MATH. SCAND. 54 (1984), 279-294

ONE-DIMENSIONAL K-TYPES IN FINITE DIMENSIONAL REPRESENTATIONS OF SEMISIMPLE LIE GROUPS: A GENERALIZATION OF HELGASON'S THEOREM

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1. Introduction.

Let G be a semisimple connected noncompact real Lie group, and let K be a maximal compact subgroup. Let (π, V) be a finite dimensional irreducible representation of G. A renowned theorem due to S. Helgason gives the condition in terms of the highest weight of π under which π is class one. This means that V contains a nonzero vector fixed by $\pi(K)$, or in other words that π contains the trivial K-type in its decomposition into irreducible representations of K. In this paper we generalize this theorem to give a complete description in terms of the highest weight of π of all one-dimensional K-types contained in π.

Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and let $\mathfrak{j}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{j}=\mathfrak{t}^+\oplus\mathfrak{a}$ where $\mathfrak{t}^+=\mathfrak{j}\cap\mathfrak{k}$. Assume that G is contained in the complex simply connected Lie group with Lie algebra $\mathfrak{g}_C=\mathfrak{g}+i\mathfrak{g}$. Since we are dealing only with finite dimensional representations of G this assumption causes no loss of generality. Choose compatible orderings of $\mathfrak{a}$ and $\mathfrak{j}$, and let $\lambda\in\mathfrak{j}_C^*$ be a complex linear form on $\mathfrak{j}$.

The precise content of Helgason's theorem is as follows (cf. [2, III § 3]): If the restriction of λ to $\mathfrak{t}^+$ is zero and if for all positive roots α of $\mathfrak{a}$ in $\mathfrak{g}$ the number $(\lambda,\alpha)/(\alpha,\alpha)$ is a nonnegative integer, then λ is the highest weight of a finite dimensional class one representation of G, and all finite dimensional class one representations of G occur in this way.

If K is semisimple then the trivial representation is its only one-dimensional representation, and therefore no more can be said about one-dimensional K-types in π. However, if K is not semisimple, or equivalently if G/K is Hermitian symmetric, then there are other one-dimensional K-types than the trivial. It is for such groups G our generalization of Helgason's theorem applies.

Received March 10, 1983.

* Partially supported by the Danish Natural Science Research Council.

Let $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$ be the semisimple part of $\mathfrak{k}$. The condition on the highest weight λ of π for one-dimensional K-types to occur is very similar to that of Helgason's theorem: the restriction of λ to $\mathfrak{t}^+ \cap \mathfrak{k}_1$ has to be zero, and furthermore λ has to satisfy a certain integrality condition (see Theorem 7.2). The method of our investigation is by reduction to the rank-one case in the manner of S. G. Gindikin and F. I. Karpelevič ([1]).

The paper is organized as follows: First the restricted root theorem of C. C. Moore is stated. In Section 3 we study the structure of K and determine the complete set of one-dimensional K-types. In the next section the centralizer of $\mathfrak{a}$ in K and its intersection with the semisimple part of K are considered. In Section 5 the rank-one subgroups of G are studied, and in the succeeding section we look upon $SU(n,1)$ ($n \geqq 1$), which are the only rank-one groups in which K is not semisimple. Finally in Section 7 the main theorem is stated and proved.

The problem of generalizing Helgason's theorem in this direction emerged in [8]. Though we will not go into that here we point out, that Theorem 7.2 in combination with the results of [8] can be applied to the construction of interesting unitary representations, in particular of some exceptional groups.

2. Root structure.

Let G be a connected real simple noncompact Lie group, and let $\mathfrak{g}$ be its Lie algebra. Assume that $G \subset G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is a simply connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decompositon, and let K be the corresponding maximal compact subgroup of G. Let $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$ and assume that $\mathfrak{k}_1 \neq \mathfrak{k}$, that is G/K is Hermitian symmetric (cf. [3, Ch. VIII]).

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$, then $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta \subset i\mathfrak{t}^*$ consist of the roots of $\mathfrak{t}$ in $\mathfrak{g}$, and let $\mathfrak{g}_\gamma \subset \mathfrak{g}_{\mathbb{C}}$ for $\gamma \in \Delta$ denote the γ root space. Let Δ_c, respectively Δ_n be the set of compact, respectively noncompact roots, i.e. those roots γ for which $\mathfrak{g}_\gamma \subset \mathfrak{k}_{\mathbb{C}}$, respectively $\mathfrak{g}_\gamma \subset \mathfrak{p}_{\mathbb{C}}$. Then $\Delta = \Delta_c \cup \Delta_n$.

Let $\mathfrak{z}$ be the center of $\mathfrak{k}$, then $\dim \mathfrak{z} = 1$. As is well-known, we can choose an element $Z_0 \in \mathfrak{z}$ such that $\gamma(Z_0) = \pm i$ for all $\gamma \in \Delta_n$. Choose an ordering of Δ such that

$$\Delta_n^+ = \{\gamma \in \Delta \mid \gamma(Z_0) = i\},$$

where $\Delta_n^+ = \Delta^+ \cap \Delta_n$. Let $\Delta_c^+ = \Delta^+ \cap \Delta_c$.

For $\varphi \in \mathfrak{t}_{\mathbb{C}}^*$ let $H_\varphi \in \mathfrak{t}_{\mathbb{C}}$ be defined by $\varphi(H) = (H_\varphi, H)$ for all $H \in \mathfrak{t}$, where $(\cdot, \cdot)$ denotes the Killing form. Let $\{\gamma_1, \ldots, \gamma_r\} \subset \Delta_n$ be a maximal strongly orthogonal subset, such that γ_j is the highest element of Δ_n strongly orthogonal to $\{\gamma_{j+1}, \ldots, \gamma_r\}$, for $j = r, \ldots, 1$ (cf. [3, p. 386]). Let

$$\mathfrak{t}^- = \sum_{j=1}^{r} \mathbf{R} i H_{\gamma_j}$$

and

$$\mathfrak{t}^+ = \{H \in \mathfrak{t} \mid \gamma_j(H)=0,\ j=1,\ldots,r\}\ ,$$

then $\mathfrak{t}=\mathfrak{t}^+\oplus\mathfrak{t}^-$. Identify γ_j with its restriction to $\mathfrak{t}^-$ $(j=1,\ldots,r)$.

