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ON THE FINITENESS OBSTRUCTION OF CERTAIN PERIODIC GROUPS

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0. Introduction.

For groups G with periodic cohomology, the Tate cohomology groups $\hat{H}^{nd}(G, \mathbf{Z})$ are isomorphic to $\mathbf{Z}/|G|$ for all multiples of the period d. Each generator e of $\hat{H}^{nd}(G, \mathbf{Z})$ can be realized by a periodic resolution P_* of $\mathbf{Z}$ by finitely generated projective $\mathbf{Z}G$-modules, and the resolution can be taken to be free exactly if the element

$$\sigma_{nd}(e) := \sum_{i=1}^{nd-1} (-1)^i [P_i]$$

vanishes in $\tilde{K}_0(\mathbf{Z}G) = \mathrm{Cl}(G)$. This element $\sigma_{nd}(e)$ is called the finiteness obstruction because its vanishing is equivalent to the existence of a free simplicial action of G on a finite simplicial complex Σ, homotopy equivalent to the $(nd-1)$-dimensional sphere, such that Σ/G has e as its first k-invariant. The calculation of σ_{nd} is therefore of interest from a topological as well as from an algebraic point of view.

By results of [8] the groups of particular topological importance are the groups $Q(8a, b, c) \times \mathbf{Z}/d$. In this paper we consider the group $Q(8a, b)$ which is a semi-direct product of $\mathbf{Z}/ab$ with $Q(8)$ such that the kernel of the action of $Q(8)$ on $\mathbf{Z}/ab$ is $\mathbf{Z}/2$. This group has cohomology of period 4. By Corollary C of [4] there is a special class of generators of $\bar{H}^4(Q(8a, b), \mathbf{Z})$ that is of interest for the topological space form problem; these are the almost linear generators considered below. We take such an almost linear generator e_0 and calculate $\sigma_4(e_0)$ (Theorem 1). The result is a generalization of Milgram's result for $Q(8p, q)$ where p and q are primes, cf. [5] or [1].

Unfortunately, Theorem 1 gives the answer in terms of non-trivial number theoretical questions. If both a and b are prime powers, these questions have been almost completely settled in [1].

Before stating the main result, we fix some notation that will remain in force throughout the paper.

Like in [1] we use Fröhlich's description of $\mathrm{Cl}(G)$ in terms of character

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homomorphisms. For details on this formalism we refer to [2] and the appendix of [3], but let us briefly sketch the basic features. Let H be a Galois functor from the category of extension fields of $\mathbf{Q}$ to abelian groups. It can be extended to a functor on commutative finite-dimensional semi-simple algebras over $\mathbf{Q}$ by letting $H(\prod L_i)=\prod H(L_i)$. Then there is a basic isomorphism

$$H(Z(\mathbf{Q}G)) \cong \mathrm{Hom}_{\Omega}(R(G), H(\bar{\mathbf{Q}})) \tag{0.1}$$

for any finite group G. Here Z denotes the center, Ω is the full Galois group of $\mathbf{Q}$, and $R(G)$ is the virtual character ring of G. The left hand side is convenient for notational purposes while the right hand side is convenient for calculations, and we shall constantly change from one side to the other.

Taking the exact sequence

$$Z(\mathbf{Q}G)_+^{\times} \to J(Z(\mathbf{Q}G))_+/\mathrm{Nrd}\,(U(\mathbf{Z}G)) \xrightarrow{\partial} \mathrm{Cl}(G) \to 0 \tag{0.2}$$

of [2, Theorem 2] and using (0.1) we get

$$\mathrm{Hom}_{\Omega}^{+}(R(G), \bar{\mathbf{Q}}^{\times}) \to \mathrm{Hom}_{\Omega}^{+}(R(G), J(\bar{\mathbf{Q}}))/\mathrm{Det}\,(U(\mathbf{Z}G)) \xrightarrow{\partial} \mathrm{Cl}\,(G) \to 0\,, \tag{0.3}$$

where $U(\mathbf{Z}G)$ is the group of unit ideles:

$$U(\mathbf{Z}G) = \prod_{p} (\mathbf{Z}_pG)^{\times}$$

with p running through all primes, including the infinite, cf. [3, I.10]. Similarly there is an exact sequence

$$\begin{aligned} \mathrm{Hom}_{\Omega}^{+}(R(G), \mathcal{O}_F^{\times}) \to \mathrm{Hom}_{\Omega}\Big(R(G), \prod_{p\,||\,G|} \mathcal{O}_{F_p}^{\times}\Big)\Big/\mathrm{Det}\Big(\prod_{p\,||\,G|} \mathbf{Z}_pG\Big)^{\times}\Big) \xrightarrow{\partial} \\ \to D(G) \to 0 \end{aligned} \tag{0.4}$$

describing the kernel group, [F2, II.2]. Here F is any number field containing all character values, and $\mathrm{Hom}_{\Omega}^{+}(R(G), \mathcal{O}_F^{+})$ denotes the character homomorphisms f for which $f(\chi)$ is real and positive at every infinite prime (real or complex) of F whenever χ is symplectic. The middle term in (0.4) decomposes into

$$\prod_{p\,||\,G|} \mathrm{Hom}_{\Omega}(R(G), \mathcal{O}_{F_p}^{\times})/\mathrm{Det}\,((\mathbf{Z}_pG)^{\times})\,.$$

The component at p is called the local kernel group (at p) and is denoted $\hat{D}_p(G)$. The whole product is denoted $\hat{D}(G)$.

A basic feature of the character homomorphism formalism is that the restriction and induction maps on class groups are determined through induction and restriction of characters, cf. [3, VI and VIII].

We are considering the group $\pi = Q(8a, b)$ which is defined as follows. Let

$$Q(8) = \langle X, Y \mid X^4 = Y^4 = 1,\ X^2 = Y^2 \neq 1,\ XYX^{-1} = Y^{-1} \rangle$$

be the quaternion group of order 8. Let X act on $\mathbf{Z}/a \times \mathbf{Z}/b$ by inversion on $\mathbf{Z}/a$ and the identity on $\mathbf{Z}/b$, and let Y act by inversion on $\mathbf{Z}/b$ and the identity on $\mathbf{Z}/a$. This gives an action on $Q(8)$ on $\mathbf{Z}/ab$ and $Q(8a, b)$ is the semi-direct product $\mathbf{Z}/ab \times Q(8)$ defined through this action.

If $a = \prod_i p_i^{n_i}$, $b = \prod_i q_i^{m_i}$ is the prime decomposition of a and b we let $P_1 = \{p_i^{n_i} \mid i\}$, $P_2 = \{q_i^{m_i} \mid i\}$, and $P = P_1 \cup P_2$. With an abuse of language we shall by a statement like "$p \in P_1$" mean that for some power p^n of p we have $p^n \in P_1$. Likewise, by $P_1 \setminus \{p\}$ we shall mean $P_1 \setminus \{p^n\}$. For each set $S \subseteqq P$, let $\pi(S)$ be the subgroup of π obtained by deleting the prime powers not in S. Let further $C = C_\pi(\mathbf{Z}/ab)$ be the cyclic subgroup of π of order $2ab$, centralizing $\mathbf{Z}/ab$.

For each prime power p^n in P we choose a generator A_p of $\mathbf{Z}/p^n$, and for each A_p we let X_p and Y_p be defined by: If $X^{-1}A_pX = A_p$, we let $X_p = X$ and $Y_p = Y$. If $Y^{-1}A_pY = A_p$ we let $X_p = Y$ and $Y_p = X$. Furthermore we let for each A_p, χ_p be the generator of $\mathrm{Ch}(\mathbf{Z}/p^n)$ with $\chi_p(A_p) = e^{2\pi i/p^n}$, and we let χ_2 be the unique irreducible representation of $Q(8)$ of dimension 2.

