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ARITHMETICAL QUADRATIC SURFACES OF GENUS 0, II

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Introduction.

If $f \in \mathbf{Z}[X_0, X_1, X_2]$ is a primitive ternary quadratic form, then the singularities of the Spec (Z)-scheme $M(f) = \operatorname{Proj}(\mathbf{Z}[X_0, X_1, X_2]/(f))$ depend on the prime numbers dividing the determinant $\mathfrak{d}(f)$ of f. In the first part of this paper we were concerned with the regular schemes $M(f)$. Such schemes correspond to hereditary quadratic forms f, that is, the quadratic forms with $\mathfrak{d}(f)$ square-free. This part contains two results about arbitrary schemes $M(f)$. First of all, we prove (Theorem (2.1)) that the normalization of $M(f)$ is $M(B(f))$, where $B(f)$ is the Bass closure of f (the quadratic form $B(f)$ will be defined in section 1). In particular, $M(f)$ is normal if and only if $f = B(f)$ is a Bass form (see [7] for different characterizations of Bass forms). The second result (Theorem (2.12)) shows how to resolve the singularities of a normal scheme $M(f)$. We prove that for each Bass non-hereditary quadratic form f there exists a chain of quadratic forms $f_0 = f, f_1, \ldots, f_n$ such that $M(f_{i+1})$ is an elementary transform of $M(f_i)$ at the singular points of the fiber $M(f_i)_{p_i}$, p_i a prime number, and $M(f_n)$ is regular. Hence it is possible to consider a process of resolution of singularities of a quadratic form f which consists of its normalization $B(f)$ and a chain of elementary transformations starting with $B(f)$ and ending with a hereditary quadratic form (Proposition (1.8) and (2.24)). A similar result is true if Spec (Z) is replaced by Spec $(k[T])$ (or Proj $(k[T_0, T_1])$) for a perfect field k, and f by a ternary quadratic form whose coefficients are polynomials in T (respectively, homogeneous polynomials of the same degree in T_0, T_1). The process of resolution of singularities by using quadratic forms only is closely related to the usual process of resolution of singularities by normalizations and blowings-up. More details about this will be published in a subsequent paper by P. Salberger.

The paper consists of two sections. In the first we prove a number of results about extensions of lattices on ternary quadratic spaces. This part depends strongly on [6] and [7]. In section 2 we prove the main results formulated above in the particular case of integral quadrtic forms. These results are valid if

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Spec (Z) or Proj ($k[T_0, T_1]$) is replaced by an arbitrary perfect Dedekind scheme S (that is, an integral, noetherian and normal scheme of dimension 1 such that the residue fields $k(\mathfrak{p})$ are perfect for all closed points $\mathfrak{p} \in S$), and the quadratic forms are replaced by arbitrary S-lattices (that is, locally free S-sheaves of rank 3) on a half-regular ternary quadratic space over the field of rational functions on S. Note that considering a perfect Dedekind scheme instead of $S = \mathrm{Spec}(A)$, A a perfect Dedekind ring, we get a slight generalization of the situation in [4] (we discuss this change in the relevant places).

The paper is a continuation of [4] and we follow the notations and definitions introduced there.

1. Extensions of lattices.

Let A be a complete discrete valuation ring with maximal ideal $\mathfrak{p} = (\pi)$ and residue field $k = A/\mathfrak{p}$ which will be assumed to be perfect. Let F be the quotient field of A and (V, q) a half-regular quadratic space over F ([4, p. 185]). We follow the notations introduced in [4, sections 1 and 3]. In particular, $\mathfrak{n}(L), \mathfrak{v}(L), \mathfrak{d}(L)$ denote respectively the norm, the volume and the determinant of an A-lattice L on V (recall that $\mathfrak{d}(L) = \mathfrak{v}(L)\mathfrak{n}(L)^{-3}$). If $L = \sum Ae_i$ for an A-basis $\{e_i\}$ of L and $\mathfrak{n}(L) = (a)$, $a \in F^*$, then

$$(1.1) \qquad q_L = (1/a)\left(\sum_i q(e_i)X_i^2 + \sum_{i<j} b(e_i, e_j)X_iX_j\right),$$

where $b(x, y) = q(x+y) - q(x) - q(y)$ for $x, y \in V$, is a quadratic form corresponding to L. $\mathfrak{O}(L)$ denotes the A-order corresponding to L in the quaternion algebra $Q = C_0(V, q)$ (see [4, p. 192]). N and T denote the reduced norm and trace in Q. We recall some facts contained in [6] and [7]. An A-lattice L on V is called a Bass lattice if the order $\mathfrak{O}(L)$ is a Bass order, that is, each order in Q containing it is a Gorenstein order. An A-order Λ in Q is called a Gorenstein order if $\Lambda^{\#} = \{x \in Q\colon T(x\Lambda) \subset \Lambda\}$ is a projective left (or right) Λ-module. An A-order Λ in Q is a Gorenstein order if and only if $\Lambda = \mathfrak{O}(L)$ for an A-lattice L on V ([5, (3.4)]). If Λ is an arbitrary A-order in Q, then there is an A-ideal $\mathfrak{b}(\Lambda) \subset A$ and a uniquely determined Gorenstein order $G(\Lambda) \supset \Lambda$ such that $\Lambda = \langle 1, \mathfrak{b}(\Lambda)G(\Lambda)\rangle$ (for a subset X of an A-module we denote by $\langle X\rangle$ the submodule generated by X). We have $\mathfrak{b}(\Lambda) = \mathfrak{d}(\Lambda)N(\Lambda^{\#})$, where $\mathfrak{d}(\Lambda)$ is the discriminant of Λ ([6, (1.4)]). $G(\Lambda)$ is called the Gorenstein closure of Λ. Let $e_{\mathfrak{p}}(L) = e_{\mathfrak{p}}(\mathfrak{O}(L))$, where $e_{\mathfrak{p}}(\Lambda)$ was defined in [6, (1.8)] in such a way that

$$2^{|e_{\mathfrak{p}}(\Lambda)|} = \dim_k(\Lambda/J(\Lambda))$$

and $e_{\mathfrak{p}}(\Lambda) < 0$ if and only if $\Lambda/J(\Lambda)$ is a quadratic field extension of k ($J(\Lambda)$ is the

Jacobson radical of Λ). $|e_{\mathfrak{p}}(L)|+1$ is equal to the rank over k of q_L modulo $\mathfrak{p}$ and $e_{\mathfrak{p}}(L)<0$ if and only if q_L has rank 2 modulo $\mathfrak{p}$ and is irreducible over k ([7, section 3]). We write $c_{\mathfrak{p}}(V)=1$ if $Q=C_0(V,q)$ is non-ramified and $c_{\mathfrak{p}}(V)=-1$ if Q is ramified over F. Note that in the first case the quadratic space (V,q) may be anisotropic.

(1.2) PROPOSITION. *If* $e_{\mathfrak{p}}(L)\neq 0$ *or* $\mathfrak{d}(L)\supset \mathfrak{p}^3$, *then* L *is a Bass lattice.*

PROOF. The first part follows directly from the fact that any order Λ with $e_{\mathfrak{p}}(\Lambda)\neq 0$ is a Bass order ([6, (2.4) and (3.1)]). To prove the second part, let us note that if $\Lambda=\mathfrak{O}(L)$ is not a Bass order, then there is a unique overorder $\Lambda'\supset\Lambda$ such that $[\Lambda':\Lambda]=\mathfrak{p}$ and $\mathfrak{b}(\Lambda')=\mathfrak{p}$ ([6, (4.4)]). Thus $[G(\Lambda'):\Lambda]=\mathfrak{p}^4$, which gives a contradiction, since $\mathfrak{d}(\Lambda)=\mathfrak{p}^4\mathfrak{d}(G(\Lambda'))$.