THEOREM 2.1. (C. C. Moore). *The set of nonzero restrictions of the elements of Δ^+ to $\mathfrak{t}^-$ is one of the following two sets*:

Case I: $\{\gamma_i, \frac{1}{2}(\gamma_j \pm \gamma_k) \mid 1 \leqq i \leqq r,\ 1 \leqq k < j \leqq r\}$

Case II: $\{\frac{1}{2}\gamma_i, \gamma_i, \frac{1}{2}(\gamma_j \pm \gamma_k) \mid 1 \leqq i \leqq r,\ 1 \leqq k < j \leqq r\}$.

Furthermore the nonzero restrictions of compact roots have the form $\frac{1}{2}\gamma_i$ or $\frac{1}{2}(\gamma_j-\gamma_k)$, and the restrictions of noncompact roots have the form $\frac{1}{2}\gamma_i$, γ_i or $\frac{1}{2}(\gamma_j+\gamma_k)$.

The roots $\gamma_1,\ldots,\gamma_r$ do all belong to the set of longest roots in Δ. In Case II, *only one root length occurs in Δ.*

Unless when $\mathfrak{t}^+=0$, $\gamma_1,\ldots,\gamma_r$ are the only restricted roots of multiplicity one.

PROOF. This can be verified case by case from the diagrams [3, pp. 532–34], or it can be proved by combinatorial arguments, cf. [6].

REMARK 2.2. In [4] it is shown that Case I is necessary and sufficient for G/K to be a tube domain. Here is the classification of the possible algebras $\mathfrak{g}$ (cf. [3]):

Case I: $\mathfrak{su}(n,n)$ $(n \geqq 2)$, $\mathfrak{so}(n,2)$ $(n \geqq 5)$, $\mathfrak{so}^*(4n)$ $(n \geqq 3)$, $\mathfrak{sp}(n,\mathbf{R})$ $(n \geqq 1)$ and $\mathfrak{e}_{7(-25)}$

Case II: $\mathfrak{su}(p,q)$ $(q > p \geqq 1)$, $\mathfrak{so}^*(4n+2)$ $(n \geqq 2)$ and $\mathfrak{e}_{6(-14)}$.

Among these, $\mathfrak{t}^+=0$ only happens for $\mathfrak{sp}(n,\mathbf{R})$ $(n \geqq 1)$.

For each $\gamma \in \Delta_n$ choose $X_\gamma \in \mathfrak{g}_\gamma \setminus \{0\}$ subject to $\bar{X}_\gamma = X_{-\gamma}$, and $\gamma([X_\gamma, X_{-\gamma}])=2$, where the bar denotes conjugation with respect to the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbf{C}}$. Let

$$\mathfrak{a} = \sum_{j=1}^{r} \mathbf{R}(X_{\gamma_j}+X_{-\gamma_j})\ ,$$

then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. Let c be the automorphism of $\mathfrak{g}_{\mathbf{C}}$ given by

$$c = \operatorname{Ad}\exp\frac{\pi}{4}\sum_{j=1}^{r}(X_{\gamma_j}-X_{-\gamma_j})\ .$$

Then c maps $i\mathfrak{t}^-$ bijectively to $\mathfrak{a}$ and fixes $\mathfrak{t}^+$ (cf. [4]).

Let $\mathfrak{j}=\mathfrak{t}^+ + \mathfrak{a}$ and let $c_*\colon \mathfrak{t}_{\mathbf{C}}^* \to \mathfrak{j}_{\mathbf{C}}^*$ denote the adjoint of $c^{-1}\colon \mathfrak{j}_{\mathbf{C}} \to \mathfrak{t}_{\mathbf{C}}$. Then $c_*\Delta$ consists of the roots of the Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{g}$. Let Σ denote the set of nonzero restrictions of $c_*\Delta$ to $\mathfrak{a}$, and let Σ^+ consists of the nonzero restrictions of $c_*\Delta^+$ to $\mathfrak{a}$. Let $\alpha_j = c_*\gamma_j$, then exchanging the γ's in Case I and II above with α's, we get the two possible forms of Σ^+.

3. The structure of K.

Let K_1 denote the analytic subgroup of K with Lie algebra $\mathfrak{k}_1$.

LEMMA 3.1. *Let Φ denote the set of simple roots for Δ^+, and let $s_\varphi \in \mathbf{R}$ for each $\varphi \in \Phi$. Then*

$$\exp\left(\sum_{\varphi\in\Phi} s_\varphi \frac{2iH_\varphi}{(\varphi,\varphi)}\right) = e$$

if and only if $s_\varphi \in 2\pi\mathbf{Z}$ for all $\varphi \in \Phi$.

PROOF. Let U be the analytic subgroup of $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{u}=\mathfrak{k}+i\mathfrak{p}$. Then U is compact and simply connected, and $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{u}$. The lemma then follows from [9, Theorem 4.6.7].

COROLLARY 3.2. *K_1 is simply connected.*

PROOF. It is easily seen that $\Phi\cap\Delta_c$ consists of the simple roots for Δ_c^+. From Lemma 3.1

$$\exp\left(\sum_{\varphi\in\Phi\cap\Delta_c} s_\varphi \frac{2iH_\varphi}{(\varphi,\varphi)}\right) = e$$

if and only if $s_\varphi \in 2\pi\mathbf{Z}$ for all $\varphi \in \Phi\cap\Delta_c$, and so K_1 is simply connected, again by [9, Theorem 4.6.7].

Let a denote the length of the longest roots in Δ. The short roots, if there are any, then have length $a/\sqrt{2}$.

