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THE EXISTENCE OF GENERIC FREE RESOLUTIONS AND RELATED OBJECTS

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One of the subjects on which commutative algebra has made significant progress during the last decade is the theory of finite free resolutions. Some of the most remarkable recent results in this area, like the Buchsbaum-Eisenbud theorems [4] and the syzygy theorem of Evans-Griffith [8], concern the generic structure of finite free resolutions whose complete determination, according to Hochster [9], is the ultimate object of the structure theory of finite free resolutions.

To be specific: Hochster calls a pair (S, G) consisting of a commutative ring S and a finite free resolution (with specified bases)

$$G\colon 0 \to S^{b_n} \to S^{b_{n-1}} \to \ldots \to S^{b_1} \to S^{b_0}$$

generic of type $(b_n, \ldots, b_0)$ if every finite free resolution F of this type over a commutative ring R is a specialization of G, i.e. if there is a ring homomorphism $\varphi\colon S \to R$ such that $F = G \otimes R$ with respect to φ. Hochster conjectures that for every possible type $(b_n, \ldots, b_0)$ there exists a generic pair (S, G) and that, furthermore, S can be taken as a $\mathbf{Z}$-algebra of finite type. He proves this conjecture for $n \leqq 2$ by showing that in this case the generic pair can even be chosen to be *universal*: the extension φ is always unique then [9, Theorem 7.2].

In this article we prove the first part of Hochster's conjecture in full generality whereas we only establish a weaker result in regard to the finiteness of S: there exist generic free resolutions in which S is a countably generated $\mathbf{Z}$-algebra. For $n \leqq 2$ we reproduce Hochster's universal pairs, and show that the existence of universal free resolutions is essentially limited to this case.

The range of application of our construction is not bounded to finite free resolutions. It works for all classes of objects which can be defined by general "exactness conditions" for systems of polynomial equations, including periodic free resolutions, perfect free resolutions, and complexes like G above, which are only required to be acyclic locally over certain subsets of the spectrum of the underlying ring.

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The structure theory of finite free resolutions whose original developments were mostly restricted to noetherian rings was extended to the class of all commutative rings by Northcott [11]. If Hochster's conjecture concerning the finiteness of the rings underlying generic finite free resolutions is true, then there is nothing special about such resolutions over noetherian rings. However, not far away from finite free resolutions one can find objects for which the general and the noetherian theory differ significantly: Examples are supplied by exact sequences

$$R \to R \to R^2 \to R$$

and periodic exact sequences

$$\begin{tikzcd} R \arrow[r] & R^2 \arrow[r] & R \arrow[ll, bend left] \end{tikzcd} .$$

NOTATIONS AND TERMINOLOGY. All rings and algebras R are assumed to be commutative and to contain an identity element $1 \neq 0$. For the general theory of commutative rings we refer the reader to Matsumura's book [10]. The notion of *grade* is always used for the true grade as defined in [11]. We only attribute a *rank* to a matrix if its rank and its reduced rank in the sense of [11] coincide. For systematic reasons we allow matrices with zero rows or columns. A *u-minor* of a (m, n)-matrix x is the determinant of a (u, u)-submatrix of x. The value of a u-minor is 1, if $u=0$, and 0, if $u > \min(m, n)$. The ideal generated by the u-minors of a matrix x is denoted by $\mathrm{I}_u(x)$. The matrix x has rank r if and only if grade $\mathrm{I}_r(x) \geqq 1$ and $\mathrm{I}_{r+1}(x)=0$. A (m, n)-matrix specifies a map $R^m \to R^n$.

Let $K \subset \mathbf{Z}$ be an interval and $(b_k)_{k \in K}$ a family of non-negative integers. A *complex of type* $(b_k)_{k \in K}$ (of finitely generated free modules with specified bases) is a sequence $C = (x^k)_{k \in K}$ of matrices over R, where x^k is a (b_k, b_{k-1})-matrix and the sequence

$$\ldots \to R^{b_k} \xrightarrow{x^k} R^{b_{k-1}} \xrightarrow{x^{k-1}} R^{b_{k-2}} \to \ldots$$

is a complex in the usual sense. By abuse of language we also call the pair (R, C) a complex of type $(b_k)_{k \in K}$. Given complexes $(R, (x^k))$ and $(\tilde{R}, (\tilde{x}^k))$ of the same type, $(\tilde{R}, (\tilde{x}^k))$ is said to be a (*universal*) *specialization* of $(R, (x^k))$, if there exists a (unique) ring homomorphism $\varphi\colon R \to \tilde{R}$ such that $\varphi(x_{ij}^k) = \tilde{x}_{ij}^k$ for all k, i, j. It is understood that a *finite free resolution* of type $(b_n, \ldots, b_0)$ is an acyclic complex of type $(0, b_n, \ldots, b_0)$.

1. Generic finite free resolutions.

Let R be a commutative ring and

$$f:\ 0 \to R^{b_n} \xrightarrow{a^n} R^{b_{n-1}} \to \ldots \to R^{b^1} \xrightarrow{a^1} R^{b_0}$$

a finite free resolution of type $(b_n, \ldots, b_0)$. Since the ranks of the cokernels of the maps (represented by the matrices) a^k are non-negative, the inequality

$$r_k := \sum_{j=k}^{n} (-1)^{k-j} b_j \geqq 0$$

holds for $k=0, \ldots, n$. We refer to this set of inequalities as the *rank condition* for $(b_n, \ldots, b_0)$ and to the numbers r_k as the *ranks* associated with $(b_n, \ldots, b_0)$. If $(b_n, \ldots, b_0)$ satisfies the rank condition, then there are obviously free resolutions of type $(b_n, \ldots, b_0)$ over every commutative ring R.

