the A_3 set is smooth and contains D_4 as a codimension 1 submanifold. Transversality to the D_4 stratum now ensures the normal form above.

We now follow [9] and obtain a picture of the set of umbilics in V_3.

Recall that the umbilics are those points with Monge normal form f_2+f_3+g with $f_2=\sigma(x^2+y^2)$ for some σ. The cubic form f_3 lies in the vector space of such forms V. Using the action of the positive reals we may consider the unit sphere $S^3\subset V$ and the induced action of SO (2) on this sphere. Now V can also be thought of as the product $\mathbf{C}\times\mathbf{C}$ via the map $(v,w)\mapsto \operatorname{Re}(vz^3+wz^2\bar{z})$, where z is the complex variable $z=x+iy$, Re denotes real part and the bar complex comjugation. The action of SO (2) corresponds to that of the circle group $e^{i\theta}.(v,w)=(ve^{3i\theta},we^{i\theta})$. According to [11] we can obtain a picture of the relevant sets in S^3 by working with the solid torus $|v|=1$. (We actually lose the SO (2) orbit of $x(x^2+y^2)$ which corresponds to a transverse hyperbolic umbilic and hence is of no interest anyway.) In turn it is then enough to consider the picture in the plane $v=1$, since that in the solid torus can obtained from this via the SO (2) action.

PROPOSITION 4. *Let $p\in M$ be an umbilic with the cubic part of its Monge normal form at p, $f_3=\operatorname{Re}(z^3+wz^2\bar{z})$ and $a\in\mathbf{R}^3$ its unique centre of curvature.*

(a) *The umbilic is* not *of type D_4 if and only if $w=2e^{i\theta}+e^{-2i\theta}$ for some θ.*

(b) *The family of distance squared functions fails to versally unfold the D_4 singularity at $(p,a)\in M\times\mathbf{R}^3$ if and only if $|w|=3$ (and $w\neq 2e^{i\theta}+e^{-2i\theta}$ of course). This is precisely the condition that the Monge-Taylor map is* not *transverse to the D_4 set in V_3.*

(c) *The cubic form f_3 has orthogonal roots if and only if $|w|=1$.*

PROOF. These condition are easily found: (a) by forming the resultant of f_3 and $\partial f_3/\partial x$, (b) by writing down the condition that (x^2+y^2), $\partial f_3/\partial x$, $\partial f_3/\partial y$ are linearly dependent quadratic forms, and (c) by substituting $z=re^{i\theta}$, $se^{i(\theta+\pi/2)}$ in f_3. One easily checks that the Monge-Taylor map fails to be transverse to the D_4 set if and only if $|w|=3$ using Proposition 2.

Consequently we obtain Porteous' picture in Diagram 1.

In regions 1 and 2 we have an elliptic umbilic, in 3 and 4 a hyperbolic umbilic. Using the group action it is clear that the regions 2 are all connected (see the pictures of the umbilic bracelet in [11]). Geometrically one way of distinguishing umbilics in regions 1 and 2 is a follows. An A_3 point on M is said to be of type A_3^+ if the distance squared function is right equivalent to $+x^2+y^3$ and of type A_3^- if it is equivalent to $-x^2+y^3$. Using a broken line for ribs of type A_3^-, a solid line for those of type A_3^+ the configuration of the rib lines near an umbilic are as pictured in Diagram 2.

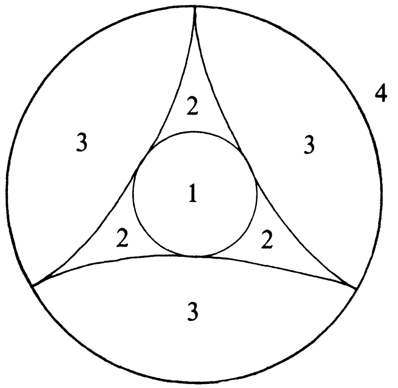

Diagram 1.