LEMMA 3.3 *Let $t_\gamma \in \mathbf{R}$ for each $\gamma \in \Delta_n^+$ and let*

$$x = \exp\left(\sum_{\gamma\in\Delta_n^+} t_\gamma \frac{2iH_\gamma}{(\gamma,\gamma)}\right).$$

Then $x \in K_1$ if and only if

$$\sum_{\gamma \in \Delta_n^+} \frac{t_\gamma}{(\gamma, \gamma)} \in \frac{2\pi}{a^2} \mathbf{Z} .$$

PROOF. Assume that $x \in K_1$. Since x centralizes $\mathfrak{t} \cap \mathfrak{k}_1$, which is a Cartan subalgebra of $\mathfrak{k}_1$, $x = \exp Y$ for some $Y \in \mathfrak{t} \cap \mathfrak{k}_1$. Then

$$\exp\left(\sum_{\gamma \in \Delta_n^+} t_\gamma \frac{2iH_\gamma}{(\gamma, \gamma)} - Y \right) = e$$

and it follows from Lemma 3.1 that

$$\sum_{\gamma \in \Delta_n^+} t_\gamma \frac{2iH_\gamma}{(\gamma, \gamma)} - Y = \sum_{\varphi \in \Phi} s_\varphi \frac{2iH_\varphi}{(\varphi, \varphi)}$$

for some $s_\varphi \in 2\pi\mathbf{Z}$ $(\varphi \in \Phi)$. Taking inner product with $\frac{1}{2}Z_0$ it follows that

$$\sum_{\gamma \in \Delta_n^+} \frac{t_\gamma}{(\gamma, \gamma)} = \frac{s_\psi}{(\psi, \psi)} \in \frac{2\pi}{a^2} \mathbf{Z}$$

where ψ is the unique simple noncompact root.

Assume conversely that

$$t = \sum_{\gamma \in \Delta_n^+} \frac{t_\gamma}{(\gamma, \gamma)} \in \frac{2\pi}{a^2} \mathbf{Z} .$$

Then

$$\sum_{\gamma \in \Delta_n^+} t_\gamma \frac{2iH_\gamma}{(\gamma, \gamma)} - ta^2 \frac{2iH_{\gamma_1}}{(\gamma_1, \gamma_1)} \in \mathfrak{k}_1$$

since it is orthogonal to Z_0. However

$$\exp\left(ta^2 \frac{2iH_{\gamma_1}}{(\gamma_1, \gamma_1)} \right) = e$$

by Lemma 3.1, and hence $x \in K_1$.

Let $N = \sum_{\gamma \in \Delta_n^+} a^2/(\gamma, \gamma)$, then N is the number of longest roots in Δ_n^+ plus twice the number of short roots, if there are any. Define $Z \in \mathfrak{t}$ by

$$Z = \frac{1}{N} \sum_{\gamma \in \Delta_n^+} \frac{2iH_\gamma}{(\gamma, \gamma)} . \tag{3.1}$$

PROPOSITION 3.4. *Let* $t \in \mathbf{R}$

(i) $Z \in \mathfrak{z} \setminus \{0\}$,
(ii) $\exp tZ \in K_1$ *if and only if* $t \in 2\pi\mathbf{Z}$.

PROOF. (i) From (3.1) it follows that

$$(3.2)\qquad (Z, Z_0) = -\frac{2}{a^2}.$$

Therefore $Z \neq 0$. Let $\varphi \in \Delta_c^+$ and $\gamma \in \Delta_n^+$. If $(\varphi, \gamma) = 0$ then γ contributes nothing to $\varphi(Z)$. If $(\varphi, \gamma) \neq 0$ then also the reflected root $\sigma_\varphi \gamma$ is positive and noncompact since $\sigma_\varphi \gamma(Z_0) = i$. From $(\varphi, \gamma + \sigma_\varphi \gamma) = 0$ it then follows that $\varphi(Z) = 0$ for all $\varphi \in \Delta_c^+$, and hence $Z \in \mathfrak{z}$.

(ii) follows immediately from Lemma 3.3.

Let $l \in \mathbf{Z}$, and define $\chi_l \colon K \to \mathbf{C}$ by $\chi_l(k) = 1$ for $k \in K_1$ and $\chi_l(\exp tZ) = e^{ilt}$ for $t \in \mathbf{R}$. From Proposition 3.4 we get that χ_l is a well defined one dimensional representation of K, and that all one dimensional representations of K have this form.

REMARK 3.5. In Case I, we can give a simpler formula for Z as follows

$$(3.3)\qquad Z = \frac{1}{r} \sum_{j=1}^{r} \frac{2iH_{\gamma_j}}{(\gamma_j, \gamma_j)}.$$

In fact, let Z' denote the right hand side of (3.3). Then $Z' \in \mathfrak{z}$ by Theorem 2.1, and $(Z, Z_0) = (Z', Z_0)$ from (3.2), so (3.3) follows. In particular we have $\mathfrak{z} \subset \mathfrak{t}^-$ in Case I (cf. also [4, Proposition 3.12]).

4. The structure of M.

Let M denote the centralizer of $\mathfrak{a}$ in K, and let $\mathfrak{m}$ be its Lie algebra. Then $\mathfrak{t}^+$ is a Cartan subalgebra of $\mathfrak{m}$. For any Lie group F, let F_0 denote its identity component. It is well-known (cf. [2, p. 75]) that

$$(4.1)\qquad M = M_0 \cdot (\exp i\mathfrak{a} \cap K)$$

and also that if $H_\alpha \in \mathfrak{a}$ for $\alpha \in \mathfrak{a}^*$ is defined by $(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{a}$, then (cf. [2, p. 77])

$$(4.2)\qquad \{H \in \mathfrak{a} \mid \exp iH \in K\} = \left\{ \sum_{\alpha \in \Sigma^+} s_\alpha \frac{H_\alpha}{(\alpha, \alpha)} \;\middle|\; s_\alpha \in 2\pi\mathbf{Z} \text{ for all } \alpha \in \Sigma^+ \right\}.$$

From the description of Σ^+ by Theorem 2.1 it follows that (4.2) can be restated as follows:

$$(4.3)\qquad \{H \in \mathfrak{a} \mid \exp iH \in K\} = \left\{ \sum_{j=1}^{r} s_j \frac{H_{\alpha_j}}{(\alpha_j, \alpha_j)} \;\middle|\; s_j \in 2\pi\mathbf{Z} \text{ for } j = 1, \ldots, r \right\}.$$

LEMMA 4.1.

$$\exp \frac{2\pi i H_{\alpha_j}}{(\alpha_j, \alpha_j)} = \exp \frac{2\pi i H_{\gamma_j}}{(\gamma_j, \gamma_j)}$$

for $j = 1, \ldots, r$.