As mentioned, π has cohomology of period 4. For each $p^n \in P$, let

$$e_p = c_2(\chi_p + \chi_p^{-1}) \in \hat{H}^4(\mathbf{Z}/p^n, \mathbf{Z})$$

and let

$$e_2 = c_2(\chi_2) \in \hat{H}^4(Q(8), \mathbf{Z}) .$$

Here c_2 denotes the second Chern class. These e's are generators of the different cohomology groups, and we define e_0 to be the generator of $\hat{H}^4(Q(8a, b), \mathbf{Z})$ that restricts to these e's under the isomorphism

$$\hat{H}^4(Q(8a, b), \mathbf{Z}) \cong \prod_{p^n \in P} \hat{H}^4(\mathbf{Z}/p^n, \mathbf{Z}) \oplus \hat{H}^4(Q(8), \mathbf{Z}) .$$

This is the almost linear generator for which we shall calculate $\sigma_4(e_0)$.

Once and for all we make a choice of roots of unity: If $n = p^m$, then $\zeta_n = e^{2\pi i/p^m}$, and if $n = \prod p_i^{n_i}$, then $\zeta_n = \prod \zeta_{p_i^{n_i}}$. Further, for any $n \in \mathbf{N}$ we let $\eta_n = \zeta_n + \zeta_n^{-1}$.

For any number ring A we let $A^\times$ denote the units of A, and, if A is real, we let A^* denote the totally positive units. For any finite prime $\mathfrak{p}$ of A we let $\varrho_\mathfrak{p}$ denote the reduction map $\varrho_\mathfrak{p}: A \to A/\mathfrak{p}$.

Our main result is

THEOREM 1. *$\sigma_4(e_0)$ is of order at most 2; it lies in the kernel group $D(\pi)$ and it is zero if and only if for any pair (α, β) of square free numbers with $\alpha \mid \beta$, $\beta \mid b$ and $\alpha, \beta \neq 1$ there exist numbers $x_{\alpha,\beta} \in \mathbf{Z}[\eta_\alpha, \eta_\beta]^\times$, $y_{\alpha,\beta} \in \mathbf{Z}[\eta_\alpha, \eta_\beta]^*$ such that*

$$\varrho_{\mathfrak{p}}(x_{\alpha,\beta}) = \begin{cases} 1 \text{ if } \mathfrak{p} \mid \alpha \\ -1 \text{ if } \mathfrak{p} \mid \beta \end{cases}$$

$$\varrho_{\mathfrak{p}}(y_{\alpha,\beta}) = 4 \quad \text{for } \mathfrak{p} \mid \alpha\beta$$

Note that since $\sigma_4(e_0)$ is 2-torsion, we can in Theorem 1 replace 4 by its 2-primary part in $(\mathbf{Z}[\eta_\alpha, \eta_\beta]/\mathfrak{p})^\times$.

1. Decomposition of the class group.

For $p \in P$, let E_p be the idempotent endomorphism of π that sends the Sylow-p part to 1 and is the identity on the rest. It factors as

$$E_p : \pi \to \pi(P \setminus \{p\}) \to \pi ,$$

where $\pi \to \pi(P \setminus \{p\})$ is the projection and $\pi(P \setminus \{p\}) \to \pi$ is the inclusion. If no power of p is in P, we let E_p be the identity. These maps are defined on all the subgroups $\pi(S)$ and C of π, and they commute with the inclusion maps. Furthermore these endomorphisms will induce commuting idempotents on the virtual character rings $R(\pi(S))$ and $R(C)$ and on the projective class groups $\mathrm{Cl}(\pi(S))$, $\mathrm{Cl}(C)$.

LEMMA 1.1. *On the class groups we have*

a) $\mathrm{Res}^{\pi}_{\pi(P\setminus\{p\})} \circ E_p = \mathrm{Res}^{\pi}_{\pi(P\setminus\{p\})}$ *for all* $p \in P$

b) $\mathrm{Res}^{\pi}_{\pi(P\setminus\{q\})} \circ E_p = E_p \circ \mathrm{Res}^{\pi}_{\pi(P\setminus\{q\})}$ *for all* $q \in P \setminus \{p\}$

c) $\mathrm{Res}^{\pi}_{C} \circ E_p = E_p \circ \mathrm{Res}^{\pi}_{C}$ *for all* $p \in P$.

PROOF. Obvious.

NOTE. Concerning (c) of Lemma 1.1 we actually have a stronger result: E_p commutes with induction of characters from C to π, so the diagram

$$\begin{array}{ccc} R(C) & \xrightarrow{\mathrm{Ind}} & R(\pi) \\ \downarrow E_p & & \downarrow E_p \\ R(C) & \xrightarrow{\mathrm{Ind}} & R(\pi) \end{array}$$

is commutative.

For any $S \subseteqq P$, let

$$\tilde{E}_S = \prod_{p \notin S} E_p \prod_{p \in S} (1 - E_p) .$$

Then, on the character ring $R(\pi)$:

$$1 = \sum_{S \subseteq P} \tilde{E}_S$$

is a decomposition of the identity into a sum of orthogonal idempotents, i.e. projections. This gives a decomposition

$$\mathrm{Cl}(\pi) = \prod_{S \subseteq P} \tilde{E}_S(\mathrm{Cl}(\pi))$$

of the class groups.

According to Lemma 1.1 we have

$$\mathrm{Res}^{\pi}_{\pi(S)} \circ \tilde{E}_S = \tilde{E}_S \circ \mathrm{Res}^{\pi}_{\pi(S)}$$

on the class groups. Hence we get

COROLLARY 1.2. *The restriction to $\pi(S)$ of the S-component is the top component of the restriction.*

By the top component of $\mathrm{Cl}(\pi)$ is meant $\tilde{E}_P(\mathrm{Cl}(\pi))$.

2. Detecting the components.

We begin by detecting the 2-primary part of the non top components.

PROPOSITION 2.1. *If $S \not\subseteq P$, then*

$$\mathrm{Res}^{\pi}_{\pi(S)} : \tilde{E}_S(\mathrm{Cl}(\pi)) \to \mathrm{Cl}(\pi(S))$$

is injective on the 2-primary part.

PROOF. In Fröhlich's description of the class group

$$\mathrm{Cl}(\pi) = \mathrm{Hom}_{\Omega}(R(\pi), J(\bar{\mathbf{Q}}))/\mathrm{Det}(U(\mathbf{Z}(\pi))\mathrm{Hom}_{\Omega}(R(\pi), \bar{\mathbf{Q}}^{\times}),$$

and $\mathrm{Res}^{\pi}_{\pi(S)} \circ \prod_{p \notin S} E_p$ is induced by the map

$$f: R(\pi(S)) \xrightarrow{\mathrm{Ind}} R(\pi) \xrightarrow{\mathrm{Res}} R(\pi(P \setminus \{p_1\})) \xrightarrow{\mathrm{lift}} R(\pi) \xrightarrow{\mathrm{Res}} R(\pi(P \setminus \{p_2\})) \xrightarrow{\mathrm{lift}} R(\pi) \to \text{etc.}$$

where $P \setminus S = \{p_1, p_2, \ldots\}$. Obviously the diagram

$$\begin{array}{ccccc} R(\pi(P \setminus \{p_i\})) & \xrightarrow{\mathrm{lift}} & R(\pi) & \xrightarrow{\mathrm{Res}} & R(\pi(P \setminus \{p_j\})) \\ & \searrow & & & \nearrow \\ & & R(\pi(P \setminus \{p_i, p_j\})) & & \end{array}$$

is commutative. Hence f is the composite map:

$$R(\pi(S)) \xrightarrow{\text{Ind}} R(\pi) \xrightarrow{\text{Res}} R(\pi(S)) \xrightarrow{\text{lift}} R(\pi) .$$

To prove the proposition it suffices to prove that Res∘Ind: $R(\pi(S)) \to R(\pi(S))$ induces a map on the class group $\text{Cl}(\pi(S))$ that is injective on the 2-primary part. We prove this by proving that this induced map is of the form $\text{id}+g$, where $g \circ g = 2 \cdot g_0$ for some map g_0. If x is 2-primary and $x+g(x)=0$, then $x = 2^i \cdot g_0^i(x)$ for all $i>0$, and hence $x=0$.