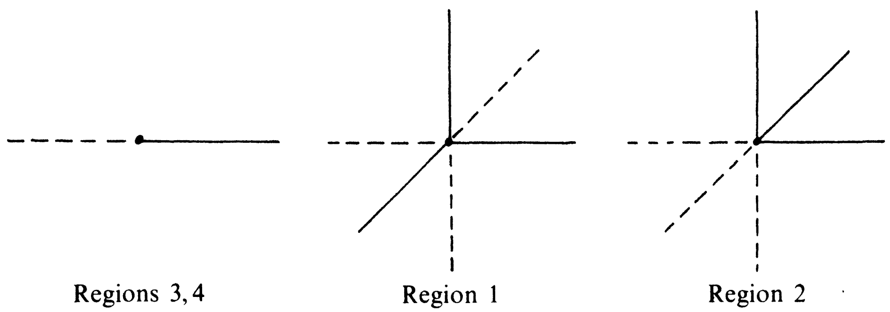

Diagram 2.

One final result.

PROPOSITION 5. *For a generic surface* **M** *the rib lines self intersect transversally at non umbilics.*

PROOF. We use the same notation as above. Consider the set of expansions f_2+f_3 in V_3 corresponding to points of M whose spheres of curvature both have A_3 contact with M.

Using the action of SO (2) we can reduce f_2 to $\frac{1}{2}(ax^2+by^2)$, $a \neq b$ and the condition for an $A_{\geqq 3}$ is

$$\varrho a = 1,\ f_3(1,0) = 0 \quad \text{or} \quad \varrho b = 1,\ f_3(0,1) = 0\,.$$

So we have a self intersection of ribs if $f_3(1,0)=f_3(0,1)=0$ in this case. (In *invariant* terms if $f_3=0$ has two orthogonal roots.) Working in a transversal $(\frac{1}{2}(a+A)x^2+(b+B)y^2)$, $f_3+F_3)$, where F_3 is a general cubic the set of forms yielding $A_{\geqq 3}$ points is given by $(f_3+F_3)(1,0)=0$ or $(f_3+F_3)(0,1)=0$. These are transversally intersecting hyperplanes. Since the Monge-Taylor map generically will be transverse to the double rib stratum, corresponding to the intersection of these two hyperplanes, generically the rib lines self intersect transversally at non umbilics.

This result was first obtained in [2]. The method given here is considerably simpler, and circumvents the problems involved in proving the required multi-transversality results, say for hypersurfaces in $\mathbf{R}^4$, discussed in [2]. The Monge-Taylor method reduces these questions of self intersections of geometric subjects of the manifold to local problems again. Indeed it seems well suited to describing features on the surface itself. The usual techniques only describe the geometry of sets (say the focal set) in some auxiliary space.

3.

In this section we classify all orthogonal projections of generic surfaces onto planes which, as remarked in the introduction, was first done by Gaffney and Ruas and later by Arnold.

Given a linear surjection $\Pi\colon \mathbf{R}^3 \to \mathbf{R}^2$ its restriction to any surface $M\subset\mathbf{R}^3$, up to changes of co-ordinates on M and $\mathbf{R}^2$, clearly only depends on the kernel of Π. As usual we write our surface $M\subset\mathbf{R}^3$ locally, at 0 say, as $z=f(x,y)$. Any orthogonal projection Π whose kernel vector has first component zero clearly gives a local diffeomorphism $M,0 \to \mathbf{R}^2,0$. Consequently we can conveniently replace the natural two parameter family of orthogonal projections by the two parameter family of linear surjections $\Pi_{(u,v)}\colon \mathbf{R}^3 \to \mathbf{R}^2$ given by $\Pi_{(u,v)}(x,y)=(ux-y,vx-z)$. The restriction of these surjections to M yields a two parameter family of map germs $M,0 \to \mathbf{R}^2,0$, also denoted by $\Pi_{(u,v)}$, with $\Pi_{(u,v)}(x,y)=(ux-y,vx-f(x,y))$.