PROOF. There is a homomorphism $\mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{g}_{\mathbf{C}}$ for which

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \frac{2H_{\alpha_j}}{(\alpha_j, \alpha_j)} \quad \text{and} \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \to \frac{2iH_{\gamma_j}}{(\gamma_j, \gamma_j)} .$$

The lemma then follows, since in SL $(2, \mathbf{C})$

$$\exp \pi i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \exp \pi \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} .$$

LEMMA 4.2. *In* Case II, M *is connected.*

PROOF. By (4.3) and Lemma 4.1, it suffices to prove that for $j = 1, \ldots, r$:

$$\exp \frac{2\pi i H_{\gamma_j}}{(\gamma_j, \gamma_j)} \in M_0 .$$

Let $\varphi \in \Delta^+$ be a root whose restriction to $\mathfrak{t}^-$ is $\frac{1}{2}\gamma_j$. Then obviously $H_{\gamma_j} - 2H_\varphi \in \mathfrak{t}^+$. From Lemma 3.1 we have

$$\exp \left(2\pi \frac{2iH_\varphi}{(\varphi, \varphi)} \right) = e$$

and therefore

$$\exp \frac{2\pi i H_{\gamma_j}}{(\gamma_j, \gamma_j)} = \exp \frac{2\pi i}{a^2} (H_{\gamma_j} - 2H_\varphi) \in M_0 .$$

Let W be the Weyl group of Δ, and let $\theta = \sigma_{\gamma_1} \cdot \ldots \cdot \sigma_{\gamma_r} \in W$. Then $\theta(S+T) = S - T$ for $S \in \mathfrak{t}^+$, $T \in \mathfrak{t}^-$. Let $\Xi \subset \Delta^+$ denote the set of roots whose restriction to $\mathfrak{t}^-$ is $\frac{1}{2}\gamma_j$ for some j, and let $\Xi_n = \Xi \cap \Delta_n$ and $\Xi_c = \Xi \cap \Delta_c$. Note that $\theta(\Xi_n) = -\Xi_c$, since if $\xi \in \Xi_n$ with $\xi|_{\mathfrak{t}^-} = \frac{1}{2}\gamma_j$, then $\theta\xi = \xi - \gamma_j \in -\Xi_c$.

Let R denote the number of elements of Ξ_n (or Ξ_c). Define $X \in \mathfrak{t}$ by $X = 0$ in Case I and in Case II:

$$X = \frac{1}{R} \left[\sum_{\xi \in \Xi_n} \frac{2iH_\xi}{(\xi, \xi)} - \sum_{\xi \in \Xi_c} \frac{2iH_\xi}{(\xi, \xi)} \right] . \tag{4.4}$$

LEMMA 4.3. *In* Case II, *the following holds*:

(i) $Z+\theta Z = \frac{R}{N}X$.

(ii) $X \in \mathfrak{t}^+$, $X \perp \mathfrak{t}^+ \cap \mathfrak{k}_1$

(iii) $X \neq 0$.

(iv) $Z - X \in \mathfrak{k}_1$.

(v) *For* $t \in \mathbf{R}$: $\exp tX \in K_1$ *if and only if* $t \in 2\pi\mathbf{Z}$.

PROOF. (i) is clear from (4.4) since $\theta(\Xi_n) = -\Xi_c$.
(ii) follows from (i). From (4.4) it follows that

$$(X, Z_0) = -\frac{2}{a^2} \tag{4.5}$$

and from this (iii) is obvious.
(iv) follows from (4.5) and (3.2).
(v) is obvious from (iv) and Proposition 3.4 (ii).

Note that it follows that $\mathfrak{z} \not\subset \mathfrak{t}^-$ in Case II (cf. also [4, Section 4]).

LEMMA 4.4. *In* Case II, $M \cap K_1$ *is connected.*

PROOF. Since M is connected and $\mathfrak{m} = (\mathfrak{m} \cap \mathfrak{k}_1) \oplus \mathbf{R}X$ by the preceding lemmas, it suffices to prove that $\exp \mathbf{R}X \cap K_1 \subset (M \cap K_1)_0$. By Lemma 4.3 (v) it then suffices to prove that $\exp 2\pi X \in (M \cap K_1)_0$.

Choose $\xi \in \Xi_n$, then $\theta\xi \in \Delta_c$. Therefore $(H_\xi + \theta H_\xi, Z_0) = i$, and from (4.5) it follows that

$$X - \frac{2i}{a^2}(H_\xi + \theta H_\xi) \in \mathfrak{t}^+ \cap \mathfrak{k}_1 .$$

However, by Lemma 3.1

$$\exp\left[2\pi \frac{2i}{a^2}(H_\xi + \theta H_\xi)\right] = e$$

and hence $\exp 2\pi X \in \exp \mathfrak{t}^+ \cap \mathfrak{k}_1$.

We can now state an analogue of (4.1) and (4.3).

PROPOSITION 4.5. (i) *$M \cap K_1 = (M \cap K_1)_0(\exp i\mathfrak{a} \cap K_1)$.*

(ii)
$$\{H \in \mathfrak{a} \mid \exp iH \in K_1\}$$
$$= \left\{ \sum_{j=1}^{r} s_j \frac{H_{\alpha_j}}{(\alpha_j, \alpha_j)} \;\middle|\; s_j \in 2\pi \mathbf{Z} \text{ for } j=1,\dots,r \text{ and } \sum_{j=1}^{r} s_j \in 4\pi\mathbf{Z} \right\}.$$

PROOF. (i) In Case I, $M_0 \subset K_1$ and hence $(M \cap K_1)_0 = M_0$. Therefore (i) follows from (4.1). In Case II (i) is obvious from Lemma 4.4.