We apply the Mackey decomposition to Res∘Ind. We have the double cosets

$$\begin{aligned} \pi \;=\; & \pi(S) \cup \pi(S)A_{p_1}\pi(S) \cup \ldots \cup \pi(S)A_{p_1}^{(p_1^{n_1}-1)/2}\pi(S) \\ & \cup \pi(S)A_{p_2}\pi(S) \cup \ldots \cup \pi(S)A_{p_2}^{(p_2^{n_2}-1)/2}\pi(S) \\ & \cup \text{etc.} \end{aligned}$$

and

$$A_p^j \pi(S) A_p^{-j} \cap \pi(S) \;=\; \langle A_l, X_p \;\big|\; l \in S \rangle \;=\; \tau(S,p) .$$

Therefore

$$(\text{Res} \circ \text{Ind})(\chi) \;=\; \chi + \sum_{p \notin S} ((p^n-1)/2) \cdot N_p(\chi) ,$$

where N_p is the map

$$N_p \;=\; \text{Ind}_{\tau(S,p)}^{\pi(S)} \circ \text{Res}_{\tau(S,p)}^{\pi(S)} .$$

If $p \in P_1$, $q \in P_2$, the Mackey decomposition gives

$$\begin{aligned} N_p \circ N_q \;&=\; \text{Ind}_{\tau(S,p)}^{\pi(S)} \circ \text{Res}_{\tau(S,p)}^{\pi(S)} \circ \text{Ind}_{\tau(S,q)}^{\pi(S)} \circ \text{Res}_{\tau(S,q)}^{\pi(S)} \\ &=\; \text{Ind}_{\tau(S,p)}^{\pi(S)} \circ \text{Ind}_{\gamma}^{\tau(S,p)} \circ \text{Res}_{\gamma}^{\tau(S,q)} \circ \text{Res}_{\tau(S,q)}^{\pi(S)} \\ &=\; \text{Ind}_{\gamma}^{\pi(S)} \text{Res}_{\gamma}^{\pi(S)} , \end{aligned}$$

where $\gamma = \langle A_l, X_p^2 \mid l \in S \rangle$.

If $p,q \in P_1$ or $p,q \in P_2$, the Mackey decomposition gives

$$N_p \circ N_q \;=\; \text{Ind}_{\tau(S,p)}^{\pi(S)} \circ (\text{id} + \mathscr{C}(Y_p)) \circ \text{Res}_{\tau(S,p)}^{\pi(S)} ,$$

where $\mathscr{C}(Y_p)$ is conjugation by Y_p. Hence, the N_p's are commuting operators satisfying $N_p \circ N_p = 2 \cdot N_p$. Putting

$$N \;=\; \sum_{p \notin S} ((p^n-1)/2) \cdot N_p ,$$

there exists an operator N_0 such that $N \circ N = 2 \cdot N_0$. This completes the proof of the Proposition.

Now we turn to the top component. We shall not detect the top component

of the class group. Instead we detect the top component of the middle term in the exact sequence (0.2).

We shall prove

PROPOSITION 2.2. Res_C *is injective on*

$$[\tilde{E}_p(J(Z(\mathbf{Q}\pi))_+/\mathrm{Nrd}\,(U(\mathbf{Z}\pi)))]_{(2,f)}\,,$$

where the subscript $(2, f)$ *denotes the 2-primary part plus the free part.*

To prove Proposition 2.2 we shall actually have to calculate the 2-primary parts of the local kernel groups $\hat{D}_p(\pi)$. To state the result, we need some terminology. For p a prime and x an integer, we let x'_p denote the p-free part of x. For two pairs (α, β) and (x, y) of integers we call (α, β) full with respect to (x, y) if $\alpha \mid x$, $\beta \mid y$ and any prime that divides x (respectively y) also divides α (respectively β). Then

PROPOSITION 2.3. *The top component of* $\hat{D}_p(\pi)_{(2)}$ $(p \mid a)$ *splits into a direct sum* $D_{p,+} \oplus D_{p,-}$, *where both* $D_{p,+}$ *and* $D_{p,-}$ *are isomorphic to*

$$\prod_{(\alpha,\beta)} \prod_{i=1}^{n(p)} (F_p \otimes \mathbf{Z}[\eta_\alpha, \eta_\beta])^\times_{(2)}\,.$$

Here (α, β) *runs through the pairs of (positive) integers that are full with respect to* (a'_p, b) *and* $p^{n(p)}$ *is the exact power of* p *dividing* a. *Furthermore, the map*

$$\tilde{E}_P(\Phi_p) : \tilde{E}_P(\mathcal{O}^\times_{Z(\mathbf{Q}\pi),+}) \to \tilde{E}(\hat{D}_p(\pi))_{(2)}$$

in the exact sequence describing $\tilde{E}_P(D(\pi))_{(2)}$ *factors into* $\Phi_{p,+} \oplus \Phi_{p,-}$, *where*

$$\Phi_{p,+} : \prod_{(\alpha,\beta)} \prod_{i=1}^{n(p)} \mathbf{Z}[\eta_{\alpha p^i}, \eta_\beta]^\times \to \prod_{(\alpha,\beta)} (F_p \otimes \mathbf{Z}[\eta_\alpha, \eta_\beta])^\times_{(2)}$$

and

$$\Phi_{p,-} : \prod_{(\alpha,\beta)} \prod_{i=1}^{n(p)} \mathbf{Z}[\eta_{\alpha p^i}, \eta_\beta]^* \to \prod_{(\alpha,\beta)} \prod_{i=1}^{n(p)} (F_p \otimes \mathbf{Z}[\eta_\alpha, \eta_\beta])^\times_{(2)}$$

are the natural reduction maps. Furthermore, the top component of $\hat{D}_2(\pi)_{(2)}$ *is trivial.*

PROOF (of Propositions 2.2 and 2.3). First of all we have the exact sequence

$$0 \to U(Z(\mathbf{Q}\pi))_+/\mathrm{Nrd}\,(U(\mathbf{Z}\pi)) \to J(Z(\mathbf{Q}\pi))_+/\mathrm{Nrd}\,(U(\mathbf{Z}\pi)) \to I(\pi) \to 0\,.$$