Our aim is to classify, up to right left equivalence (local changes of co-ordinates on M at 0 and $\mathbf{R}^2$ at 0) the types of germ which appear in this two parameter family for a generic surface. The classification is done initially, in Part A, using k-jets of the form $\Pi_{(u,v)}$ at 0. The conditions for this projection to have a k-jet of a given type will be algebraic in the u,v co-ordinates and Monge-Taylor coefficients of M at 0, lying in V_k. If these conditions determine a (semi-algebraic) set of codimension >4, then projecting to V_k one obtains a set of codimension >2. By Theorem 1.1 this can be generically avoided. Any jet involving greater than 4 conditions will be marked with a sharp ✻, and these, together with any higher jet with such a truncation, can consequently be discarded.

On the other hand certain jets, marked with a star $*$, will be normal forms. That is, any projection having this jet is smoothly equivalent to the germ determined by the jet itself.

The proof that these jets are indeed normal forms is the only difficult step in the classification, done in Part B, and depends entirely on work of A. du Plessis. Again there is clearly no need to consider higher jets whose truncation is a normal form. Thus as we increase the order of the jets, from k to $k+1$ say, we need only consider $(k+1)$-jets whose truncation to degree k is one of the *unmarked* k-jets.

Now to work. First note that setting $-y'=ux-y$ our projection becomes $(x,y)\to(-y',vx-f(x,ux+y'))$ which dropping primes is clearly equivalent to $(x,y)\to(y,vx-f(x,ux+y))$. We write $f(x,y)=f_2(x,y)+f_3(x,y)+\ldots$, where f_j is homogeneous degree j. More explicitly we shall set $f_2(x,y)=\frac{1}{2}(\varkappa x^2+\tau y^2)$, $f_3=a_0x^3+\ldots+a_3y^3$, $f_4=b_0x^4+\ldots+b_4y^4$, etc.

Part A.

1. *The* 1-*jet*. Clearly if $v\neq 0$ we can reduce to (y,x) $(*)$. Remains to consider the case $v=0$, with 1-jet $(y,0)$.

2. *The* 2-*jet*. The 2-jet is $(y,\frac{1}{2}(\varkappa+\tau u^2)x^2+2\tau uxy+\tau y^2))$. By a change of coordinates in the target we can always lose powers of y in the second term, and from now on we shall do so without comment. This leaves three possibilities

(a) 2-jet is equivalent to $(\sim)$ $(y,0)$ iff $\tau u=\varkappa+\tau u^2=0$ so either $u=\varkappa=0$ or $\varkappa=\tau=0$. This latter is non generic, the former is next considered in 3.I. From now on that part of the analysis below, which further investigates a given unmarked jet, will be indicated by a go to statement.

(b) 2-jet $\sim(y,xy)$ iff $\varkappa+\tau u^2=0$, $\tau u\neq 0$; go to 3.II.

(c) 2-jet $\sim(y,x^2)$ iff $\varkappa+\tau u^2\neq 0$ $(*)$.

3. *The* 3-*jet*. We have two separate cases depending on the 2-jet.

3.I. 2-jet is $(y,0)$ means the 3 -jet is $(y,f_3(x,y))$ since $u=0$, that is $(y,a_0x^3+a_1x^2y+a_2xy^2)$.

(a) If $a_0=a_1=a_2=0$, 3-jet $\sim(y,0)$ $(\circledast)$.

(b) If $a_0=a_1=0$, $a_2\neq 0$, 3-jet $\sim(y,xy^2)$ $(\circledast)$.

(c) If $a_0=0$, $a_1\neq 0$, 3-jet $\sim(y,x^2y)$; go to 4.II.

(d) If $a_0\neq 0$, $3a_0a_2=a_1^2$, 3-jet $\sim(y,x^3)$; go to 4.I.

(e) If $a_0\neq 0$, $3a_0a_2\neq a_1^2$, 3-jet $\sim(y,x^3\pm xy^2)$ $(*)$.