Now we are going to prove a number of results about minimal extensions of lattices which are direct consequences of the corresponding results for orders discussed in [6].

(1.3) PROPOSITION. *If* L *is a non-Bass lattice on* (V,q), *then there exists exactly one lattice* $L'\supset L$ *such that* $[L':L]=\mathfrak{p}^2$ *and* $\mathfrak{d}(L)=\mathfrak{p}^4\mathfrak{d}(L')$. *Moreover,* $\mathfrak{p}L'\subset L$ *and* $\mathfrak{n}(L')=\mathfrak{n}(L)$.

PROOF. Scaling the quadratic space (V,q) we may assume that $\mathfrak{d}(L)=\mathfrak{n}(L)=\mathfrak{p}^n$, where $n\geq 4$ by (1.2). The order $\Lambda=\mathfrak{O}(L)$ in $Q=C_0(V,q)$ is a Gorenstein non-Bass order, so $e_{\mathfrak{p}}(\Lambda)=0$ by (1.2). Let Λ' be the minimal overorder of Λ (see [6, (4.4)]), and $G(\Lambda')$ its Gorenstein closure. Since $\mathfrak{b}(\Lambda')=\mathfrak{p}$, we have $[G(\Lambda'):\Lambda']=\mathfrak{p}^3$ and $\mathfrak{d}(G(\Lambda'))=\mathfrak{p}^{n-4}$. According to [6, (4.4)], $G(\Lambda')=\langle 1,x_1,x_2,x_3\rangle$, $\Lambda'=\langle 1,\pi x_1,\pi x_2,\pi x_3\rangle$ and $\Lambda=\langle 1,\pi x_1,\pi x_2,\pi^2 x_3\rangle$, where x_i are suitable elements of Q. Let $G(\Lambda')^{\natural}=\langle y_0,y_1,y_2,y_3\rangle$. Using the correspondance of [4, (3.6)], we get

$$L = \mathfrak{L}(\Lambda) = \pi^{n-2}\langle \pi y_1,\pi y_2,y_3\rangle .$$

Choose

$$L' = \pi^{n-2}\langle y_1,y_2,y_3\rangle = \mathfrak{p}^2\mathfrak{L}(G(\Lambda')) .$$

It is easy to see that $L'\supset L$, $\mathfrak{p}L'\subset L$ and

$$\mathfrak{d}(L') = \mathfrak{d}(\mathfrak{L}(G(\Lambda'))) = \mathfrak{d}(G(\Lambda')) = \mathfrak{p}^{n-4} .$$

The equality $\mathfrak{n}(L')=\mathfrak{n}(L)$ is also satisfied since $\mathfrak{d}(L)=\mathfrak{p}^4\mathfrak{d}(L')$ and $\mathfrak{v}(L)=\mathfrak{p}^4\mathfrak{v}(L')$.

In order to prove the uniqueness of L' let us choose a basis y_1,y_2,y_3 of V such that $L'=Ay_1+Ay_2+Ay_3$ and $L=A\pi y_1+A\pi y_2+Ay_3$. The equalities $\mathfrak{d}(L)$

$=\mathfrak{p}^4\mathfrak{d}(L')$ and $\mathfrak{v}(L)=\mathfrak{p}^4\mathfrak{v}(L')$ imply $\mathfrak{n}(L)=\mathfrak{n}(L')$. Hence $\mathfrak{O}(L')=\langle 1, x_1, x_2, x_3\rangle$ and $\mathfrak{O}(L)=\langle 1, \pi x_1, \pi x_2, \pi^2 x_3\rangle$, where $x_i=\pi^{-n}[y_j, y_k]$ and i,j,k is a cyclic permutation of 1, 2, 3 (see the notation of [4, p. 192]). We have $\Lambda=\mathfrak{O}(L)$, $\Lambda'=\langle 1, \pi\mathfrak{O}(L')\rangle$ is the minimal overorder of Λ and $G(\Lambda')=\mathfrak{O}(L')$. Let $L''=\mathfrak{p}^{-2}L'$. Then $\mathfrak{n}(L'')=\mathfrak{d}(L'')=\mathfrak{p}^{n-4}$, so $L''=\mathfrak{L}(\mathfrak{O}(L''))$ by [4, (3.6)]. Since $G(\Lambda')=\mathfrak{O}(L')=\mathfrak{O}(L'')$ is uniquely determined by L, we get that $L'=\mathfrak{p}^2\mathfrak{L}(G(\Lambda'))$ is also uniquely determined by L.

If L is a non-Bass Λ-lattice on (V, q), then using (1.3) we can construct a uniquely determined chain of lattices

$$L = L_0 \subset L_1 \subset \dots \subset L_n \tag{1.4}$$

such that $[L_{i+1}: L_i]=\mathfrak{p}^2$, $\mathfrak{d}(L_i)=\mathfrak{p}^4\mathfrak{d}(L_{i+1})$ and L_i are non-Bass lattices for $i=0, 1, \dots, n-1$, while L_n is a Bass lattice. The lattice L_n will be denoted by $B(L)$ and called the Bass closure of L (or the normal closure of L—for this term see (2.1)). Let us note that the chain (1.4) corresponds in terms of orders (see [6, (4.4)]) to the chain:

$$\Lambda = \Lambda_0 \subset \Lambda_1 \subset \dots \subset \Lambda_n, \tag{1.4$'$}$$

where Λ_i are Gorenstein non-Bass orders and $\Lambda_{i+1}=G(\Lambda_i')$ is the Gorenstein closure of the minimal overorder Λ_i' of Λ_i for $i=0, 1, \dots, n-1$, while Λ_n is a Bass order. We shall call Λ_n the Bass closure of Λ and denote it by $B(\Lambda)$. The lattices in (1.4) and the orders in (1.4)′ are related by the correspondence of [4, (3.6)], that is, $L_i=\mathfrak{L}(\Lambda_i)$ and $\Lambda_i=\mathfrak{O}(L_i)$. In particular, $B(L)=\mathfrak{L}(B(\mathfrak{O}(L)))$.

Now we shall look at the minimal extensions of Bass lattices. We omit some details of the proofs since the arguments are similar to those in the proof of (1.3). We start with a generalization of [4, (1.8)]:

(1.5) PROPOSITION. *Let L be a Bass lattice on (V, q) such that $e_{\mathfrak{p}}(L)=1$ and $\mathfrak{d}(L)=\mathfrak{p}^n$, $n\geqq 1$. Then there are exactly two lattices L' and L'' containing L such that $[L':L]=[L'':L]=\mathfrak{p}^2$ and $\mathfrak{d}(L')=\mathfrak{d}(L'')=\mathfrak{p}^{n-1}$. Moreover, $\mathfrak{p}L'\subset L$, $\mathfrak{p}L''\subset L$, $\mathfrak{n}(L')=\mathfrak{n}(L'')=\mathfrak{p}^{-1}\mathfrak{n}(L)$, and if $n\geqq 2$, then $e_{\mathfrak{p}}(L')=e_{\mathfrak{p}}(L'')=1$.*