Here $U(Z(\mathbf{Q}\pi))$ is the group of unit ideles of $Z(\mathbf{Q}\pi)$, and $I(\pi)$ is nothing but the

product of the ideal groups of the field components of $Z(\mathbf{Q}\pi)$. Since $I(\pi)$ is free and $U(Z(\mathbf{Q}\pi))_+/Nrd(U(\mathbf{Z}\pi))$ is finite, we get the split exact sequence

$$0 \to [U(Z(\mathbf{Q}\pi))_+/Nrd(U(\mathbf{Z}\pi))]_{(2)} \to [J(Z(\mathbf{Q}\pi))_+/Nrd(U(\mathbf{Z}\pi))]_{(2,f)} \to$$
$$\to I(\pi) \to 0 .$$

Here,

$$U(Z(\mathbf{Q}\pi))_+/\mathrm{Nrd}\,(U(\mathbf{Z}\pi)) = \prod_{l\mid|\pi|} [(\mathcal{O}_{Z(\mathbf{Q}\pi)_l})^\times/\mathrm{Nrd}\,(\mathbf{Z}_l\pi)^\times)] = \hat{D}(\pi)$$

since $\mathbf{Z}_l\pi$ is a maximal order (respectively $\mathbf{Q}_l\pi$) for $l \nmid |\pi|$ (respectively l infinite). We get the diagram

$$\begin{array}{ccccccc} 0 \to & \hat{D}(\pi) & \to & J(Z(\mathbf{Q}\pi))_+/\mathrm{Nrd}\,(U(\mathbf{Z}\pi)) & \to & I(\pi) & \to 0 \\ & \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} & & & \\ 0 \to & \hat{D}(C) & \longrightarrow & J(\mathbf{Q}C)/\mathrm{Nrd}\,(U(\mathbf{Z}C)) & \longrightarrow & I(C) & \to 0 \end{array}$$

and Proposition 2.2 will be proved if we prove that Res is injective on $(\tilde{E}_P\hat{D}(\pi))_{(2)}$ and on $\tilde{E}_PI(\pi)$.

The group algebra decomposes as

$$\begin{aligned} \mathbf{Q}\pi = {} & \mathbf{Q}_{++} \oplus \mathbf{Q}_{+-} \oplus \mathbf{Q}_{-+} \oplus \mathbf{Q}_{--} \oplus \mathbf{H}_{\mathbf{Q}} \\ & \oplus \prod_{\substack{\alpha \mid a \\ \alpha \neq 1}} (M_2(\mathbf{Q}(\eta_\alpha))_{Y=1} \oplus M_2(\mathbf{Q}(\eta_\alpha))_{Y=-1} \oplus \mathbf{H}_{\mathbf{Q}(\eta_{4\alpha})}) \\ & \oplus \prod_{\substack{\beta \mid b \\ \beta \neq 1}} (M_2(\mathbf{Q}(\eta_\beta))_{X=1} \oplus M_2(\mathbf{Q}(\eta_\beta))_{X=-1} \oplus \mathbf{H}_{\mathbf{Q}(\eta_{4\beta})}) \\ & \oplus \prod_{\substack{\alpha \mid a,\ \beta \mid b \\ \alpha,\beta \neq 1}} (M_4(\mathbf{Q}(\eta_\alpha, \eta_\beta)) \oplus A(\mathbf{Q}(\zeta_{\alpha\beta})/\mathbf{Q}(\eta_\alpha, \eta_\beta))) . \end{aligned}$$

Here $A(\mathbf{Q}(\zeta_{\alpha\beta})/\mathbf{Q}(\eta_\alpha, \eta_\beta))$ is a central simple algebra over $\mathbf{Q}(\eta_\alpha, \eta_\beta)$. This we shall not specify further; we only note that it does not split at $\mathbf{R}$. Furthermore, $\mathbf{Q}_{++} \oplus \mathbf{Q}_{+-} \oplus \mathbf{Q}_{-+} \oplus \mathbf{Q}_{--} \oplus \mathbf{H}_{\mathbf{Q}}$ is the decomposition of $\mathbf{Q}Q(8)$, where $\mathbf{Q}_{++}$ (say) is the summand corresponding to $X=1$, $Y=1$.

We choose representatives for $\mathrm{Irr}\,(\pi)/\Omega$ corresponding to this decomposition, and we choose them in accordance with our choice of roots of unity (cf. section 0). Let us denote these characters

$$\begin{array}{lll} x_1, & x_2, & x_3, x_4, x_5 \\ x_{\alpha,1}, & x_{\alpha,-1}, & x_{\alpha,0} \\ y_{\beta,1}, & y_{\beta,-1}, & y_{\beta,0} \\ z_{\alpha,\beta,1}, & z_{\alpha,\beta,-1} & . \end{array}$$

Then we can tabulate the action of the idempotent E_p on $R(\pi)$ as in Table 2.4, where σ is the generator of Gal $(\mathbf{Q}(i)/\mathbf{Q})$.

χ		$E_p(\chi)$	
x_i	$i=1,\ldots,5$	x_i	
$x_{\alpha,1}$		$x_{\alpha'_p,1}$	(or x_1+x_3 if α is a power of p)
$x_{\alpha,-1}$		$x_{\alpha'_p,-1}$	(or x_2+x_4 if α is a power of p)
$x_{\alpha,0}$		$x_{\alpha'_p,0}$	(or x_5 if α is a power of p)
$y_{\beta,i}$	$i=-1,0,1$	$y_{\beta,i}$	
$z_{\alpha,\beta,1}$		$z_{\alpha'_p,\beta,1}$	(or $y_{\beta,1}+y_{\beta,-1}$ if α is a power of p)
$z_{\alpha,\beta,-1}$		$z_{\alpha'_p,\beta,-1}$	(or $y_{\beta,0}+y^{\sigma}_{\beta,0}$ if α is a power of p)

Table 2.4

From this table we see that $\tilde{E}_P=\prod_{l\in P}(1-E_l)$ maps all characters to zero, except those $z_{\alpha,\beta,1}$ and $z_{\alpha,\beta,-1}$, where (α,β) is full with respect to (a,b). Hence the top component of $R(\pi)/\Omega$ is equal to

$$\prod_{\substack{(\alpha,\beta)\\ \text{full}}} \mathbf{Z}\cdot z_{\alpha,\beta,1} \oplus \prod_{\substack{(\alpha,\beta)\\ \text{full}}} \mathbf{Z}\cdot z_{\alpha,\beta,-1}\,.$$

To calculate Res^{π}_{C} on $\mathrm{Hom}_{\Omega}(R(\pi),J(\bar{\mathbf{Q}}))$ we must induce characters from C to π. We decompose

$$\mathbf{Q}C = \prod_{\substack{\alpha\,|\,a\\ \beta\,|\,b}} (\mathbf{Q}(\zeta_{\alpha\beta})_{T=1}\oplus\mathbf{Q}(\zeta_{\alpha\beta})_{T=-1})\,,$$

where T is the involution of C. We choose representatives of Irr $(C)/\Omega$ according to our choice of roots of unity, and we denote these

$$w_{\alpha,\beta,1}, w_{\alpha,\beta,-1} \qquad \alpha\,|\,a,\ \beta\,|\,b\,.$$

These characters will induce as tabulated in Table 2.5.

χ		$\mathrm{Ind}_C(\chi)$
$w_{1,1,1}$		$x_1+x_2+x_3+x_4$
$w_{1,1,-1}$		$2\cdot x_5$
$w_{\alpha,1,1}$	$\alpha\neq 1$	$x_{\alpha,1}+x_{\alpha,-1}$
$w_{\alpha,1,-1}$	$\alpha\neq 1$	$x_{\alpha,0}+x^{\sigma}_{\alpha,0}$
$w_{1,\beta,1}$	$\beta\neq 1$	$y_{\beta,1}+y_{\beta,-1}$
$w_{1,\beta,-1}$	$\beta\neq 1$	$y_{\beta,0}+y^{\sigma}_{\beta,0}$
$w_{\alpha,\beta,1}$	$\alpha,\beta\neq 1$	$z_{\alpha,\beta,1}$
$w_{\alpha,\beta,-1}$	$\alpha,\beta\neq 1$	$z_{\alpha,\beta,-1}$

Table 2.5.