3.II. 2-jet $\sim (y, xy)$, so the 3-jet is $(y, 2\tau uxy + f_3(x, ux+y))$ with $\varkappa + \tau u^2 = 0$, $\tau u \neq 0$. But by changes of co-ordinates of the form $x' = x + p(x, y)$, where degree $p \geqq 2$ we can reduce d-jet $(y, xy + g(x, y))$ where degree $g \geqq 3$ to (y, xy) or $(y, xy + x^d)$ for some $3 \leqq d \leqq k$. (We say this is a change of co-ordinates of type α.) So

(a) $f_3(1, u) = 0$, 3-jet $\sim (y, xy)$; go to 4.III.
(b) $f_3(1, u) \neq 0$, 3-jet $\sim (y, xy + x^3)$ (*).

4. *The* 4-*jet*. We have

4.I. 3-jet $\sim (y, x^3)$ so 4-jet is $(y, a_0x^3 + a_1x^2y + (a_1^2/3a_0)xy^2) + f_4(x, y))$ $\sim (y, a_0x^3 + f_4(x - (a_1/3a_0)y, y))$. A change of co-ordinates of type α above eliminates x^4, x^3y, x^2y^2 terms.

(a) If $\partial f_4/\partial x(-a, 3a_0) = 0$, 4-jet $\sim (y, x^3)$ (⊛).
(b) $\partial f_4/\partial x(-a_1, 3a_0) \neq 0$, 4-jet $\sim (y, x^3 + xy^3)$ (*).

4.II. 3-jet $\sim (y, x^2y)$ so 4-jet is $(y, a_1x^2y + a_2xy^2 + f_4(x, y))$ $(a_1 \neq 0)$ which $\sim (y, a_1x^2y + f_4(x - (a_2/2a_1)y, y))$. A change of co-ordinates of type α now gets rid of x^3y, x^2y^2, xy^3 terms. This leaves $(y, a_1x^2y + b_0x^4)$.

(a) If $b_0 = 0$, 4-jet $\sim (y, x^2y)$ (⊛).
(b) If $b_0 \neq 0$, 4-jet $\sim (y, x^2y + x^4)$; go to 5.II.

4.III. 3-jet $\sim (y, xy)$ so 4-jet is $(y, 2\tau uxy + f_3(x, ux+y) + f_4(x, ux+y))$, with $f_3(1, u) = 0$.

(a) If $f_4(1, u) = 0$, 4-jet $\sim (y, xy)$; goto 5.I.
(b) If $f_4(1, u) \neq 0$, 4-jet $\sim (y, xy + x^4)$ (*).

5. *The* 5-*jet*. For the 5-jet we need only consider the cases when the 4-jet $\sim (y, xy)$ or $(y, x^2y + x^4)$.

5.I. 4-jet $\sim (y, xy)$.

(a) If $f_5(1, u) = 0$, 5-jet $\sim (y, xy)$ (⊛).
(b) If $f_5(1, u) \neq 0$, 5-jet $\sim (y, xy + x^5)$; goto 6.

5.II. 4-jet $\sim (y, x^2y + x^4)$ so 5-jet is $(y, a_1x^2y + a_2xy^2 + f_4(x, y) + f_5(x, y))$ with $a_1 \neq 0$, which $\sim (y, a_1x^2y + f_4(x - (a_2/2a_1)y, y) + f_5(x - (a_2/2a_1)y, y))$. Again it is easy to reduce to the case $(y, a_1x^2y + b_0x^4 + c_0x^5)$, with $b_0 \neq 0$.

(a) If $c_0 = 0$, $\sim (y, x^2y + x^4)$ (⊛).
(b) If $c_0 \neq 0$, $\sim (y, x^2y + x^4 + x^5)$ (*).

6. *The* 6-*jet*. For the 6-jet we now need only consider the case when the 5-jet

$\sim (y, xy+x^5)$. One easily checks that the process which leads to this 5-jet does not introduce any new powers of x. So the x^6 coefficient is $f_6(1,u)$.

(a) If $f_6(1,u)=0$, 6-jet $\sim (y, xy+x^5)$.
(b) If $f_6(1,u)\neq 0$, 6-jet $\sim (y, xy+x^5 \pm x^6)$.