(ii) follows from (4.3), Lemma 4.1, and Lemma 3.3.

Later on, we need the following lemma:

LEMMA 4.6. (i) *If $\gamma \in \Delta^+ \setminus \Xi$, then $\gamma(X) = 0$.*
(ii) *If $\gamma \in \Xi_n$, then $\gamma(X) = -a^2(X, X)i/4$,*
(iii) *If $\gamma \in \Xi_c$, then $\gamma(X) = a^2(X, X)i/4$.*

PROOF. Assume Case II. Let $b \in \mathbf{R}$ be given by $Z = bZ_0$.

(i) Let $\gamma \in \Delta^+ \setminus \Xi$. If γ is compact, then $\theta\gamma$ is also compact by Theorem 2.1, and hence $(\gamma + \theta\gamma)(Z) = 0$. If γ is noncompact, then $\theta\gamma$ is also noncompact but negative, and hence $(\gamma + \theta\gamma)(Z) = ib - ib = 0$. Then $\gamma(X) = 0$ by Lemma 4.3 (i).

(ii)–(iii) If $\gamma \in \Xi_n$, then $\theta\gamma \in -\Xi_c$ and hence $\gamma(Z + \theta Z) = ib$. Therefore $\gamma(X) = ibN/R$, and since $\theta X = X$, we then have $-\theta\gamma(X) = -ibN/R$. But then by (4.4)

$$(X, X) = \frac{1}{R}\left[\sum_{\xi \in \Xi_n} \frac{2i\xi(X)}{(\xi, \xi)} - \sum_{\xi \in \Xi_c} \frac{2i\xi(X)}{(\xi, \xi)}\right] = -\frac{4bN}{Ra^2}$$

so $bN/R = -a^2(X, X)/4$.

5. The rank-one reduction.

Let $\alpha \in \Sigma^+ \setminus 2\Sigma^+$ and let $\mathfrak{g}^\alpha$ be the subalgebra of $\mathfrak{g}$ generated by the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. Let G^α be the corresponding analytic subgroup of G. Then G^α is a simple Lie group of real rank one, and $\mathfrak{g}^\alpha = \mathfrak{k}^\alpha \oplus \mathfrak{p}^\alpha$ is a Cartan decomposition, where $\mathfrak{k}^\alpha = \mathfrak{k} \cap \mathfrak{g}^\alpha$ and $\mathfrak{p}^\alpha = \mathfrak{p} \cap \mathfrak{g}^\alpha$. Therefore $K^\alpha = K \cap G^\alpha$ is a maximal compact subgroup of G^α. (For these well-known facts, see [3, pp. 407–409].) Let $m(\alpha)$ denote the multiplicity of α.

LEMMA 5.1. *In* Case I, *G^α/K^α is Hermitian symmetric if and only if $m(\alpha) = 1$, and then $\mathfrak{g}^\alpha \cong \mathfrak{su}(1, 1)$. In* Case II, *$G^\alpha/K^\alpha$ is Hermitian symmetric if and only if $\alpha = \frac{1}{2}\alpha_j$ for some $j \in \{1, \dots, r\}$, and then $\mathfrak{g}^\alpha \cong \mathfrak{su}(n, 1)$, where $n = 1 + \frac{1}{2}m(\alpha)$.*

PROOF. According to classification the only rank one Hermitian symmetric spaces are $SU(n,1)/S(U(n)\times U(1))$ ($n\in \mathbf{N}$). If G^α/K^α is Hermitian symmetric, it follows therefore that $\mathfrak{g}^\alpha\cong\mathfrak{su}(n,1)$ for some $n\in\mathbf{N}$. In Case I, 2α is not a root, and hence $\mathfrak{g}^\alpha\cong\mathfrak{su}(1,1)$. Obviously this happens if and only if $m(\alpha)=1$. In Case II, if α is one of the roots $\frac{1}{2}(\alpha_i\pm\alpha_j)$, then $m(\alpha)>1$ and $2\alpha\notin\Sigma$, so $\mathfrak{g}^\alpha$ cannot be isomorphic to $\mathfrak{su}(n,1)$ for any $n\in\mathbf{N}$. On the other hand, if $\alpha=\frac{1}{2}\alpha_j$, then $m(2\alpha)=1$, and therefore $\mathfrak{g}^\alpha\cong\mathfrak{su}(n,1)$ with $n+1\frac{1}{2}m(\alpha)$ by the classification of real rank one algebras.

Let $l\in\mathbf{Z}$. We will determine the restriction of χ_l to K^α, and assume therefore that K^α is not semisimple.

In Case II, $\mathfrak{g}^\alpha\cong\mathfrak{su}(1,1)$ and in this identification

$$\frac{2ic^{-1}H_\alpha}{(\alpha,\alpha)}=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}.$$

By Remark 3.5

$$\frac{1}{(Z,Z)}\left(\frac{2ic^{-1}H_{\alpha'}}{(\alpha,\alpha)},Z\right)=\sum_{j=1}^{r}\frac{(\alpha,\alpha_j)}{(\alpha,\alpha)}$$

and therefore

$$\chi_l\left(\exp t\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}\right)=\begin{cases}1 & \text{if } \alpha=\frac{1}{2}(\alpha_i-\alpha_j)\\ e^{ilt} & \text{if } \alpha=\alpha_i\\ e^{i2lt} & \text{if } \alpha=\frac{1}{2}(\alpha_i+\alpha_j)\ \ (i\neq j)\,.\end{cases}\tag{5.1}$$

In Case II we have $\alpha=\frac{1}{2}\alpha_j$ and $\mathfrak{g}^\alpha\cong\mathfrak{su}(n,1)$. Let $\Delta_n^+(\alpha)$ denote the set of noncompact positive roots of $\mathfrak{t}\cap\mathfrak{g}^\alpha$ in $\mathfrak{g}^\alpha$. Note that the cardinality of $\Delta_n^+(\alpha)$ is n, and put

$$Z(\alpha)=\frac{1}{n}\sum_{\gamma\in\Delta_n^+(\alpha)}\frac{2iH_\gamma}{(\gamma,\gamma)}\,.\tag{5.2}$$