It follows that on $\tilde{E}_P(J(Z(\mathbf{Q}\pi)))$, Res is the inclusion

$$\prod_{\substack{(\alpha,\beta)\\ \text{full}}} (J(\mathbf{Q}(\eta_\alpha,\eta_\beta))\oplus J(\mathbf{Q}(\eta_\alpha,\eta_\beta))) \to \prod_{\substack{(\alpha,\beta)\\ \text{full}}} (J(\mathbf{Q}(\zeta_{\alpha\beta}))\oplus J(\mathbf{Q}(\zeta_{\alpha\beta})))$$

and on $\tilde{E}_P(I(\pi))$, Res is the inclusion

$$\prod_{\substack{(\alpha,\beta)\\ \text{full}}} (I(\mathbf{Q}(\eta_\alpha,\eta_\beta))\oplus I(\mathbf{Q}(\eta_\alpha,\eta_\beta))) \to \prod_{\substack{(\alpha,\beta)\\ \text{full}}} (I(\mathbf{Q}(\zeta_{\alpha\beta}))\oplus I(\mathbf{Q}(\zeta_{\alpha\beta}))),$$

Now we turn to $\tilde{E}_P(\hat{D}(\pi))$. Let $p\,|\,|\pi|$, $p\neq 2$. We shall calculate $\tilde{E}_P(\hat{D}_p(\pi))_{(2)}$, so we decompose $\mathbf{Z}_p\pi$ and calculate the p'-part of $\tilde{E}_P(\hat{D}_p(\pi))$. Let us assume $p\,|\,a$.

$$\begin{aligned}
\mathbf{Z}_p\pi \;=\; & 2\cdot \mathbf{Z}_p[\mathbf{Z}/p^n]^t[X\,|\,X^2=1]_{Y=\pm 1}\\
& \oplus\; 2\cdot \prod_{\substack{\alpha\,|\,a'_p\\ \alpha\neq 1}} ((\mathbf{Z}_p\otimes \mathbf{Z}[\zeta_\alpha])[\mathbf{Z}/p^n])^t[X\,|\,X^2=1]\\
& \oplus \prod_{\substack{\alpha\,|\,a'_p,\,\beta\,|\,b\\ \beta\neq 1}} ((\mathbf{Z}_p\otimes \mathbf{Z}[\zeta_\alpha,\zeta_\beta])[\mathbf{Z}/p^n])^t[X,Y\,|\,X^2=Y^2=1]\\
& \oplus \prod_{\alpha\,|\,a'_p,\,\beta\,|\,b} ((\mathbf{Z}_p\otimes \mathbf{Z}[\zeta_\alpha,\zeta_\beta])[\mathbf{Z}/p^n])^t[X,Y\,|\,X^2=Y^2=-1]\,.
\end{aligned}$$

We suppress everywhere the relations $X\zeta_\alpha X^{-1}=\zeta_\alpha^{-1}$, $Y\zeta_\beta Y^{-1}=\zeta_\beta^{-1}$. The top component involves only the blocks

$$B_{\alpha,\beta,+} \;=\; ((\mathbf{Z}_p\otimes \mathbf{Z}[\zeta_\alpha,\zeta_\beta])[\mathbf{Z}/p^n])^t[X,Y\,|\,X^2=Y^2=1]$$

and

$$B_{\alpha,\beta,-} \;=\; ((\mathbf{Z}_p\otimes \mathbf{Z}[\zeta_\alpha,\zeta_\beta])[\mathbf{Z}/p^n])^t[X,Y\,|\,X^2=-1]\,,$$

where (α,β) is full with respect to (a'_p,b). We shall have to distinguish between the two cases: (1) when P_1 contains at least two prime powers, and (2) when P_1 contains only p^n. We first consider (1). We assume that $\mathbf{Q}(\zeta_{\alpha\beta})/\mathbf{Q}(\eta_\alpha,\eta_\beta)$ does not split at p (leaving the split case to the reader), so we assume that

$$B_{\alpha,\beta,+} \;=\; \hat{\mathbf{Z}}_p[\zeta_\alpha,\zeta_\beta][\mathbf{Z}/p^n]^t[X,Y\,|\,X^2=Y^2=1]\,,$$

and must calculate the p'-part of

$$\hat{D}(B_{\alpha,\beta,+}) \;=\; \frac{\mathbf{Z}_p[\eta_\alpha,\eta_\alpha]^\times\oplus \mathbf{Z}_p[\eta_{\alpha p},\eta_\beta]^\times\oplus\ldots\oplus \mathbf{Z}_p[\eta_{\alpha p^n},\eta_\beta]^\times}{\mathrm{Nrd}\,((B_{\alpha,\beta,+})^\times)}\,.$$

Let J be the ideal

$$J \;=\; (A_p-1)B_{\alpha,\beta,+}\,.$$

This ideal is topologically nilpotent. Therefore we get an exact sequence

$$0 \to (1+J)^{\times} \to B^{\times}_{\alpha,\beta,+} \to \mathbf{Z}_p[\zeta_\alpha, \zeta_\beta]'[X, Y]X^2 = Y^2 = 1]^{\times} \to 0 .$$

By a result of Auslander-Goldman-Rim, cf. [7, Theorem 40.14] the rightmost term is a maximal order, so by taking reduced norms we get the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & (1+J)^{\times} & \longrightarrow & B^{\times}_{\alpha,\beta,+} & \longrightarrow & \mathbf{Z}_p[\zeta_\alpha, \zeta_\beta]'[X, Y \mid X^2 = Y^2 = 1]^{\times} \to 0 \\
 & & \downarrow Nrd & & \downarrow Nrd & & \downarrow \\
 & & \prod_{i=1}^{n} \mathbf{Z}_p[\eta_{\alpha p^i}, \eta_\beta]^{\times} & \to & \prod_{i=0}^{n} \mathbf{Z}_p[\eta_{\alpha p^i}, \eta_\beta]^{\times} & \longrightarrow & \mathbf{Z}_p[\eta_\alpha, \eta_\beta]^{\times} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & C_J & \longrightarrow & \hat{D}_p(B_{\alpha,\beta,+}) & \longrightarrow & 0 .
\end{array}$$

If we let $N_0, N_1, \ldots, N_n$ denote the components of the reduced norm map on

$$\mathbf{Q}_p(\zeta_\alpha, \zeta_\beta)[\mathbf{Z}/p^n]'[X, Y \mid X^2 = Y^2 = 1] = \prod_{i=0}^{n} M_4(\mathbf{Q}_p(\eta_{\alpha p^i}, \eta_\beta)) ,$$

then for $x \in B_{\alpha,\beta,+}$ we have the relations

$$N_i(x) \equiv N_j(x) \quad \text{in} \quad F_p \otimes \mathbf{Z}[\eta_\alpha, \eta_\beta], \quad \text{all } i, j .$$