In fact these jets are actually equivalent, as the following sequence of explicit (and self explanatory) transformations shows.

$$\begin{aligned}
&(y, xy+x^5) \sim (y+x^5, xy+x^5+x^6) \sim (y-xy-x^6, xy+x^5+x^6)\\
&\sim (y, x(y+x^6)(1-x)^{-1}+x^5+x^6) \sim (y, x(1-x)^{-1}y+x^5+x^6)\\
&\sim (y, xy+(x(1+x)^{-1})^5+(x(1+x^{-1}))^6) \sim (y, xy+x^5-4x^6)\\
&\sim (y, xy+x^5-x^6)\,.
\end{aligned}$$

(To get $(y, xy+x^5+x^6)$ replace the first component of the second form by $y-x^5$.) Go to 7.

7. *The 7-jet*. For the 7-jet we need only consider the case when the 6-jet $\sim (y, xy+x^5)$. Again one easily gets rid of any homogeneous terms of degree 7 divisible by y, and the condition that the x^7 term is non zero can be written down in terms of the co-efficients of the normal form (not quite so easily now because of the (inverse) transformations involved in 6 to get rid of the x^6 terms.) So we reduce to

(a) x^7 coefficient $=0$, 7-jet $\sim (y, xy+x^5)$ (⁂).
(b) x^7 coefficient $\neq 0$, 7-jet $\sim (y, xy+x^5 \pm x^7)$ (*).

Part B.

We now want to show that the k-jets marked (*) above are in fact normal forms. For some of the germs it is fairly straightforward to show that they are at least formal normal forms (!). That is by formal changes of co-ordinates one can reduce any formal power series whose k-jet coincides with one of the above to its k-jet.

Example. Consider $(y, x^3 \pm xy^2)$. Clearly it is enough to show that by changes of co-ordinates (in source and target) one can reduce any k-jet $(y+p(x,y), x^3 \pm xy^2 + a(x,y))$, where p and q are homogeneous of degree $k \geqq 4$ to $(y, x^3 \pm xy^2)$. Note that a change of co-ordinates $y_1 = y + p(x,y)$ immediately reduces us to the case $p=0$ since inverting this equation we find $y = y_1 + p_1(x, y_1)$ with degree $p_1 \geqq k$.

Now consider the changes of co-ordinates $(x, y) \mapsto (x + r(x, y), y)$ in source, where r is homogeneous of degree $k-2$, $(u, v) \mapsto (u, v + avu^{k-3})$ in target, where a is a constant.

Modulo terms of degree $\geqq k+1$ this gives the jet

$$(y, x^3 \pm xy^2 + (3x^2 \pm y^2)r + ax^3y^{k-3} \pm axy^{k-1} + q) .$$

Trying to solve $(3x^2 \pm y^2)r + ax^3y^{k-3} \pm axy^{k-1} + q = 0$ leads to a matrix equation $A(r, a) = q$, with A clearly invertible, whence the result. One can now appeal to a result of Mather ([4, (2.5)]) to deduce that this is in fact a smooth normal form.

Unfortunately the rather simple minded approach used above runs into problems with more complicated examples. Thankfully most of the examples have been done in a paper by du Plessis [4] on determinacy. In that paper examples (3.11) covers $(y, xy + x^3)$, $(y, x^3 \pm xy^2)$, $(y, x^3 + xy^3)$. Example (3.32) covers $(y, xy + x^4)$. For $(y, x^2y + x^4 + x^5)$ we note that this is equivalent at the 5-jet level to $(y, x^2y + xy^2 + x^4)$ (killing the xy^2 term in the latter produces an x^3y term, and killing this with another change of x co-ordinate produces an x^5 term.) In the remarks following (4.6) in [4] du Plessis notes that this is 5-determined.

Finally we have $(y, xy + x^5 \pm x^7)$ to consider.