PROOF. We may assume that $\mathfrak{d}(L)=\mathfrak{n}(L)=\mathfrak{p}^n$, $n\geqq 1$. The order $\Lambda=\mathfrak{O}(L)$ has exactly two minimal overorders since $e_{\mathfrak{p}}(\Lambda)=1$ by [6, (2.3)]. These orders Λ' and Λ'' satisfy $[\Lambda':\Lambda]=[\Lambda'':\Lambda]=\mathfrak{p}$ and $e_{\mathfrak{p}}(\Lambda')=e_{\mathfrak{p}}(\Lambda'')=1$ if $n\geqq 2$. Using [6, (5.4) and (2.3)] we get $\Lambda=\langle 1, x_1, x_2, x_3\rangle$, $\Lambda'=\langle 1, x_1, \pi^{-1}x_2, x_3\rangle$ and $\Lambda''=\langle 1, x_1, x_2, \pi^{-1}x_3\rangle$, where x_i are suitable elements of Q. Let $L'=\mathfrak{L}(\Lambda')$ and $L''=\mathfrak{L}(\Lambda'')$ (see [4, (3.6)]). Then it is easy to check that the lattices L' and L'' satisfy all the requirements. In order to prove the uniqueness let us note that the equalities $[L': L]=[L'': L]=\mathfrak{p}^2$ and $\mathfrak{d}(L)=\mathfrak{p}\mathfrak{d}(L')=\mathfrak{p}\mathfrak{d}(L'')$ imply that

$\mathfrak{n}(L')=\mathfrak{n}(L'')=\mathfrak{p}^{-1}\mathfrak{n}(L)$. Hence $\mathfrak{n}(L')=\mathfrak{d}(L')$ and $\mathfrak{n}(L'')=\mathfrak{d}(L'')$, so by [4, (3.6)], we get $L'=\mathfrak{L}(\mathfrak{O}(L'))$ and $L''=\mathfrak{L}(\mathfrak{O}(L''))$. But it is easy to check that $\mathfrak{O}(L')$ and $\mathfrak{O}(L'')$ are the two minimal overorders of $\mathfrak{O}(L)$, so L' and L'' are uniquely determined by L.

(1.6) PROPOSITION. *Let L be a Bass lattice on (V,q) such that $e_{\mathfrak{p}}(L)=-1$ and $\mathfrak{d}(L)=\mathfrak{p}^n$, $n\geqq 2$. Then there is exactly one lattice L' containing L such that $[L'\colon L]=\mathfrak{p}$ and $\mathfrak{d}(L')=\mathfrak{p}^{n-2}$. Moreover, $\mathfrak{p}L'\subset L$ and $\mathfrak{n}(L')=\mathfrak{n}(L)$. If $c_{\mathfrak{p}}(V)=1$, then n is even and $e_{\mathfrak{p}}(L')=-1$ when $n\geqq 4$. If $c_{\mathfrak{p}}(V)=-1$, then n is idd and $e_{\mathfrak{p}}(L')$ $=-1$.*

PROOF. As earlier, we assume that $\mathfrak{d}(L)=\mathfrak{n}(L)=\mathfrak{p}^n$. If $\Lambda=\mathfrak{O}(L)$, then $e_{\mathfrak{p}}(\Lambda)=-1$, so there exists a unique minimal overorder Λ' of Λ such that $[\Lambda'\colon\Lambda]=\mathfrak{p}^2$ (see [6, (3.1)]). By [6, (3.2)], $\Lambda'=\langle 1,x_1,x_2,x_3\rangle$ and $\Lambda=\langle 1,x_1,\pi x_2,\pi x_2\rangle$ for suitable elements $x_i\in Q$. Now it is easy to check that the lattice $L'=\mathfrak{p}\mathfrak{L}(\Lambda')$ satisfies the requirements. To prove the uniqueness, note that the assumptions $[L'\colon L]=\mathfrak{p}$ and $\mathfrak{d}(L)=\mathfrak{p}^2\mathfrak{d}(L')$ imply that $\mathfrak{n}(L')=\mathfrak{n}(L)$. If we define $L''=\mathfrak{p}^{-1}L'$, then

$$\mathfrak{d}(L'') = \mathfrak{d}(L') = \mathfrak{p}^{n-2}$$

and

$$\mathfrak{n}(L'') = \mathfrak{p}^{-2}\mathfrak{n}(L') = \mathfrak{p}^{n-2},$$

so $\mathfrak{d}(L'')=\mathfrak{n}(L'')$. Thus $L''=\mathfrak{L}(\mathfrak{O}(L''))$ by [4, (3.6)]. Now it is easy to check that $\Lambda'=\mathfrak{O}(L')=\mathfrak{O}(L'')$ is the minimal overorder of $\Lambda=\mathfrak{O}(L)$. Hence $L'=\mathfrak{p}\mathfrak{L}(\mathfrak{O}(L'))$ is uniquely determined by L.

(1.7) PROPOSITION. *Let L be a Bass lattice on (V,q) such that $e_{\mathfrak{p}}(L)=0$ and $\mathfrak{d}(L)=\mathfrak{p}^n$, $n\geqq 2$. Then there exists exactly one lattice L' containing L such that $[L'\colon L]=\mathfrak{p}^2$ and $\mathfrak{d}(L')=\mathfrak{p}^{n-1}$. Moreover, $\mathfrak{p}L'\subset L$, $\mathfrak{n}(L')=\mathfrak{p}^{-1}\mathfrak{n}(L)$, L' is also a Bass lattice, $e_{\mathfrak{p}}(L')=0$ if $n\geqq 3$ and $e_{\mathfrak{p}}(L')=c_{\mathfrak{p}}(V)$ if $n=2$.*

PROOF. Assume, as earlier, that $\mathfrak{d}(L)=\mathfrak{n}(L)=\mathfrak{p}^n$, $n\geqq 2$. This time $\Lambda=\mathfrak{O}(L)$ is a Bass order with $e_{\mathfrak{p}}(\Lambda)=0$ and $\mathfrak{d}(\Lambda)=\mathfrak{p}^n$, $n\geqq 2$. Hence there is exactly one minimal overorder Λ' of Λ and $[\Lambda'\colon\Lambda]=\mathfrak{p}$ by [6, (4.1)]. We have, $\Lambda'=\langle 1,x_1,x_2,x_3\rangle$ and $\Lambda=\langle 1,x_1,x_2,\pi x_3\rangle$, where $x_i\in Q$. Defining $L'=\mathfrak{L}(\Lambda')$ it is easy to check that we get a required extension of L. To prove its uniqueness, we note as in the previous cases, that the assumptions $[L'\colon L]=\mathfrak{p}^2$ and $\mathfrak{d}(L)=\mathfrak{p}\mathfrak{d}(L')$ imply that $\mathfrak{n}(L)=\mathfrak{p}\mathfrak{n}(L')$. Hence $\mathfrak{d}(L')=\mathfrak{n}(L')$, which implies $L'=\mathfrak{L}(\mathfrak{O}(L'))$ by [4, (3.6)]. But we check easily that $\Lambda'=\mathfrak{O}(L')$ is the minimal overorder of $\Lambda=\mathfrak{O}(L)$, so L' is uniquely determined by L. The two last statements of the Proposition follow at once from [6, (4.1) and (1.2)].