This is because ζ_{p^i} maps to 1 by reduction modulo primes above p. This relation implies that

$$\mathrm{Nrd}\,((1+J)^{\times}) \subseteqq \prod_{i=1}^{n} U_1(\mathbf{Z}_p[\eta_{p^i\alpha}, \eta_\beta]) ,$$

where U_1 denotes the first unit group. Since $\mathrm{Nrd}\,(1+J)^{\times})$ is open, there exist integers $m_1, \ldots, m_n$ such that

$$\prod_{i=1}^{n} U_{m_i}(\mathbf{Z}_p[\eta_{\alpha p^i}, \eta_\beta]) \subseteqq Nrd(1+J)^{\times}) .$$

Hence the p'-part of C_J must be equal to

$$\prod_{i=1}^{n} (F_p \otimes \mathbf{Z}[\eta_\alpha, \eta_\beta])^{\times}$$

through reduction modulo the prime in $\mathbf{Z}_p[\eta_{\alpha p^i}, \eta_\beta]$. Furthermore, the above relation implies that on the p'-part, the map $C_J \to \hat{D}_P(B_{\alpha,\beta,+})$ is injective and hence

$$\hat{D}_p(B_{\alpha,\beta,+})_{(p')} \cong \prod_{i=1}^{n} (F_p \otimes \mathbf{Z}[\eta_\alpha, \eta_\beta])^{\times}$$

via the map

$$(x_0, x_1, \ldots, x_n) \to \prod_{i=1}^{n} (\varrho_p(x_i x_0^{-1})) ,$$

where ϱ_p means reduction modulo the appropriate prime over p.

If P_1 contains only p^n, then the group algebra splits in a little different way:

$$\mathbf{Q}(\zeta_\beta)[\mathbf{Z}/p^n]^t[X, Y \mid X^2 = Y^2 = 1]$$
$$= M_2(\mathbf{Q}(\eta_\beta))_{x=1} \oplus M_2(\mathbf{Q}(\eta_\beta))_{x=-1} \oplus \prod_{i=1}^{n} M_4(\mathbf{Q}(\eta_{p^i}, \eta_\beta)) .$$

If we let $N_{0,1}, N_{0,-1}, N_1, \ldots, N_n$ denote the components of the reduced norm corresponding to this decomposition, we get the relations

$$N_{0,1}(x)N_{0,-1}(x) \equiv N_i(x) \quad \text{in } (F_p \otimes \mathbf{Z}[\eta_\beta])$$

for $x \in \mathbf{Z}_p[\zeta_\beta][\mathbf{Z}/p^n]^t[X, Y \mid X^2 = Y^2 = 1]$, $i > 0$. This is essentially contained in section of [1]. Now we can argue like in the previous case and find that the p'-part of $\hat{D}_p(B_{\alpha,\beta,+})$ is equal to $\prod_{i=1}^{n} (F_p \otimes \mathbf{Z}[\eta_\alpha, \eta_\beta])^\times$ via the map

$$(x_{0,1}, x_{0,-1}, x_1, \ldots, x_n) \to \prod (\varrho_p(x_i x_{0,1}^{-1} x_{0,-1}^{-1})) .$$

When using the same technique to calculate $\hat{D}_p(B_{\alpha,\beta,-})$ we get the twisted group ring $\mathbf{Z}_p[\zeta_\alpha, \zeta_\beta]^t[X, Y \mid X^2 = Y^2 = -1]$, where P_1 contains at least two prime powers, and we must prove that the reduced norm is surjective on the units of this ring. But $\mathbf{Q}_p(\zeta_\alpha, \zeta_\beta)/\mathbf{Q}(\eta_\alpha, \eta_\beta)$ is unramified, so the ring is a hereditary order by a result of Williamson-Harada, cf. [7, Theorem 40.15]. From the structure theorem for local hereditary orders, [7, Theorem 39.14], it is obvious that the reduced norm is surjective. One could also argue that since $\mathrm{Trd}(X) = \mathrm{Trd}(Y) = \mathrm{Trd}(XY) = 0$, the proof of [7, Theorem 40.14] carries over to show that the above group ring is actually a maximal order. In any case, the above method applies and we find that the p'-part of $\hat{D}_p(B_{\alpha,\beta,-})$ is equal to $\prod_{i=1}^{n} (F_p \otimes \mathbf{Z}[\eta_\alpha, \eta_\beta])^\times$ if P_1 contains at least two prime powers. If P_1 contains only p^n, the calculation is a little more complicated. The group algebra splits like

$$\mathbf{Q}_p(\zeta_{4\beta})^t[Y \mid Y^2 = -1] \oplus \prod_{i=1}^{n} \mathbf{Q}_p(\zeta_{p^i}, \zeta_\beta)^t[X, Y \mid X^2 = Y^2 = -1] .$$

Letting $N_0, N_1, \ldots, N_n$ denote the components of the reduced norm, we get the relation

$$(N \circ N_0)(x) \equiv N_i(x), \quad i > 0 ,$$

for $x \in \mathbf{Z}_p[\zeta_\beta][\mathbf{Z}/p^n]^t[X, Y \mid X^2 = Y^2 = -1]^\times$. Here N is the norm from $\mathbf{Q}(\eta_{4\beta})$ to $\mathbf{Q}(\eta_\beta) \subseteq \mathbf{Q}(\eta_{p^i}, \eta_\beta)$. This is essentially contained in section 2 of [1]. Therefore, also in this case the p'-part of $\hat{D}_p(B_{\alpha,\beta,-})$ is equal to $\prod_{i=1}^{n} (F_p \otimes \mathbf{Z}[\eta_\beta])^\times$ via the map

$$(x_0, x_1, \ldots, x_n) \to \prod_{i=1}^{n} (\varrho_p(x_i N(x_0)^{-1})) .$$

Finally, it is easy to see that $\tilde{E}_P(D_2(\pi))_{(2)}$ is trivial. This proves Proposition 2.3.

We turn to the cyclic group C. If $p \,||\, C|$, $p \neq 2$, we decompose

$$\mathbf{Z}_p C = \prod_{\substack{\alpha \mid a'_p \\ \beta \mid b}} ((\mathbf{Z}_p \otimes \mathbf{Z}[\zeta_\alpha, \zeta_\beta])[\mathbf{Z}/p^n]_{T=1} \oplus (\mathbf{Z}_p \otimes \mathbf{Z}[\zeta_\alpha, \zeta_\beta])[\mathbf{Z}/p^n]_{T=-1}) \cdot$$

The top component involves only the blocks

$$A_{\alpha,\beta,+} = (\mathbf{Z}_p \otimes \mathbf{Z}[\zeta_\alpha, \zeta_\beta])[\mathbf{Z}/p^n]_{T=1}$$

$$A_{\alpha,\beta,-} = (\mathbf{Z}_p \otimes \mathbf{Z}[\zeta_\alpha, \zeta_\beta])[\mathbf{Z}/p^n]_{T=-1},$$

where (α, β) is full with respect to (a'_p, b). By the same method as above we find that

$$\tilde{E}_P(\hat{D}_p(A_{\alpha,\beta,\pm}))_{(p')} \cong \prod_{i=1}^{n} (\boldsymbol{F}_p \otimes \mathbf{Z}[\zeta_\alpha, \zeta_\beta])^\times$$

via reduction modulo primes above p. Furthermore it is easy to see that $\tilde{E}_P(\hat{D}_2(A_{\alpha,\beta,\pm}))_{(2)} = 0$.