Clearly this is equivalent to $f = (y, xy \pm x^3y + x^5)$ as a 7-jet. The following proof that this is 7-determined is due to Andrew du Plessis. With the notation of [4] we have $tf(\theta_N) + wf(\theta_P) = (C_N \div \{x, x^2\}, C_N \div \{x, x^2, x^3\}) + C_N(1, x + x^2)$. So the C_N-module $D = (\{y, x^3\} . C_N, \{y, x^4\} . C_N) \subset tf(\theta_N) + wf(\theta_P)$. Now $f^*m_P . D = \{y, x^5\} . D$ (since D is a C_N-module) so $f^*m_P . D = (\{y^2, yx^3, x^8\} . C_N, \{y^2, yx^4, x^9\} . C_N)$. Now $(0, x^8) = tf(\frac{1}{5}x^4 . \partial/\partial x) - \frac{1}{5}(0, yx^4 + 3yx^6)$, so $m_N^8\theta_f \subset tf(m_N^2\theta_N) + f^*m_P . D$, it is also true that $m_N^5\theta f \subset tf(m_N\theta_N) + wf(m_P\theta_P)$. The 7-determinacy of f now follows from these inclusions and (3.15) of [4].

In conclusion then we have

THEOREM 4. *The following are normal forms for the projections of a generic surface M in $\mathbf{R}^3$ onto planes.*

$$(x, y) \to (y, x), (y, x^2), (y, x^3 \pm xy^2), (y, xy + x^3), (y, x^3 + xy^3) ,$$
$$(y, xy + x^4), (y, x^2y + x^4 + x^5), (y, xy + x^5 \pm x^7) .$$

4. Generic Reflections.

One general method for studying the geometry of generic curves and surfaces in Euclidean space is that outlined by Gaffney in [5]. Briefly, if one is interested in the contact of the submanifolds with spheres, planes, lines or some other interesting family of submaniolds then one choose a family of submersions $\mathbf{R}^3 \to \mathbf{R}^k$ having these submanifolds as fibres and studies the contact by considering the restriction of this family to the curve or surface. Thus the family of projections discussed in section 4 can be viewed as a way of studying the contact of generic surfaces in $\mathbf{R}^3$ with lines.

Another method of producing families of mappings which hopefully are of some geometric significance is by selecting a smooth mapping f: $\mathbf{R}^n \to X$ and if E denotes the group of Euclidean motions of $\mathbf{R}^n$ then considering the family $F: \mathbf{R}^n \times E \to X$ defined by $F(x, \varphi) = f(\varphi(x))$, and its restriction to submanifolds. If H is the isotropy subgroup $\{\varphi \in E : f \circ \varphi = f\}$ then to avoid redundancy in the parameter space E one might replace it with the coset space E/H. The natural families of distance squared functions, height functions and projections to planes can be obtained by this method.

In this section we are going to consider the family of mappings obtained in this way from the fold map $f: \mathbf{R}^3 \to \mathbf{R}^3$, given by $(x, y, z) \mapsto (x, y^2, z)$. The parameter space here is the Grassmannian of 2-planes in $\mathbf{R}^3$. The geometric significance of this family is that it will relate to the infinitesimal reflective symmetry of the surface in a plane; the more symmetric a surface is at some point about a plane through that point, the more degenerate will be the restriction of the fold map determined by this plane. Our aim is to obtain a list of normal forms for the restriction of these fold maps to a generic surface. To do this we first note that if the surface M is given locally at $0 \in \mathbf{R}^3$ as the image of a map $g(x, y) = (g_1(x, y), g_2(x, y), g_3(x, y))$, then $f \circ g(x, y) = (g_1, g_2^2, g_3)$ is an immersion with normal form $(x, y) \mapsto (x, y, 0)$ unless the normal at M contains the reflecting plane $\{y = 0\}$. Thus at each point we need to consider the reflections in the pencil of planes containing the normal. Writing the surface locally as $(x, y, f(x, y))$ as usual, with $j^1 f(0) = 0$, we need to consider the k-jets of these reflections in terms of the k-jet of f. To simplify matters we *fix* the fold map to be $(x, y, z) \mapsto (x, y^2, z)$ and obtain conditions on the coefficients of the k-jet of f at 0 for the restriction to have various smooth types. These conditions give us semi-algebraic subsets W of V_k. Now the orthogonal group $0(2)$ acts on V_k via its variables; the semi-algebraic subsets of V_k obtained from the orbits under this $0(2)$ action of points in the W's will be the relevant sets for the whole family. Since $0(2)$ is one dimensional this means that we can ignore sets W of codimension > 3 in V_k. That is any normal form for a jet requiring more than 3 conditions on the k-jet of f at 0 can be discarded. As before our derivation of the list is in two parts. Part A where the computations are done with jets, and Part B where the relevant jets are shown to give normal forms. Again discarded jets are labelled by (✻) and jets yielding normal forms by (∗). Again we write $j^k f(0) = f_2(x, y) + f_3(x, y) + \ldots + f_k(x, y)$ with f_j homogeneous of degree j. We shall write $f_2 = a_0x^2 + a_1xy + a_2y^2$, $f_3 = b_0x^3 + \ldots + b_3y^3$ etc. One initial simplification: when considering the map $(x, y) \mapsto (x, y^2, f(x, y)) = (u, v, w)$ and its k-jet note that by changes of co-ordinates of the form $w' = w - p(u, v)$, where p is some polynomial we can ignore terms in f of the form x^iy^{2j} and do so from now on.