We end this section with some remarks concerning global versions of the above notions and results. Let S be an arbitrary perfect Dedekind scheme (see the Introduction), F the field of rational functions on S, and (V, q) a half-regular ternary quadratic space over F. An S-lattice L on V is a locally free S-subsheaf of rank 3 of the constant S-sheaf V. The group operation in the divisor group Div (S) will be denoted multiplicatively. The determinant $\mathfrak{d}(L)$ and the norme $\mathfrak{n}(L)$ are the elements of Div (S) such that $\mathfrak{d}(L)_{\mathfrak{p}} = \mathfrak{d}(L_{\mathfrak{p}})$ and $\mathfrak{n}(L)_{\mathfrak{p}} = \mathfrak{n}(L_{\mathfrak{p}})$ for each closed point $\mathfrak{p} \in S$. If L and L' are S-lattices on V, then $[L' : L]$ is the divisor on S defined by the condition $[L' : L]_{\mathfrak{p}} = [L'_{\mathfrak{p}} : L_{\mathfrak{p}}]$ for each closed point $\mathfrak{p} \in S$. If L is an S-lattice and $\mathfrak{p}$ a closed point of S, then we shall say that L is a Bass lattice at $\mathfrak{p}$ if $L_{\mathfrak{p}}$ is a Bass $O_{S,\mathfrak{p}}$-lattice. L is a Bass S-lattice if L is a Bass lattice at $\mathfrak{p}$ for each $\mathfrak{p} \in S$. This means that each stalk $\mathfrak{O}(L)_{\mathfrak{p}} = \mathfrak{O}(L_{\mathfrak{p}}) \subset C_0(V, q)$ of the sheaf of S-orders $\mathfrak{O}(L)$ is a Bass $O_{S,\mathfrak{p}}$-order (note that $\mathfrak{O}(L)(U) = \bigcap \mathfrak{O}(L_{\mathfrak{p}})$, $\mathfrak{p} \in U$, for any open subset U of S). If $\mathfrak{a}$ is an invertible sheaf on S considered as an S-subsheaf of the constant sheaf F, then we shall denote by $\mathfrak{a}L$ the tensor product $\mathfrak{a} \otimes L$ considered as an S-lattice on V. If $\mathfrak{p} \in S$ is a closed point, define $e_{\mathfrak{p}}(L) = e_{\mathfrak{p}}(L_{\mathfrak{p}})$. By the well-known "local-global principle" describing lattices on V in terms of their local components $L_{\mathfrak{p}}$ (see e.g. [2, Chapter 7, § 4, Theorem 3 and Proposition 4, Cor.], (1.3) is valid if $L_{\mathfrak{p}}$ is not a Bass lattice at $\mathfrak{p}$, (1.5) if $e_{\mathfrak{p}}(L) = 1$, (1.6) if $e_{\mathfrak{p}}(L) = -1$, and (1.7) if $e_{\mathfrak{p}}(L) = 0$. The Bass closure of L can be defined as the uniquely determined S-lattice $B(L)$ such that $B(L)_{\mathfrak{p}} = B(L_{\mathfrak{p}})$ for each closed point $\mathfrak{p} \in S$. These remarks in connection with the local results of this Section imply the following result:

(1.8) PROPOSITION. *Let S be a perfect Dedekind scheme and (V, q) a half-regular quadratic space over the field of rational functions on S.*

(a) *If L is a non-Bass lattice on V, then there is a chain*

(1.9) $$L = L_0 \subset L_1 \ldots \subset L_m = B(L)$$

such that $L_i \subset L_{i+1}$ satisfies Proposition (1.3) (*with $L = L_i$, $L' = L_{i+1}$, $\mathfrak{p} = \mathfrak{p}_i$ a closed point of S dividing $\mathfrak{d}(L_i)$*) *and L_i is not a Bass lattice at $\mathfrak{p}_i$ for $i = 0, 1, \ldots, m-1$, while $L_m = B(L)$ is the Bass closure of L.*

(b) *If L is a Bass non-hereditary lattice on V, then there is a chain*

(1.10) $$L = L_0 \subset L_1 \subset \ldots \subset L_n$$

such that $L_i \subset L_{i+1}$ satisfies one of the Propositions (1.5), (1.6) *or* (1.7) (*with $L = L_i$, $L' = L_{i+1}$, $\mathfrak{p} = \mathfrak{p}_i$ a closed point of S dividing $\mathfrak{d}(L_i)$*) *and L_i is a Bass non-hereditary lattice for $i = 0, 1, \ldots, n-1$, while L_n is a hereditary lattice.*

(c) *If $\mathfrak{p}_i^2 \mid \mathfrak{d}(L_i)$, then L_{i+1} in* (1.9) *or* (1.10) *is uniquely determined by the* *condition* $\mathfrak{p}_i \mid [L_{i+1}: L_i]$ *unless* $e_{\mathfrak{p}_i}(L_i)=1$ *in which case there are two possibilities for* L_{i+1} *(this is only possible in* (1.10)).

(1.11) REMARK. Let L be a given lattice on V and $\mathfrak{d}(L)=\prod \mathfrak{p}^{\alpha(\mathfrak{p})}$, $\mathfrak{p} \in S$. In order to construct the chains (1.9) and (1.10), we can check whether $L_{\mathfrak{p}}$ is a Bass lattice at $\mathfrak{p}$ or not using the characterizations of Bass lattices given in [7, (3.4) and (3.5)]. If $L=L_0$ is not a Bass lattice at $\mathfrak{p} \mid \mathfrak{d}(L)$, then by the proof of (1.3) it is possible to construct $L_1=L'$. If $L=L_0$ is a Bass lattice and $\alpha(p) \geqq 2$, it is possible to compute $e_{\mathfrak{p}}(L)$ and afterwards to construct $L_1=L'$ by the methods described in the proofs of (1.5), (1.6) or (1.7). Note however that the same point $\mathfrak{p}$ can appear several times.

2. Resolution of singularities.

Let S be a perfect Dedekind scheme with field of rational functions F and E a finitely generated regular extension of F of genus 0. Let (V,q) be a half-regular ternary quadratic space corresponding to the extension E/F (see [4, p. 195]). Recall that an S-model of E is an integral S-scheme $\alpha: M \to S$, where α is a proper, dominant morphism and the induced map $\alpha^*: R(S) \to R(M)$ of the fields of rational functions on S and M is an embedding of $F=R(S)$ into $E=R(M)$. In [4, p. 183] we defined regular quadratic S-models of E and proved that each model of that type is S-isomorphic to an S-model $M(L)$, where L is a hereditary S-lattice on V ([4, (4.5)]). Now we shall consider models $\alpha_L: M(L) \to S$ corresponding to arbitrary S-lattices L on V (see [4, p. 189] for the defintion of α_L). In order to underline the presence of L already in the definition of $M=M(L)$ we shall call M a conic bundle (S-)surface. We shall identify the S-models M of E with the corresponding sets of local subrings $O_{M,x} \subset E$ for $x \in M$. If $M=M(L)$, define $\mathfrak{d}(M)=\mathfrak{d}(L)$—this generalizes the definition of $\mathfrak{d}(M)$ given in [4, p. 183]. Define also $e_{\mathfrak{p}}(M(L))=e_{\mathfrak{p}}(L)$.

If A is a discrete valuation ring with maximal ideal $\mathfrak{p}=(\pi)$ and X an A-module, then we denote by $\hat{X}$ the completion of X in the $\mathfrak{p}$-adic topology. If $q_{\hat{L}} \in A[X_0, X_1, X_2]$ is a quadratic form corresponding to the $\hat{A}$-lattices $\hat{L}$ on $(\hat{V},q)$ (see (1.1)), where L is an A-lattice on (V,q), then for each integer $N>0$ there is a quadratic form q_L corresponding to L such that $q_L=q_{\hat{L}}+\pi^N q$ with $q \in A[X_0, X_1, X_2]$. Throughout the proofs in this section it will be convenient to write $q_L \equiv q_{\hat{L}}$ without specifying N which should be chosen sufficiently large in order to make the arguments correct.