According to the remark after Lemma 1.1 the maps Res^π_C and $\tilde{E}_P$ commute on the character rings. Hence $\mathrm{Res}_C(\tilde{E}_P(\hat{D}(\pi))) \subseteq \tilde{E}_P(\hat{D}(C))$. Comparing the results just obtained on the structure of $\tilde{E}_P(\hat{D}(\pi))$ and $\tilde{E}_P(\hat{D}(C))$ with the tabulation of the induction map $R(C) \to R(\pi)$ (Table 2.5), we see that on $\tilde{E}_P(\hat{D}(\pi))_{(2)}$ the restriction map Res_C is the inclusion

$$\prod_{(\alpha,\beta)} \prod_{i=1}^{n} (\boldsymbol{F}_p \otimes \mathbf{Z}[\eta_\alpha, \eta_\beta)^\times_{(2)} \to \prod_{(\alpha,\beta)} \prod_{i=1}^{n} (\boldsymbol{F}_p \otimes \mathbf{Z}[\zeta_\alpha, \zeta_\beta])^\times_{(2)}$$

and therefore injective. Since $\tilde{E}_P(\hat{D}_2(\pi))_{(2)} = 0$, this completes the proof of Proposition 2.2.

3. Proof of the main theorem.

LEMMA 3.1. *$\sigma_4(e_0)$ is of order at most 2. If $S \cap P_1$ or $S \cap P_2$ is empty, then $\tilde{E}_S(\sigma_4(e_0)) = 0$.*

PROOF. Let χ_C be the character on the cyclic group C that restricts to χ_p on the p-part ($p \mid ab$) and to the non-trivial character on $\mathbf{Z}/2$. Let $\psi_C = \mathrm{Ind}^\pi_C(\chi_C)$. By the Mackey decomposition theorem ψ_C restricts to $2 \cdot (\chi_p + \chi_p^{-1})$ on the p-part of π, for $p \mid ab$, and to $2 \cdot \chi_2$ on $\mathbf{Q}(8)$. In other words, $c_4(\psi_C)$ restricts like the cup product square $e_0^2 \in H^8(\pi, \mathbf{Z})$ so $e_0^2 = c_4(\psi_C)$. Hence $0 = \sigma_8(e_0^2) = 2 \cdot \sigma_4(e_0)$. This proves half of the lemma. To prove the second half it suffices to prove that $\tilde{E}_S(\sigma_4(e_0)) = 0$ when $S = P_1$. By the naturality of σ_4 with respect to restriction

and Proposition 2.1 it suffices to prove that $\sigma_4(\mathrm{Res}^{\pi}_{\pi(P_1)}(e_0))=0$. Let $e=\mathrm{Res}^{\pi}_{\pi(P_1)}(e_0)$. Then e maps to $c_2(\chi_p+\chi_p^{-1})$ by restriction to the p-part ($p\,|\,a$) and to $c_2(\chi_2)$ by restriction to $Q(8)$. The group $Q(8a,1)$ contains the cyclic subgroup C_1 of order $4a$. Let χ be the character on C_1 that restricts to χ_p (for $p\,|\,a$) on the p-part and to a faithful character on $\mathbf{Z}/4$. Let ψ be the representation of $Q(8a,1)$ induced from χ. Then ψ restricts to $\chi_p+\chi_p^{-1}$ on the p-part ($p\,|\,a$) and to χ_2 on $Q(8)$. Hence $c_2(\psi)=e$ and therefore $\sigma_4(e)=0$.

Now we turn to the top component of $\sigma_4(e_0)$. In the previous section we found that

$$\tilde{E}_P(\hat{D}(\pi))_{(2)} = \prod_{p\,|\,ab} (D_{p,+}\oplus D_{p,-}) .$$

Let Σ be the element of $\tilde{E}_P(\hat{D}(\pi))_{(2)}$ that decomposes as $\prod_{p|a}(1,4)\times\prod_{p|b}(-1,4)$ where we consider $F_p^{\times}$ mapped diagonally into $(F_p\otimes\mathbf{Z}[\eta_\alpha,\eta_\beta])^{\times}_{(2)}$.

PROPOSITION 3.2. *The top component $\tilde{E}_P(\sigma_4(e_0))$ is contained in the kernel group $D(\pi)$ and under the map $\partial\colon \tilde{E}_P(\hat{D}(\pi))_{(2)}\to\tilde{E}_P(D(\pi))$ it is the image of Σ.*

PROOF. Corresponding to e_0 there is a periodic projective resolution of $\mathbf{Z}$ over $\mathbf{Z}\pi$. Following [8], this resolution defines an idele

$$\hat{\tau}_4(e_0)\in J(Z(\mathbf{Q}\pi))_+/Nrd(U(\mathbf{Z}\pi))$$

and $\partial\hat{\tau}_4(e_0)=\sigma_4(e_0)$, cf. [8, Theorem 9.1]. Our calculations center around the diagram

$$\begin{array}{ccccccccc}
0 & \to & \tilde{E}_P\left(\dfrac{Z(\mathbf{Q}\pi)_+^{\times}}{Nrd(U(\mathbf{Z}\pi))\cap Z(\mathbf{Q}\pi)^{\times}}\right)_{(2,f)} & \to & \tilde{E}_P\left(\dfrac{J(Z(\mathbf{Q}\pi))_+}{Nrd(U(\mathbf{Z}\pi))}\right)_{(2,f)} & \xrightarrow{\partial} & \tilde{E}_P(\mathrm{Cl}\,(\pi))_{(2)} & \to & 0\\
 & & \downarrow\mathrm{Res} & & \downarrow\mathrm{Res} & & \downarrow\mathrm{Res} & & \\
0 & \longrightarrow & \tilde{E}_P\left(\dfrac{QC^{\times}}{Nrd(U(\mathbf{Z}C))\cap QC^{\times}}\right) & \longrightarrow & \tilde{E}_P\left(\dfrac{J(QC)}{Nrd(U(\mathbf{Z}C))}\right) & \xrightarrow{\partial} & \tilde{E}_P(\mathrm{Cl}\,(C)) & \longrightarrow & 0
\end{array}$$

remembering that Res on $\tilde{E}_P(J(Z(\mathbf{Q}\pi))_+/Nrd(U(\mathbf{Z}\pi)))_{(2,f)}$ is just inclusion of ideal groups and residue fields. When restricting to C we have $\mathrm{Res}\,(e_0)=c_2(\chi_C+\chi_C^{-1})$, where χ_C is the faithful character on C from Lemma 3.1. Hence $\mathrm{Res}\,(\tilde{E}_P(\sigma_4(e_0)))=0$ and

$$\tilde{E}_P(\hat{\tau}_4(\mathrm{Res}\,(e_0))) = \tilde{E}_P(\tau_4(\mathrm{Res}\,(e_0))) ,$$

where τ_4 is the Reidemeister torsion:

$$\tau_4(\mathrm{Res}\,(e_0))\in QC^{\times}/QC^{\times}\cap Nrd(U(\mathbf{Z}C)) .$$

According to [6], $\tau_4(c_2(\chi_C+\chi_C^{-1}))$ is the class of

$$w_{\alpha,\beta,\pm 1} \to 2\mp\eta_{\alpha\beta} \qquad (\alpha\,|\,a,\ \beta\,|\,b;\ \alpha\cdot\beta\neq 1)$$
$$w_{1,1,1} \to (4a^2b^2)^{-1}$$
$$w_{1,1,-1} \to 4$$

in $\mathbf{Q}C^\times/\mathbf{Q}C^\times \cap U(\mathbf{Z}C)$. The action of $\tilde{E}_P$ on $J(\mathbf{Q}C)$ is easily calculated by restriction of characters, and we get:

If (α,β) is not full with respect to (a,b), then $\tilde{E}_P w_{\alpha,\beta,\pm 1}=0$, while

$$\tilde{E}_P w_{\alpha,\beta,1} = w_{\alpha,\beta,1} - W^+_{(1)} + W^+_{(2)} - \ldots + (-1)^i W^+_{(i)} + \ldots$$
$$\tilde{E}_P w_{\alpha,\beta,-1} = w_{\alpha,\beta,-1} - W^-_{(1)} + W^-_{(2)} - \ldots + (-1)^i W^-_{(i)} + \ldots,$$

where $W^\pm_{(i)}$ is the sum of all $w_{\gamma,\delta,\pm 1}$ with $\gamma\delta$ containing all prime powers in $\alpha\beta$ except i of them.