Then, in the identification with $\mathfrak{su}(n,1)$, $Z(\alpha)$ is the diagonal matrix with i/n in the first n entries and $-i$ in the last entry. From (5.2) and (3.2) it follows that $Z-Z(\alpha)\in\mathfrak{k}_1$. Therefore

$$\chi_l(\exp tZ(\alpha))=e^{ilt}\,.\tag{5.3}$$

6. A lemma concerning SU $(n,1)$.

Let $G=KAN$ be the Iwasawa decomposition of G corresponding to Σ^+, and define maps $\varkappa\colon G\to K$ and $H\colon G\to\mathfrak{a}$ by

$$x\in\varkappa(x)\exp H(x)N$$

for $x \in G$. Let $\varrho = \frac{1}{2}\sum_{\alpha \in \Sigma^+} m(\alpha)\alpha$ and let $\bar{N}$ be the group opposite to N, i.e.

$$\bar{N} = \exp\left(\sum_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}\right).$$

Let $d\bar{n}$ denote some Haar measure on $\bar{N}$. Let $n \in \mathbf{N}$ and $k \in]0,\infty[$. Let $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$.

LEMMA 6.1. *Assume $G = \mathrm{SU}(n,1)$, and let $\beta \in \Sigma^+$ be the root for which $2\beta \notin \Sigma$. Put $\nu = \frac{1}{2}k\beta$. Then the integral*

$$\int_{\bar{N}} e^{\langle -\nu-\varrho, H(\bar{n})\rangle}\chi_l(\varkappa(\bar{n}))\, d\bar{n} \tag{6.1}$$

converges absolutely for all $l \in \mathbf{Z}$, and it is nonzero if and only if $|l| \notin k+n+2\mathbf{Z}_+$.

PROOF. We have

$$K = \left\{\begin{pmatrix} U & 0 \\ 0 & \det U^{-1}\end{pmatrix} \;\middle|\; U \in U(n)\right\}$$

and

$$\chi_l\begin{pmatrix} U & 0 \\ 0 & \det U^{-1}\end{pmatrix} = (\det U)^l .$$

We put $H = E_{1,n+1} + E_{n+1,1}$, where E_{ij} denotes the $n+1$ square matrix with 1 on the i,jth entry and all other entries 0. With $\mathfrak{a} = \mathbf{R}H$ we get $\beta(H) = 2$ and $\varrho(H) = n$.

The root spaces are given as follows

$$\mathfrak{g}_\beta = \mathbf{R}i(E_{1,1} - E_{1,n+1} + E_{n+1,1} - E_{n+1,n+1})$$

$$\mathfrak{g}_{\frac{1}{2}\beta} = \left\{\sum_{j=1}^{n-1} (z_jE_{1,j+1} - \bar{z}_jE_{j+1,1} + \bar{z}_jE_{j+1,n+1} + z_jE_{n+1,j+1}) \;\middle|\; z_1,\ldots,Z_{n-1} \in \mathbf{C}\right\}$$

$$\mathfrak{g}_{-\frac{1}{2}\beta} = \left\{\sum_{j=1}^{n-1} (z_jE_{1,j+1} - \bar{z}_jE_{j+1,1} - \bar{z}_jE_{j+1,n+1} - z_jE_{n+1,j+1}) \;\middle|\; z_1,\ldots,z_{n-1} \in \mathbf{C}\right\}$$

$$\mathfrak{g}_{-\beta} = \mathbf{R}i(E_{1,1} + E_{1,n+1} - E_{n+1,1} - E_{n+1,n+1}) .$$

From the first two of these equations it is easily seen that $H(x)$ and $\chi_l(\varkappa(x))$ for $x \in \mathrm{SU}(n,1)$ can be computed as follows: Let $\eta(x)$ denote the sum of the first and the last element in the last row of x, then

$$H(x) = \log|\eta(x)|H$$

$$\chi_l(\varkappa(x)) = \left(\frac{\eta(x)}{|\eta(x)|}\right)^{-l}.$$

From the expressions for $\mathfrak{g}_{-\frac{1}{2}\beta}$ and $\mathfrak{g}_{-\beta}$, it then follows that the integral (6.1) except for a constant (nonzero) factor equals

$$\int_{\mathbf{C}^{n-1}} \int_{\mathbf{R}} [(1+|z|^2)^2+s^2]^{-\frac{k+n}{2}} \left(\frac{1+|z|^2-is}{|1+|z|^2-is|}\right)^{-l} ds dz . \tag{6.2}$$

If $n>1$ we use polar coordinates in $\mathbf{C}^{n-1}$ and get

$$\int_0^\infty \int_{\mathbf{R}} [(1+r^2)^2+s^2]^{-\frac{k+n}{2}} \left(\frac{1+r^2-is}{|1+r^2-is|}\right)^{-l} ds r^{2n-3} dr .$$

Let $c_n = \int_0^\infty (1+r^2)^{-k-n+1} r^{2n-3} dr$ if $n>1$ and $c_1 = 1$ (note that this integral converges because $k>0$). Substitution of $s=(1+r^2)tgt$ if $n>1$ and $s=tgt$ if $n=1$ gives the following integral instead of (6.2):

$$c_n \int_{-\pi/2}^{\pi/2} (\cos t)^{k+n-2} e^{ilt} dt .$$

This integral can in fact be computed in terms of the gamma function since $k>n+1$, and the result is

$$c_n \frac{\pi\Gamma(k+n-1)}{2^{k+n-2}\Gamma(\frac{1}{2}(k+n+l))\Gamma(\frac{1}{2}(k+n-l))}$$

(cf. [7, p. 158 (5)–(7)]). The lemma now follows since the denominator has poles precisely when $|l| \in k+n+2\mathbf{Z}_+$.

7. The main theorem.

Let $\lambda \in \mathfrak{j}_{\mathbf{C}}^*$, let $m_0 = \lambda(iX)$ and

$$m_j = \frac{2(\lambda, \alpha_j)}{(\alpha_j, \alpha_j)} \quad (j=1,\ldots,r) .$$

Note that in Case I, $m_0 = 0$.