Part A.

1. *The 2-jet.* Clearly if $a_1 \neq 0$ we can reduce to (x, y^2, xy) $(*)$. It remains to consider the case $a_1 = 0$ with 2-jet $(x, y^2, 0)$.

2. *The 3-jet.* The 2-jet $\sim (x, y^2, 0)$, so the 3-jet $\sim (x, y^2, b_1x^2y + b_3y^3)$.

(a) If $b_1 = b_3 = 0$, 3-jet $\sim (x, y^2, 0)$; go to 3.I.
(b) If $b_1 \neq 0$, $b_3 = 0$, 3-jet $\sim (x, y^2, x^2y)$; go to 3.II.
(c) If $b_1 = 0$, $b_3 \neq 0$, 3-jet $\sim (x, y^2, y^3)$; go to 3.III.
(d) If $b_1b_3 \neq 0$, 3-jet $\sim (x, y^2, x^2y \pm y^3)$ $(*)$.

3. *The 4-jet.* We have three separate cases depending on the 3-jet.

3.I. The 3-jet $\sim (x, y^2, 0)$, so the 4-jet is $(x, y^2, c_1x^3y + c_3xy^3)$.

(a) If $c_1 = c_3 = 0$, 3-jet $\sim (x, y^2, 0)$ $(⁂)$
(b) If $c_1 = 0$, $c_3 \neq 0$, 3-jet $\sim (x, y^2, xy^3)$ $(⁂)$
(c) If $c_1 \neq 0$, $c_3 = 0$, 3-jet $\sim (x, y^2, x^3y)$ $(⁂)$
(d) If $c_1c_3 \neq 0$, 3-jet $\sim (x, y^2, x^3y \pm xy^3)$ $(*)$.

3.II. 3-jet $\sim (x, y^2, x^2, y)$ so the 4-jet is $(x, y^2, b_1x^2y + c_1x^3y + c_3xy^3)$. Now consider the changes of co-ordinates (i) $x' = x - ay^{2k}$, $u' = u + av^k$ (a change of co-ordinates of type α say), (ii) $w' = w - bu^jv^kw$ (a change of co-ordinates of type β). By choosing $k = 1$, $a = c_3/2b_1$, and $j = 1$, $k = 0$, $b = c_1/b_1$ we can reduce to (x, y^2, x^2y) again. Go to 4.I.

3.III. 3-jet $\sim (x, y^2, y^3)$ so the 4-jet is

$$(x,\ y^2, b_3y^3 + c_1x^3y + c_3xy^3)\,.$$

A change of co-ordinates of type β kills the xy^3 term.

(a) If $c_1 = 0$ clearly the 4-jet $\sim (x, y^2, y^3)$; go to 4.II.
(b) If $c_1 \neq 0$ the 4-jet $\sim (x, y^2, y^3 + x^3y)$ $(*)$.

4. *The 5-jet.* There are two cases.