Hence $\tilde{E}_P(\hat{\tau}_4(\mathrm{Res}\, e_0))$ is the class of

$$w_{\alpha,\beta,\pm 1} \to (2\mp\eta_{\alpha\beta})\bar{W}^\pm_{(1)}(\alpha,\beta)^{-1}\bar{W}^\pm_{(2)}(\alpha,\beta)\ldots\bar{W}^\pm_{(i)}(\alpha,\beta)^{(-1)^i}\ldots$$

in $J(\mathbf{Q}C)/U(\mathbf{Z}C)$. Here $\bar{W}^\pm_{(i)}(\alpha,\beta)$ is (for $i<n$) the product of all $2\mp\eta_{\gamma\delta}$ with $\gamma\delta$ containing all prime powers in $\alpha\beta$ except i of them and $\bar{W}^+_{(n)}(\alpha,\beta)=1/4a^2b^2$ and $\bar{W}^-_{(n)}(\alpha,\beta)=4$ ($n=\mathrm{card}\,(P)$). A typical factor in $\bar{W}^\pm_{(n-1)}(\alpha,\beta)$ looks like $(2\mp\eta_{p^m})$, and since $\tilde{E}_P(Z(\mathbf{Q}\pi)^\times_+)$ contains $\mathbf{Q}(\eta_\alpha,\eta_\beta)^*\supseteq\mathbf{Q}(\eta_{p^m})^*$, we see that $\tilde{E}_P(\sigma_4(e_0))$ is the image under ∂ of the element ξ of $\tilde{E}_P(J(Z(\mathbf{Q}\pi))_+/\mathrm{Nrd}\,(U(\mathbf{Z}\pi)))$ that restricts to the element given by

$$w_{\alpha,\beta,\pm 1} \to (2\mp\eta_{\alpha\beta})\bar{W}^\pm_{(1)}(\alpha,\beta)^{-1}\ldots\bar{W}^\pm_{(n-2)}(\alpha,\beta)^{(-1)^{n-2}} = X^\pm_{\alpha,\beta}.$$

Obviously $X^\pm_{\alpha,\beta}$ is a unit, so $\xi\in\hat{D}(\pi)$, and therefore $\tilde{E}_P(\sigma_4(e_0)\in D(\pi)$. Since ζ_{p^m} maps to 1 by reduction module (primes above) p, we find that by reduction modulo p, most terms in $X^\pm_{\alpha,\beta}$ cancel out, and we are left with

$$Y^+_{\alpha,\beta} = \left(\prod_{\substack{q^i\|\alpha\beta\\ q\neq p}}(2-\eta_{q^i})\right)^{\pm 1} \in (\boldsymbol{F}_p\otimes\mathbf{Z}[\eta_{\alpha'_p},\eta_\beta])^\times$$

and

$$Y^-_{\alpha,\beta} = \left(\prod_{\substack{q^i\|\alpha\beta\\ q\neq p}}(2+\eta_{q^i})\right)^{\pm 1} \in (\boldsymbol{F}_p\otimes\mathbf{Z}[\eta_{\alpha'_p},\eta_\beta])^\times.$$

Here $\|$ means "exactly dividing". Adjusting $X^-_{\alpha,\beta}$ by the number $\prod_{q^i\|\alpha\beta}(2+\eta_{q^i})\in\mathbf{Z}[\eta_\alpha,\eta_\beta]^*$, we find that the $D_{p,-}$-components of $\tilde{E}_P(\sigma_4(e_0))$ are exactly the $D_{p,-}$-components of Σ, namely 4. To get the $D_{p,+}$-components we need the following lemma, guaranteeing the existence of a certain unit:

LEMMA 3.3. *Let S_1 and S_2 be two sets of (relatively prime) prime powers. For each set $\mathcal{M}$ of prime powers, define*

$$\eta_{\mathcal{M}} = \begin{cases} 2 & \text{if } \mathcal{M} = \varnothing \\ \eta_x & \text{if } x = \prod p^i, \text{ where } p^i \text{ runs through } \mathcal{M},\ \mathcal{M} \neq \varnothing . \end{cases}$$

Let

$$t_{S_1,S_2,2i} = \begin{cases} \prod_{\substack{S \subseteq S_1 \cup S_2 \\ \operatorname{card}(S)=2i}} (\eta_{S\cap S_1} - \eta_{S\cap S_2}) & \text{if } 2 \leqq 2i \leqq \operatorname{card}(S_1 \cup S_2) \\ 1 & \text{otherwise} \end{cases}$$

$$n_{S_1,S_2,2i+1} = \begin{cases} \prod_{\substack{S \subseteq S_1 \cup S_2 \\ \operatorname{card}(S)=2i+1}} (\eta_{S\cap S_1} - \eta_{S\cap S_2}) & \text{if } 3 \leqq 2i+1 \leqq \operatorname{card}(S_1 \cup S_2) \\ 1 & \text{otherwise} . \end{cases}$$

Put

$$t_{S_1,S_2} = \prod_{i=1}^{\infty} t_{S_1,S_2,2i}$$

$$n_{S_1,S_2} = \prod_{i=1}^{\infty} n_{S_1,S_2,2i+1} .$$

Then t_{S_1,S_2} and n_{S_1,S_2} are units of $\mathbf{Z}[\eta_{S_1}, \eta_{S_2}]$ and

$$\varrho_{\mathfrak{p}}\left(\frac{t_{S_1,S_2}}{n_{S_1,S_2}}\right) = \begin{cases} (-1)^{\operatorname{card}(S_1)-1} \prod_{\substack{q^i \in S_1 \cup S_2 \\ q \neq p}} (2 - \eta_{q^i}) & \text{if } p \in S_1 \\ (-1)^{\operatorname{card}(S_1)} \prod_{\substack{q^i \in S_1 \cup S_2 \\ q \neq p}} (2 - \eta_{q^i}) & \text{if } p \in S_2 . \end{cases}$$

Here $\mathfrak{p}$ is any prime of $\mathbf{Z}[\eta_{S_1}, \eta_{S_2}]$ lying above p.

The proof of this lemma goes by induction on card $(S_1 \cup S_2)$ and is left to the reader. Applying this lemma to adjust $X^{+}_{\alpha',\beta'}$ we get $D_{p,+}$-components exactly as stated in Proposition 3.

Now we can easily prove the main theorem: The choice of almost linear generator is natural with respect to restriction to the subgroups $\pi(S)$. By Corollary 1.2, Proposition 2.1, and Proposition 3.2 we can therefore conclude that $\tilde{E}_S(\sigma_4(e_0))$ is zero for all S with $S \cap P_1 \neq \varnothing$, $S \cap P_2 \neq \varnothing$ exactly when the conditions (1) and (2) of the theorem are fulfilled. By Lemma 3.1 the other components of $\sigma_4(e_0)$ are always zero. This completes the proof of the main theorem.

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