PROPOSITION 7.1. *If* $\lambda|_{\mathfrak{t}^+ \cap \mathfrak{t}_1} = 0$, *then* λ *is dominant integral (with respect to* $c_* \Delta^+$) *if and only if the following three conditions hold*:

(i) $m_0, m_1, \ldots, m_r$ *are integers satisfying* $|m_0| \leqq m_1 \leqq \ldots \leqq m_r$.

(ii) *In* Case I, *if* $\mathfrak{t}^+ \neq 0$, *then* $(-1)^{m_1} = \ldots = (-1)^{m_r}$.

(iii) *In* Case II, $(-1)^{m_0} = (-1)^{m_1} = \ldots = (-1)^{m_r}$.

PROOF. If $\mathfrak{t}^+=0$ the statement is obvious. Assume $\mathfrak{t}^+\neq 0$ and Case I, then $\mathfrak{t}^+\subset\mathfrak{t}_1$. Let β be a root of $\mathfrak{j}$ in $\mathfrak{g}_C$ which is not supported on $\mathfrak{a}$ and has restriction $\frac{1}{2}(\alpha_i\pm\alpha_j)$. Then necessarily β is a long root, and hence

$$\frac{2(\lambda,\beta)}{(\beta,\beta)} = \tfrac{1}{2}(m_i\pm m_j)\,. \tag{7.1}$$

The statement then follows from (7.1) and Theorem 2.1.

Assume next Case II. From Lemma 4.6, if β is a root of $\mathfrak{j}$ in $\mathfrak{g}_C$ and if $\beta|_{\mathfrak{a}}=\frac{1}{2}(\alpha_i\pm\alpha_j)$, then (7.1) holds again. On the other hand if $\beta|_{\mathfrak{a}}=\frac{1}{2}\alpha_i$ then

$$\frac{2(\lambda,\beta)}{(\beta,\beta)} = \frac{2(\lambda|_{\mathfrak{a}},\beta|_{\mathfrak{a}})}{(\beta,\beta)}+\frac{2\lambda(X)\beta(X)}{(X,X)(\beta,\beta)} = \begin{cases}\frac{1}{2}(m_i+m_0) & \text{if } \beta\in c_*\Xi_n\\ \frac{1}{2}(m_i-m_0) & \text{if } \beta\in c_*\Xi_c\,.\end{cases}$$

With that the proposition follows.

Assume now that π is a finite dimensional irreducible representation of G having λ as its highest weight. It is well known (cf. [9, Lemma 8.5.3]) that the space of N-fixed vectors for π is invariant under M, and that this representation δ of M is irreducible. Note that $\lambda|_{\mathfrak{t}^+}$ is a highest weight of δ.

THEOREM 7.2. *The following three conditions are equivalent.*

(i) $\lambda|_{\mathfrak{t}^+\cap\mathfrak{t}_1}=0$ *and* $(-1)^{m_1}=\ldots=(-1)^{m_r}$.

(ii) $\delta|_{M\cap K_1}$ *is trivial.*

(iii) π *has nonzero* K_1*-fixed vectors.*

If these conditions hold, then π *contains precisely the following one dimensional* K*-types, each contained once*:

In Case I: χ_l *for* $l=-m_1,-m_1+2,\ldots,m_1-2,m_1$.
In Case II: χ_{m_0}.

PROOF. First the equivalence of (i) and (ii) is proved. Obviously δ is trivial on $(M\cap K_1)_0$ if and only if $\lambda|_{\mathfrak{t}^+\cap\mathfrak{t}_1}=0$. We have

$$\delta\left(\exp\frac{2\pi i H_{\alpha_j}}{(\alpha_j,\alpha_j)}\right) = \exp\left(\frac{2\pi i(\lambda,\alpha_j)}{(\alpha_j,\alpha_j)}\right) = (-1)^{m_j} \tag{7.2}$$

and from Proposition 4.5 it follows therefore that (i) and (ii) are equivalent.

Let $\nu=\varrho+\lambda|_{\mathfrak{a}}\in\mathfrak{a}^*$ and let V be the representation space of δ. Consider the principal series representation $I^P_{\delta,\nu}$ of G, the definition of which we recall:

Let $C^P_{\delta,\nu}$ denote the space of V valued C^∞-functions f on G satisfying $f(gman)$

$=a^{-\nu-\varrho}\delta(m^{-1})f(g)$ for all $g \in G$, $m \in M$, $a \in A$, and $n \in N$. $I^P_{\delta,\nu}$ is then the representation by left translation on this space.

For the opposite minimal parabolic subgroup $\bar{P}=MA\bar{N}$ we define similarly $I^{\bar{P}}_{\delta,\nu}$ as the representation of G on

$$C^{\bar{P}}_{\delta,\nu} = \{f \in C^\infty(G,V) \mid f(gma\bar{n}) = a^{-\nu+\varrho}\delta(m^{-1})f(g),\ \forall g \in G,\ m \in M,\ a \in A,\ \bar{n} \in \bar{N}\}.$$

It is well known that π is equivalent to a subrepresentation of $I^P_{\delta,\nu}$ (cf. [9, Lemma 8.5.7]), and also that this subrepresentation can be realized as follows:

For each $f \in C^P_{\delta,\nu}$ and $x \in G$ the integral

$$Af(x) = \int_{\bar{N}} f(x\bar{n})\,d\bar{n}$$

is absolutely convergent and defines a G-homomorphism $A\colon C^P_{\delta,\nu} \to C^{\bar{P}}_{\delta,\nu}$ (cf. [9, Section 8.10]), whose image is equivalent to π (cf. [5, p. 75]).

Let $l \in \mathbf{Z}$. By Frobenius reciprocity χ_l occurs in the K-decomposition of $I^P_{\delta,\nu}$ if and only if $\delta=\chi_l|_M$. If χ_l occurs in π it also occurs in $C^P_{\delta,\nu}$, and therefore (iii) implies (ii), and also χ_l has multiplicity at most one in π.