(2.1) THEOREM. (a) *$M(L)$ is normal if and only if L is a Bass lattice.*

(b) *The surface $M(B(L))$, where $B(L)$ is the Bass closure of L, is the normalization of $M(L)$. In particular, the normalization of a conic bundle surface is again a conic bundle surface.*

PROOF. By its definition [4, p. 189], $M(L)$ is a Cohen–Maccaulay scheme since $M(L)$ is a local complete intersection in the projective 2-space over S (see e.g. [10, pp. 108–109]). Thus, in order to prove that $M(L)$ is normal, it suffices to check that it is regular in codimension 1 (see e.g. [10, p. 125]). Since the generic fiber of the morphism $\alpha_L\colon M(L) \to S$ is regular for any S-lattice L, we have to prove that if L is a Bass lattice, then the local rings on $M(L)$ of non-closed points in the fiber $M(L)_{\mathfrak{p}}$ are discrete valuation rings for each closed $\mathfrak{p} \in S$. We already know this if $\mathfrak{d}(M(L))$ is square-free ([4, (2.3)]). We may assume that $S = \operatorname{Spec}(A)$, where A is a discrete valuation ring and $\mathfrak{p}^2 \mid \mathfrak{d}(M(L))$, where $\mathfrak{p} = (\pi)$ is the maximal ideal of A. We shall consider 3 cases:

CASE 1. If $e_{\mathfrak{p}}(L) = 1$ and $\Lambda = \mathfrak{O}(L)$, then $\hat{\Lambda} \cong E_n^{(1)}$, $n \geqq 2$ by [6, (2.1) and (5.4)]. With the choice of the basis for $\hat{\Lambda}$ in [6, (5.4)], we have $L = \mathfrak{L}(\Lambda) = Ae_0 + Ae_1 + Ae_2$, where e_0, e_1, e_2 is a basis for L such that (see (1.1)):

$$q_L \equiv \pi^n X_0^2 + X_1 X_2 . \tag{2.2}$$

Hence the ideal in $A[X_0, X_1, X_2]/(q_L) = A[x_0, x_1, x_2]$ defining $M(L)_{\mathfrak{p}}$ is (π) and there are two projective primes in the fiber: $l_1 = (\pi, x_1)$ and $l_2 = (\pi, x_2)$. Since $x_0 \notin l_i$ $(i = 1, 2)$ and $l_1 \cap l_2 = (\pi)$, the local ring at l_i on $M(L)$ is regular with maximal ideal generated by π. Note that the only singular point of the fiber (on $M(L)$) is the intersection point (π, x_1, x_2) of the two projective primes.

CASE 2. If $e_{\mathfrak{p}}(L) = -1$ and $\Lambda = \mathfrak{O}(L)$, then $\hat{\Lambda} \cong E_n^{(-1)}$ or $\Gamma_n^{(-1)}$, $n \geqq 1$ by [6, (5.4), (5.6), (5.8)]. Using the bases for $\hat{\Lambda}$ given in [6, (5.4), (5.6) or (5.8)] we get $L = \mathfrak{L}(\Lambda) = Ae_0 + Ae_1 + Ae_2$ and

$$q_L \equiv \pi^n X_0 X_1 + X_1^2 - X_1 X_2 + \varepsilon X_2^2 , \tag{2.3}$$

$$q_L \equiv \pi^{2n+1} \eta X_0^2 - X_1^2 + X_1 X_2 - \varepsilon X_2^2 , \tag{2.4}$$

$$q_L \equiv \pi^{2n} X_0^2 - 2\alpha\pi^n X_0 X_1 - 2\beta\pi^n X_0 X_2 + (\alpha^2 - \varepsilon) X_1^2 + 2\alpha\beta X_1 X_2 + (\beta^2 - \eta) X_2^2 , \tag{2.5}$$

where we have (2.3) if $\hat{\Lambda} \cong E_n^{(n-1)}$, (2.4) if $\hat{\Lambda} \cong \Gamma_n^{(-1)}$ and $c_{\mathfrak{p}}(V) = -1$ and (2.5) if $\hat{\Lambda} \cong \Gamma_n^{(-1)}$ and $c_{\mathfrak{p}}(V) = 1$. In (2.3) and (2.4), $\varepsilon \in A$ is such that $X^2 - X + \varepsilon$ is irreducible over $k(\mathfrak{p})$. In (2.4), $\eta \in A^*$. In (2.5), $\alpha, \beta \in A$, $\varepsilon, \eta \in A^*$ and the quaternion algebra (ε, η) is a skewfield over $k(\mathfrak{p})$. Hence the ideal in $A[X_0, X_1, X_2]/(q_L) = A[x_0, x_1, x_2]$ defining the fiber $M(L)_{\mathfrak{p}}$ is $l = (\pi, \mu) = (\pi)$, where $\mu = x_1^2 - x_1 x_2 + \varepsilon x_2^2$ in (2.3) and (2.4), $\mu = (\alpha^2 - \varepsilon) x_1^2 + 2\alpha\beta x_1 x_2 + (\beta^2 - \eta) x_2^2$ in (2.5) (so the fiber is a form of two intersecting projective primes). Since e.g. $x_0 \notin l$, the local ring at l on $M(L)$ is regular with maximal ideal generated by π. Note that the only singular point of the fiber (on $M(L)$) is the $k(\mathfrak{p})$-rational point (π, x_1, x_2).

CASE 3. If $e_{\mathfrak{p}}(L)=0$ and $\Lambda=\mathfrak{O}(L)$, then $\hat{\Lambda}\cong E_n^{(0)}$, $n\geqq 2$ or $\Gamma_n^{(0)}$, $n\geqq 1$, by [6, (5.4), (5.6), (5.8)]. With the bases for $\hat{\Lambda}$ chosen according to [6, (5.4) with $\alpha_1x_1+\alpha_2x_2+\pi x_3$ replaced by $\pi\alpha x_1+x_2+\pi\beta x_3$, where $\alpha\in A$, $\beta\in A^*$] or [6, (5.6) with $\alpha_1=\pi\alpha$, $\alpha_2=\beta$, $\alpha_3=1$, where $\alpha,\beta\in A$], we get $L=\mathfrak{L}(\Lambda)=Ae_0+Ae_1+Ae_2$ and

$$(2.6)\quad q_L\equiv\pi^rX_1X_2-\pi\alpha X_0X_1-\pi\beta X_1^2+X_0^2\,,$$

$$(2.7)\quad q_L\equiv\pi^rX_0X_2-\pi\alpha X_0X_1+\pi X_1^2-\beta X_0^2\,,$$

$$(2.8)\quad q_L\equiv\pi^{2r-1}X_2^2-2\pi^r\alpha\eta X_0X_2-\pi^r(1+2\beta)X_1X_2+\pi\alpha(1+2\beta)\eta X_0X_1+ \\ +\pi(\beta^2+\beta+\varepsilon)X_1^2+(\pi\alpha^2-\eta)\eta X_0^2\,,$$

$$(2.9)\quad q_L\equiv\pi^{2r}X_2^2-2\pi^{r+1}\alpha\eta X_1X_2-\pi^r(1+2\beta)X_0X_2+\pi\alpha(1+2\beta)\eta X_0X_1+ \\ +\pi(\pi\alpha^2-\eta)\eta X_1^2+(\beta^2+\beta+\varepsilon)X_0^2\,,$$

where we have (2.6) if $\hat{\Lambda}\cong E_{2r}^{(0)}$, $r\geqq 1$, (2.7) if $\hat{\Lambda}\cong E_{2r+1}^{(0)}$, $r\geqq 1$, (2.8) if $\hat{\Lambda}\cong\Gamma_{2r-1}^{(0)}$, $r\geqq 1$, and (2.9) if $\hat{\Lambda}\cong\Gamma_{2r}^{(0)}$, $r\geqq 1$. In (2.8) and (2.9), $\varepsilon,\eta\in A^*$ and $X^2-X+\varepsilon$ is irreducible over $k(\mathfrak{p})$. Let $A[x_0,x_1,x_2]=A[X_0,X_1,X_2]/(q_L)$. The ideal defining $M(L)_{\mathfrak{p}}$ is $l=(\pi,x_0)$ so the fiber is a double projective prime). It is easy to check that the local ring at l on $M(L)$ is regular with maximal ideal generated by e.g. x_0/x_1. Note, however, that there is exactly one singular point (on $M(L)$) in the fiber $M(L)_{\mathfrak{p}}$ in all the cases but one: If $r=1$ in (2.6), then there are exactly two singular points (π,x_0,x_1) and $(\pi,x_0,x_2-\beta x_1)$. If $r=1$ in (2.8), then the only singular point is "quadratic" (a form of two singular points): $(\pi,x_0,(\beta^2+\beta+\varepsilon)x_1^2-(1+2\beta)x_1x_2+x_2^2)$. In all the remaining cases the singular point is (π,x_0,x_1).