4.I. The 4-jet $\sim (x, y^2, x^2y)$ so the 5-jet $\sim (x, y^2, b_1x^2y + c_1x^3y + c_3xy^3 + d_1x^4y$

$+d_3x^2y^3+d_5y^5)$ with $b_1 \neq 0$. By changes of co-ordinates of type α and β we can reduce to $(x, y^2, b_1x^2y+(d_5+c_3^2/4b_1)y^5)$.

(a) If $4b_1d_5+c_3^2=0$ the 5-jet$\sim(x, y^2, x^2y)$; go to 5.
(b) If $4b_1d_5+c_3^2 \neq 0$ the 5-jet$\sim(x, y^2, x^2y \pm y^5)$ (*).

4.II. The 4-jet$\sim(x, y^2, y^3)$ so the 5-jet$\sim(x, y^2, b_3y^3+c_3xy^3+d_1x^4y+d_3x^2y^3+d_5y^5)$ with $b_3 \neq 0$. Changes of co-ordinates of type α and β reduce this to $(x, y^2, b_3y^3+d_1x^4y)$.

(a) If $d_1=0$ the 5-jet$\sim(x, y^2, y^3)$ (⊛).
(b) If $d_1 \neq 0$ the 5-jet$\sim(x, y^2, y^3 \pm x^4y)$ (*).

5. *The 6-jet*. The 5-jet$\sim(x, y^2, x^2y)$ so the 6-jet$\sim$

$$(x, y^2, b_1x^2y+c_1x^3y+c_3xy^3+d_1x^4y+d_3x^2y^3+d_5y^5+e_1x^5y+e_3x^3y^3+e_5xy^5)$$
$$\text{with } 4b_1d_5+c_3^2 = 0\,.$$

Changes of co-ordinates of type α and β reduce to $(x, y^2, b_1x^2y)\sim(x, y^2, x^2y)$. Go to 6.

6. *The 7-jet*. Here the 6-jet (x, y^2, x^2y) and the 7-jet$\sim$

$$(x, y^2, b_1x^2y+c_1x^3y+c_3xy^3+d_1x^4y+d_3x^2y^3+d_5y^5+e_1x^5y$$
$$+e_3x^3y^3+e_5xy^5+f_1x^6y+f_3x^4y^3+f_5x^2y^5+f_7y^7)$$

with $4b_1d_5+c_3^2=0$. By changes of co-ordinates of type β one reduces to $(x, y^2, b_1x^2y+c_3xy^3+d_5y^5+e_5xy^5+f_7y^7)$. Changes of co-ordinates of type α now reduces this to $(x, y^2, b_1x^2y+f_7y^7)$

(a) If $f_7=0$ the 7-jet$\sim(x, y^2, x^2y)$ (⊛)
(b) If $f_7 \neq 0$ the 7-jet$\sim(x, y^2, x^2y \pm y^7)$ (*).

Part B.

Again we want to show that the k-jets marked (*) above are normal forms. Again as in section 3 one can by elementary (but rather messy) manipulations of co-ordinates prove that some of the above are formal normal forms (and then appeal to Mather's theorem). For example $(x, y^2, x^3y \pm xy^3)$ can be proved to be 4-determined in this way. (Hint: one easily gets rid of homogeneous terms of degree $k \geq 5$ in the first two components of the k-jet. For the third component one uses changes of co-ordinates of type α and β, together with a change of co-ordinates $x'=c-cx^{k-3}$, $u'=u+cu^{k-3}$. Again there is a resulting matrix equation which is solvable.) The determinacy of all of the other forms is actually covered by the paper of du Plessis [4] in his Example (3.12) (see (3.12), (3.16), (3.20) and (3.29)).

In conclusion then we have

THEOREM 5. *The following are normal forms for reflections of a generic surface in* $\mathbf{R}^3$ *in a plane.*

$$(x,y)\mapsto(x,y,0),\ (x,y^2,xy),\ (x,y^2,x^2y\pm y^3),\ (x,y^2,x^3y\pm xy^3)(x,y^2,y^3+x^3y)\ ,$$
$$(x,y^2,x^2y\pm y^5)\ (x,y^2,y^3\pm x^4y),\ (x,y^2,x^2y\pm y^7)\ .$$

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