Assume (ii). We will first determine the set of $l \in \mathbf{Z}$ such that $\delta=\chi_l|_M$. In Case I, $\mathfrak{m} \subset \mathfrak{k}_1$ and hence $\delta|_{M_0}=\chi_l|_{M_0}\equiv 1$ for all $l \in \mathbf{Z}$. By (4.3) and Lemma 4.1, $\delta=\chi_l|_M$ if and only if

$$\delta\left(\exp\frac{2\pi i H_{\gamma_j}}{(\gamma_j,\gamma_j)}\right) = \chi_l\left(\exp\frac{2\pi i H_{\gamma_j}}{(\gamma_j,\gamma_j)}\right). \tag{7.3}$$

By (5.1) the right hand side of (7.3) is $(-1)^l$. Comparing with (7.2) we see that $\delta=\chi_l|_M$ if and only if l has the same parity as $m_1,\ldots,m_r$. In Case II, M is connected, and since both δ and χ_l are trivial on $M \cap K_1$ it follows from Lemma 4.3 that $\delta=\chi_l|_M$ if and only if

$$\delta(\exp tX) = \chi_l(\exp tX) \qquad \text{for all } t \in \mathbf{R}.$$

However $\delta(\exp tX)=e^{im_0t}$ and $\chi_l(\exp tX)=e^{ilt}$ by Lemma 4.3 (iv). Therefore $\delta=\chi_l|_M$ if and only if $l=m_0$.

Assume now that l is such that $\delta=\chi_l|_M$. As mentioned χ_l then occurs in $I^P_{\delta,\nu}$. In fact, if we define for $g \in G$

$$f_l(g) = e^{-(\nu+\varrho)(H(g))}\chi_l(\varkappa(g))^{-1},$$

then $f_l \in C^P_{\delta,\nu}$ and $f_l(k^{-1}g)=\chi_l(k)f_l(g)$ for $k \in K$, $g \in G$, so f_l generates the K-type χ_l in $I^P_{\delta,\nu}$. Therefore π contains χ_l if and only if $Af_l \neq 0$.

From Iwasawa decomposition $G=KA\bar{N}$ it follows that $Af_l \neq 0$ if and only if $Af_l(e) \neq 0$, i.e. if and only if

$$(7.4) \qquad \int_{\bar N} e^{-(\nu+\varrho)(H(\bar n))}\chi_l(\varkappa(\bar n))^{-1} d\bar n \neq 0 .$$

Note that if $l=0$, (7.4) is obvious. This implies that π contains the trivial K-type χ_0 if and only if δ is trivial (this is the main step in the proof of Helgason's theorem, cf. [2, III Corollary 3.8]).

By the method of Gindikin and Karpelevič (see [9, Proof of Theorem 8.10.16]) the problem of proving (7.4) is reduced to the real rank-one case. Thus (7.4) holds if and only if

$$(7.5) \qquad \int_{\bar N^\alpha} e^{-(\nu+\varrho)(H(\bar n))}\chi_l(\varkappa(\bar n))^{-1} d\bar n \neq 0$$

for all $\alpha \in \Sigma^+ \setminus 2\Sigma^+$, where $\bar N^\alpha = G^\alpha \cap \bar N$.

When K^α is semisimple (7.5) is clear, so we may assume that $\mathfrak{g}^\alpha = \mathfrak{su}(n, 1)$ (cf. Lemma 5.1).

In Case I, we have $\mathfrak{g}^\alpha \cong \mathfrak{su}(1, 1)$ and $\chi_l|_{K^\alpha}$ is determined by (5.1). If $\alpha = \frac{1}{2}(\alpha_i - \alpha_j)$, then $\chi_l|_{K^\alpha}$ is trivial and (7.5) is obvious. If $\alpha = \alpha_j$, then by Lemma 6.1 we get that (7.5) holds precisely when

$$(7.6) \qquad |l| \notin \frac{2(\nu, \alpha)}{(\alpha, \alpha)} + 1 + 2\mathbf{Z}_+ = m_j + 2\mathbf{N} .$$

(It is easily seen that the conclusion of Lemma 6.1 also holds for any group covered by $\mathrm{SU}(n, 1)$, as long as χ_l is well defined on this group.) Finally, if $\alpha = \frac{1}{2}(\alpha_i + \alpha_j)$ we get that (7.5) holds when

$$(7.7) \qquad |2l| \notin \frac{2(\nu, \alpha)}{(\alpha, \alpha)} + 1 + 2\mathbf{Z}_+ = m_i + m_j + 2\mathbf{N} .$$

By Proposition 7.1 (i), we see that (7.6) and (7.7) holds precisely when $|l| \leqq m_1$, and thus the theorem follows in Case I.

In Case II we have $\alpha = \frac{1}{2}\alpha_j$ and $\chi_l|_{K^\alpha}$ is determined by (5.3). From Lemma 6.1 we get that (7.5) holds when

$$|l| \leqq \frac{2(\nu, \alpha_j)}{(\alpha_j, \alpha_j)} = m_j .$$

However by our assumption that $\delta = \chi_l|_M$ we have $l = m_0$, and therefore (7.5) holds by Proposition 7.1 (i).

REMARK 7.3. From the proof of Lemma 6.1 it follows that the integral

$$c(\nu, l) = \int_{\bar N} e^{-(\nu+\varrho)(H(\bar n))}\chi_l(\varkappa(\bar n))^{-1} d\bar n$$

for $G = \mathrm{SU}(n,1)$ takes the following value

$$\frac{2^{n-k}\Gamma(n)\Gamma(k)}{\Gamma(\frac{1}{2}(k+n+l))\Gamma(\frac{1}{2}(k+n-l))}. \tag{7.8}$$

Here $d\bar{n}$ is so normalized that $c(\varrho, 0)$ equals one. From the proof of Theorem 7.2 it then follows that $c(\nu, l)$ for arbitrary G can be given an explicit formula as product of expressions like (7.8) and the usual factors in the product formula for the c-function (cf. [1]).

ACKNOWLEDGEMENTS. The author is grateful to D. Vogan and H. P. Jakobsen for helpful discussions.

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