It remains to show that the model $M(L)$ is not normal if L is not a Bass form and to prove the second part of the Theorem. Both results follow at once from the following Lemma:

(2.10) LEMMA. *If L is not a Bass lattice at $\mathfrak{p}$ and L' is the lattice satisfying* (1.3), *then there is a finite morphism (not an isomorphism)* $M(L')\to M(L)$.

PROOF. If $\mathfrak{q}\neq\mathfrak{p}$, $\mathfrak{q}$ a closed point of S, then there is a neighbourhood $U_{\mathfrak{q}}$ of $\mathfrak{q}$ in S such that $\alpha_L^{-1}(U_{\mathfrak{q}})$ and $\alpha_{L'}^{-1}(U_{\mathfrak{q}})$ are $U_{\mathfrak{q}}$-isomorphic (since $L_{\mathfrak{q}}=L'_{\mathfrak{q}}$). We want to show that there is a finite morphism $\alpha_{L'}^{-1}(U_{\mathfrak{p}})\to\alpha_L^{-1}(U_{\mathfrak{p}})$ for a neighbourhood $U_{\mathfrak{p}}$ of $\mathfrak{p}$ in S. Let us choose a neighbourhood $U_{\mathfrak{p}}$ and a basis e_0,e_1,e_2 for V such that $L'_{\mathfrak{q}}=A_{\mathfrak{q}}e_0+A_{\mathfrak{q}}e_1+A_{\mathfrak{q}}e_2$, $L_{\mathfrak{p}}=A_{\mathfrak{q}}e_0+A_{\mathfrak{q}}\pi e_1+A_{\mathfrak{q}}\pi e_2$ for each $\mathfrak{q}\in U_{\mathfrak{p}}$. Such a choice is possible since $[L':L]=\mathfrak{p}^2$ and $\mathfrak{p}L'\subset L$ according to (1.3). Hence we can assume that $S=\mathrm{Spec}\,(A)$, where A is a discrete valuation ring with maximal ideal $\mathfrak{p}=(\pi)$. By (1.3), if $q_{L'}=\sum a_{ij}X_iX_j$, where $a_{ij}=b(e_i,e_j)$ for $i\neq j$ and $a_{ii}=q(e_i)$, then $q_L=\sum b_{ij}Y_iY_j$, where $b_{ij}=\pi^2a_{ij}, b_{0i}=\pi a_{0i}$ for $i,j\in\{1,2\}$ and $b_{00}=a_{00}$ (since $\mathfrak{n}(L')=\mathfrak{n}(L)$). Hence $a_{00}\in A^*$. Let $A[x_0,x_1,x_2]=A[X_0,X_1,X_2]/(q_{L'})$ and $A[y_0,y_1,y_2]=A[Y_0,Y_1,Y_2]/(q_L)$. Consider the birational map

$$\sigma\colon \operatorname{Proj}(A[x_0,x_1,x_2]) \to \operatorname{Proj}(A[y_0,y_1,y_2])$$

induced by the homomorphism

$$\sigma_0\colon A[y_0,y_1,y_2] \to A[x_0,x_1,x_2]$$

such that $\sigma_0(y_0)=\pi x_0$, $\sigma_0(y_1)=x_1$, $\sigma_0(y_2)=x_2$. It is easy to see that σ is a morphism, since $a_{00}\in A^*$ implies that σ is defined at all the points of $M(L')$. The equality $q_{L'}=0$ and $a_{00}\in A^*$ show at the same time that σ is finite.

We note as a corollary the observations made in the course of the proof of (2.1) concerning the singular points on normal models $M(L)$. Recall that $\mathfrak{d}_{E/S}=\mathfrak{d}(M)$, where M is a relatively minimal S-model of E.

(2.11) Corollary. *If $M=M(L)$ is a normal S-model of E, then all the singular points on M are in the fibers $M_\mathfrak{p}$ such that $\mathfrak{p}^2\mid\mathfrak{d}(M)$ and each fiber of that type contains exactly one singular point unless $\mathfrak{p}^3\nmid\mathfrak{d}(M)$, $e_\mathfrak{p}(M)=0$ and $\mathfrak{p}\nmid\mathfrak{d}_{E/S}$, when there are exactly two such points. The singular points are $k(\mathfrak{p})$-rational unless $\mathfrak{p}^3\nmid\mathfrak{d}(M)$, $e_\mathfrak{p}(M)=0$ and $\mathfrak{p}\mid\mathfrak{d}_{E/S}$, when the singular point is a form of two different rational points.*

(2.12) Remark. In order to prove that $M(L)$ is normal if L is a Bass lattice, it is possible to replace the local ring A by its strict Henselization (see e.g. [8, p. 54]). This gives an essential simplification of the proof of Theorem (2.1): Case 2 reduces to Case 1, while Case 3 can be limited to (2.6) and (2.7). Unfortunately, in the proof of the next theorem we need all the quadratic forms listed in the proof of (2.1).

Now we are going to discuss the question of resolution of singularities of normal conic bundle surfaces by a kind of elementary transformations which replace a surface of that type by a new one with improved singularities. Let M be a normal conic S-model of E and $\mathfrak{p}$ a closed point of S such that there is a singular point on M in $M_\mathfrak{p}$. We shall say that M' is an elementary transform of M at the singular points in $M_\mathfrak{p}$ if there exist an S-model M^* of E (not a conic bundle surface), a blowing-up $\tau\colon M^*\to M'$ at a regular closed point P' in $M'_\mathfrak{p}$ and a composition of blowings-up $\sigma\colon M^*\to M$ at the singular points in $M_\mathfrak{p}$ (more exactly, if there are two singular points in $M_\mathfrak{p}$, then $\sigma=\sigma_2\circ\sigma_1$, where σ_1

is a blowing-up of M at one singular point, and σ_2 a blowing-up of $\sigma_1^{-1}(M)$ at the inverse image of the second singular point) such that $\tau\circ\sigma^{-1}(l)=P'$ for (exactly) one non-closed point l in $M_{\mathfrak{p}}$.

(2.13) THEOREM. *If M is a normal non-regular conic bundle S-surface, then there is a chain $M_0=M, M_1,\ldots,M_n$ of normal conic bundle S-surfaces such that M_{i+1} is an elementary transform of M_i at the singular points (on M_i) in one of its fibers for $i=0,1,\ldots,n-1$ and M_n is regular.*

PROOF. Let $M=M(L)$ be a normal conic bundle S-surface and $\mathfrak{d}(M)=\prod \mathfrak{p}^{\alpha(\mathfrak{p})}$. Assume that $\mathfrak{p}$ is a closed point of S such that $\alpha(\mathfrak{p})\geqq 2$ and let $L'\supset L$ be an S-lattice satisfying one of the Propositions (1.5), (1.6) or (1.7). We shall show that $M'=M(L')$ is an elementary transform of M at the singular points in $M_{\mathfrak{p}}$. The Theorem follows then by an easy induction on $\sum \alpha(\mathfrak{p})$, because $\mathfrak{d}(M)=\mathfrak{d}(M')\mathfrak{p}^{\varepsilon}$, where $\varepsilon=1$ or 2.

Since $L'_{\mathfrak{q}}=L_{\mathfrak{q}}$ for each closed point $\mathfrak{q}\neq\mathfrak{p}$, we may assume that $S=\operatorname{Spec}(A)$, where A is a discrete valuation ring with maximal ideal $\mathfrak{p}=(\pi)$. Let $\alpha(\mathfrak{p})=n\geqq 2$. Let

$$A[x_0,x_1,x_2] = A[X_0,X_1,X_2]/(q_L)$$

and

$$A[y_0,y_1,y_2] = A[Y_0,Y_1,Y_2]/(q_{L'}),$$

where q_L and $q_{L'}$ are quadratic forms corresponding to L and L'. As earlier, we denote by E the (common) field of rational functions on M and M'. We refer to [9, Chapter II, 7] for general properties of blowings-up and to [1] for the way in which we apply them calculating inside of E. We have to consider 4 cases:

CASE 1. If $e_{\mathfrak{p}}(M)=1$, then by (1.5) there is a basis e_0,e_1,e_2 for L' such that $L'=Ae_0+Ae_1+Ae_2$, $L=A\pi e_0+A\pi e_1+Ae_2$, q_L satisfies (2.2) and

$$q_{L'} \equiv \pi^{n-1}Y_0^2+Y_1Y_2\,. \tag{2.14}$$

Note that the homomorphism $\varphi_0\colon A[Y_0,Y_1,Y_2]\to A[X_0,X_1,X_2]$ such that $\varphi_0(Y_0)=\pi X_0$, $\varphi_0(Y_1)=\pi X_1$, $\varphi_0(Y_2)=X_2$ induces a birational map $\varphi\colon \operatorname{Proj}(A[x_0,x_1,x_2])\to\operatorname{Proj}(A[y_0,y_1,y_2])$. Let σ be the blowing-up of $M=\operatorname{Proj}(A[x_0,x_1,x_2])$ at the singular point $P=(\pi,x_1,x_2)$ and τ the blowing-up of $M'=\operatorname{Proj}(A[y_0,y_1,y_2])$ at the regular point $P'=(\pi,y_0,y_1)$. Since $y_0/y_2=x_0/x_2$ and $y_1/y_2=\pi x_1/x_2$ in E, we check easily that σ and τ give the same S-model of E and $\varphi=\tau\circ\sigma^{-1}$. We have $\varphi(l_1)=P'$, where $l_1=(\pi,x_1)$ (the second prime $l_2=(\pi,x_2)$ in $M_{\mathfrak{p}}$ is mapped onto $l'_2=(\pi,y_2)$, while $l'_1=(\pi,y_1)$ is mapped on P by φ^{-1}). Note that if we choose $P'=(\pi,y_0,y_2)$, then φ maps l_2 on P' (and l_1 on l'_1).

CASE 2. If $e_p(M) = -1$, then by (1.6) there is a basis e_0, e_1, e_2 for L' such that $L' = Ae_0 + Ae_1 + Ae_2$, $L = A\pi e_0 + Ae_1 + Ae_2$, q_L satisfies one of the conditions (2.3)–(2.5), and respectively:

$$(2.15)\quad q_{L'} \equiv \pi^{n-1} Y_0 Y_1 + Y_1^2 - Y_1 Y_2 + \varepsilon Y_2^2 ,$$

$$(2.16)\quad q_{L'} \equiv \pi^{2n-1} \eta Y_0^2 - Y_1^2 + Y_1 Y_2 - \varepsilon Y_2^2 ,$$

$$(2.17)\quad q_{L'} \equiv \pi^{2n-2} Y_0^2 - 2\alpha\pi^{n-1} Y_0 Y_1 - 2\beta\pi^{n-1} Y_0 Y_2 + (\alpha^2 - \varepsilon) Y_1^2 + 2\alpha\beta Y_1 Y_2 + (\beta^2 - \eta) Y_2^2 .$$

This time the homomorphism φ_0: $A[Y_0, Y_1, Y_2] \to A[X_0, X_1, X_2]$ such that $\varphi_0(Y_0) = \pi X_0$, $\varphi_0(Y_1) = X_1$, $\varphi_0(Y_2) = X_2$ induces a birational map φ: Proj $(A[x_0, x_1, x_2]) \to$ Proj $(A[y_0, y_1, y_2])$. Let σ be the blowing-up of $M = $ Proj $(A[x_0, x_1, x_2])$ at the singular point $P = (\pi, x_1, x_2)$ and τ the blowing-up of $M' = $ Proj $(A[y_0, y_1, y_2])$ at the regular point $P' = (\pi, y_0, \mu)$, where

$$\mu = y_1^2 - y_1 y_2 + \varepsilon y_2^2$$

if $q_{L'}$ satisfies (2.15) or (2.16) and

$$\mu = (\alpha^2 - \varepsilon) y_1^2 + 2\alpha\beta y_1 y_2 + (\beta^2 - \eta) y_2^2$$

if $q_{L'}$ satisfies (2.17). Since $y_0/y_2 = \pi x_0/x_2$ and $y_1/y_2 = x_1/x_2$ in E, it is easy to check that σ and τ give the same S-model M^* of E and that $\varphi = \tau \circ \sigma^{-1}$. We have $\varphi(l) = P'$, where $l = (\pi, \mu)$ is the non-closed point of M_p (note that $\varphi^{-1}(l') = P$, where l' is the non-closed point of M'_p).

CASE 3. If $e_p(M) = 0$ and $n > 2$, then by (1.7) there is a basis e_0, e_1, e_2 for L' such that $L' = Ae_0 + Ae_1 + Ae_2$, $L = Ae_0 + A\pi e_1 + A\pi e_2$, q_L satisfies one of the conditions (2.6)–(2.9), and respectively

$$(2.18)\quad q_{L'} \equiv \pi^{r-1} Y_1 Y_2 - \pi\alpha Y_0 Y_1 + \pi Y_0^2 - \beta Y_1^2 ,$$

$$(2.19)\quad q_{L'} \equiv \pi^{r} Y_0 Y_2 - \pi\alpha Y_0 Y_1 - \pi\beta Y_0^2 + Y_1^2 ,$$

$$(2.10)\quad q_{L'} \equiv \pi^{2r-2} Y_2^2 - 2\pi^r \alpha\eta Y_0 Y_2 - \pi^{r-1}(1+2\beta) Y_1 Y_2 + \pi\alpha(1+2\beta)\eta Y_0 Y_1 + {} + \pi(\pi\alpha^2 - \eta)\eta Y_0^2 + (\beta^2 + \beta + \varepsilon) Y_1^2 ,$$

$$(2.21)\quad q_{L'} \equiv \pi^{2r-1} Y_2^2 - 2\pi^r \alpha\eta Y_1 Y_2 - \pi^r(1+2\beta) Y_0 Y_2 + \pi\alpha(1+2\beta)\eta Y_0 Y_1 + {} + \pi(\beta^2 + \beta + \varepsilon) Y_0^2 + (\pi\alpha^2 - \eta)\eta Y_1^2 .$$

The homomorphism φ_0: $A[Y_0, Y_1, Y_2] \to A[X_0, X_1, X_2]$ such that $\varphi_0(Y_0) = X_0$, $\varphi_0(Y_1) = \pi X_1$, $\varphi_0(Y_2) = \pi X_2$ induces a birational map φ: Proj $(A[x_0, x_1, x_2]) \to$ Proj $(A[y_0, y_1, y_2])$. Consider the blowing-up σ of $M = $ Proj $(A[x_0, x_1, x_2])$ at the singular point $P = (\pi, x_0, x_1)$ and the blowing-up τ

of $M'=\operatorname{Proj}(A[y_0,y_1,y_2])$ at the regular point $P'=(\pi,y_1,y_2)$. Since $y_1/y_0=\pi x_1/x_0$ and $y_2/y_0=\pi x_2/x_0$ in E, easy computations show that σ and τ give the same S-model M^* of E and that $\varphi=\tau\circ\sigma^{-1}$. We have $\varphi(l)=P'$, where $l=(\pi,x_0)$ is the ideal defining $M_{\mathfrak{p}}$ (and $\varphi^{-1}(l')=P$, where $l'=(\pi,y_1)$).

CASE 3'. If $e_{\mathfrak{p}}(M)=0$ and $n=2$, then by (1.7) and [6, (5.4) and (5.6)] there is a basis e_0,e_1,e_2 for L' such that $L'=Ae_0+Ae_1+Ae_2$, $L=Ae_0+A\pi e_1+A\pi e_2$,

(2.22) $q_L \equiv \pi X_1X_2+X_0^2$

(2.22)' $q_{L'} \equiv \pi Y_0^2+Y_1Y_2$

if $\mathfrak{p}\mid\mathfrak{d}_{E/S}$, and

(2.23) $q_L \equiv \eta X_0^2-\pi(X_1^2-X_1X_2+\varepsilon X_2^2)$

(2.23)' $q_{L'} \equiv \pi\eta Y_0^2-Y_0^2+Y_1Y_2-\varepsilon Y_2^2$

if $\mathfrak{p}\mid\mathfrak{d}_{E/S}$ (ε,η have the same meaning as in (2.4)). We have a birational map φ: $\operatorname{Proj}(A[x_0,x_1,x_2])\to\operatorname{Proj}(A[y_0,y_1,y_2])$ defined as in the previous case.

If $\mathfrak{p}\nmid d_{E/S}$, let σ_1 be the blowing-up of $M=\operatorname{Proj}(A[x_0,x_1,x_2])$ at the singular point $P_1=(\pi,x_0,x_1)$, and σ_2 the blowing-up of $\sigma_1^{-1}(M)$ at the singular point $\sigma_1^{-1}(P_2)$, where $P_2=(\pi,x_0,x_2)$. Let τ be the blowing-up of $M'=\operatorname{Proj}(A[y_0,y_1,y_2])$ at the (regular) point $P'=(\pi,y_1,y_2)$. Using the same arguments as in the previous case we get $\sigma^{-1}(M)=\tau^{-1}(M')$ and $\varphi=\tau\circ\sigma^{-1}$, where $\sigma=\sigma_2\circ\sigma_1$. We have $\varphi(l)=P$, where $l=(\pi,x_0)$ (and $\varphi^{-1}(l_i)=P_i$, where $l_i=(\pi,y_i)$ for $i=1,2$).

If $\mathfrak{p}\mid\mathfrak{d}_{E/S}$, then we take as σ the blowing-up of M at the singular point $P=(\pi,x_0,x_1^2-x_1x_2+\varepsilon x_2^2)$ and as τ the blowing-up of M' at the (regular) point $P'=(\pi,y_1,y_2)$. We repeat the above arguments noting that $\varphi(l)=P'$, where $\varphi=\tau\circ\sigma^{-1}$ and $l=(\pi,x_0)$ (while $\varphi^{-1}(l')=P$, where $l'=(\pi,y_1^2-y_1y_2+\varepsilon y_2^2)$).

(2.24) REMARK. If $M=M(L)$ is a normal conic bundle surface and M' is an elementary transform of M at the singular points of one of its fibers $M_{\mathfrak{p}}$, then $M'=M(L')$, where $L'=L_1$ is an extension of L in the sense of (1.8) (b). In fact, we know from the proof of (2.13) that for each prime l in $M_{\mathfrak{p}}$ there is an extension $L'=L_1\supset L$ satisfying (1.8) (b) such that $M(L')$ is an elementary transform of M at the singular points of $M_{\mathfrak{p}}$ and l is mapped on a (regular)point of $M(L')$. Hence, if the prime l is mapped on a point of M', then $M'=M(L')$ since M' is uniquely determined by M and the choice of a prime in $M_{\mathfrak{p}}$ whose image under the elementary transformation is a point of M'.

We end the paper with some remarks concerning possibilities to generalize the results of [4] about local behaviour of models with the same types of fibers and singularities. As we know, two regular S-models M_1 and M_2 with the same types of fibers (that is, $\mathfrak{d}(M_1)=\mathfrak{d}(M_2)$) are locally isomorphic (see [4, (4.5)]). One cannot expect that this result is true for arbitrary normal conic bundle surfaces. In fact, it is easy to give examples of normal conic S-models M_1 and M_2 of E which are not S-isomorphic even if $\mathfrak{d}(M_1)=\mathfrak{d}(M_2)$ and $e_{\mathfrak{p}}(M_1)=e_{\mathfrak{p}}(M_2)$ for each closed point $\mathfrak{p}\in S$. Let us take $S=\mathrm{Spec}(A)$, where $A=\hat{\mathbf{Z}}_3$, $M_i=\mathrm{Proj}(A[X_0,X_1,X_2]/(q_i))$, $i=1,2$, where $q_1=X_0^2-\pi X_1^2-\pi^2X_2^2$, $q_2=X_0^2+\pi X_1^2-\pi^2X_2^2$. The quadratic forms q_1 and q_2 are not A-equivalent (see e.g. [11, 92:2]) but $C_0(V,q_1)\cong C_0(V,q_2)$. Hence M_1 and M_2 are normal S-models (see (1.2)) of the same filed $E=R(M_i)$. They are not S-isomorphic because of the following result which was proved in [3, Theorem 1] for regular conic bundle surfaces without using the assumption of regularity. Note, however, thar $\mathfrak{s}(L)$ should be replaced there by $\mathfrak{n}(L)$ and ideals by corresponding divisors.

(2.25) PROPOSITION. *Let (V,q) be a half-regular quadratic space over the field of rational functions on a perfect Dedekind scheme S. If L_1 and L_2 are S-lattices on V, then the models $M(L_1)$ and $M(L_2)$ are isomorphic if and only if the lattices L_1 and L_2 are similar.*

Recall that the lattices L_1 and L_2 on (V,q) are similar if there is an invertible S-sheaf $\mathfrak{a}\subset F$ (see the end of section 1) such that L_1 and $\mathfrak{a}L_2$ are isometric, that is, there is an isometry σ of V such that $\sigma(L_1(U))=(\mathfrak{a}L_2)(U)$ for each open subset U of S.

The results of [6] imply however that sometimes two models with the same types of fibers are locally isomorphic.

(2.26) THEOREM. *Let S be a perfect Dedekind scheme with field of rational functions F and E a finitely generated regular extension of genus 0 of F. If M_1 and M_2 are conic bundle S-models of E such that $\mathfrak{d}(M_1)=\mathfrak{d}(M_2)$ and $e_{\mathfrak{p}}(M_1)=e_{\mathfrak{p}}(M_2)\neq 0$ for each closed point $\mathfrak{p}$ of S, then M_1 and M_2 are locally isomorphic.*

PROOF. Follows at once from (2.25), [4, (3.6)] and [6, (5.3)].

Let us note that [6, (5.3)] gives a possibility of a slight generalization of this Theorem to the case in which $e_{\mathfrak{p}}(M_i)=0$ if at the same time $\mathfrak{p}^3\nmid\mathfrak{d}(M_i)$ for $i=1,2$. If the residue fields $k(\mathfrak{p})$ are algebraically closed and the models M_i are normal, the last condition can be eliminated using (2.1) and the characterization of Bass lattices given in [7, (3.4)].

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