THEOREM 1. *Let $(b_n, \ldots, b_0)$ be a sequence of non-negative integers satisfying the rank condition. Then there exists a generic free resolutions (S, G) of type $(b_n, \ldots, b_0)$, in which S is a countably generated $\mathbf{Z}$-algebra.*

PROOF. We construct a direct system $(S_i, \alpha_i \colon S_i \to S_{i+1})_{i \in \mathbf{N}}$ of finitely generated $\mathbf{Z}$-algebras and ring homomorphisms α_i and a complex G_0 of type $(0, b_n, \ldots, b_0)$ over S_0 such that the following conditions are satisfied for the system (S_i, α_i) and the associated direct system (G_i) of complexes, $G_i := G_0 \otimes S_i$:

(a) The homology of G_i is mapped to zero by the induced homomorphism $\mathrm{H}(G_i) \to \mathrm{H}(G_{i+1})$ for all $i \in \mathbf{N}$.

(b) For every finite free resolution (R, F) of type $(b_n, \ldots, b_0)$ there exist ring homomorphisms $\varphi_i(R, F) \colon S_i \to R$ such that

(i) $F = G_i \otimes R$ (relative to $\varphi_i(R, F)$),

(ii) the diagram
$$\begin{tikzcd} S_i \arrow[rr, "\alpha_i"] \arrow[dr, "\varphi_i(R,F)"'] & & S_{i+1} \arrow[dl, "\varphi_{i+1}(R,F)"] \\ & R & \end{tikzcd}$$

is commutative for all $i \in \mathbf{N}$.

Let S be the direct limit of the rings S_i, and choose $G := G_0 \otimes S$. The complex G is acyclic by virtue of condition (a), and every free resolution (R, F) of type $(b_n, \ldots, b_0)$ is a specialization of (S, G) via the direct limit $\varphi(R, F)$ of the homomorphisms $\varphi_i(R, F)$. Hence (S, G) is a generic free resolution of type $(b_n, \ldots, b_0)$.

We begin the construction by choosing

$$S_0 = \mathbf{Z}[X_{ij}^k : k=1, \ldots, n,\ i=1, \ldots, b_k,\ j=1, \ldots, b_{k-1}]/\mathfrak{a}$$

where the X_{ij}^k form a system of indeterminates over $\mathbf{Z}$ and $\mathfrak{a}$ is generated by the entries of the product matrices $X^k X^{k-1}$, $k=2, \ldots, n$. There is an obvious choice for G_0:

$$G_0 : 0 \to S_0^{b_n} \xrightarrow{x^n} S_0^{b_{n-1}} \to \ldots \to S_0^{b_1} \xrightarrow{x^1} S_0^{b_0}$$

is the complex of the maps represented by the matrices x^k whose entries are the residue classes of the X^k_{ij}.

Assume that $S_0, \ldots, S_m$ have been constructed as desired. G_m is a complex of finitely generated modules over the noetherian ring S_m, hence its homology is finitely generated. Let the cycles

$$y^k_u = (y^k_{u1}, \ldots, y^k_{ub_k}) \in S^{b_k}_m, \qquad u=1,\ldots,u_k\,,$$

represent a system of generators of $\mathrm{H}_k(G_m)$, $k=1,\ldots,n$. We take

$$S_{m+1} := S_m[Z^{kl}_u\colon k=1,\ldots,n-1,\ u=1,\ldots,u_k,\ 1=1,\ldots,b_{k+1}]/\mathfrak{b}\,,$$

the Z^{kl}_u forming a system of indeterminates over S_m and the ideal $\mathfrak{b}$ being generated by the polynomials

$$y^k_{uj} - \sum_{1=j}^{b_{k+1}} Z^{kl}_u x^{k+1}_{lj}, \qquad k=1,\ldots,n-1,\ u=1,\ldots,u_k,\ j=1,\ldots,b_k\,,$$

and the elements

$$y^n_{uj}, \qquad u = 1,\ldots,u_n, \qquad j=1,\ldots,b_n\,.$$

(Here x^{k+1}_{lj} denotes the image of $x^{k+1}_{lj} \in S_0$ in S_m.) Finally we let $\alpha_m\colon S_m \to S_{m+1}$ be the natural homomorphism. For this choice of S_{m+1} and α_m condition (a) above is evidently fulfilled.

Let now R be an arbitrary commutative ring and

$$F : 0 \to R^{b_n} \xrightarrow{a^n} R^{b_{n-1}} \to \ldots \to R^{b_1} \xrightarrow{a^1} R^{b_0}$$

a finite free resolution over R. By induction hypothesis there is a ring homomorphism $\pi := \varphi_m(R,F)\colon S_m \to R$ such that $a^k_{ij} = \pi(x^k_{ij})$ for all k,i,j. The elements

$$d^k_u := (\pi(y^k_{u1}), \ldots, \pi(y^k_{ub_k})) \in R^{b_k}$$

are cycles. Since F is acyclic there exist elements

$$c^{kl}_u \in R,\ k=1,\ldots,n-1,\ u=1,\ldots,u_k,\ l=1,\ldots,b_{k+1}\,,$$

such that

$$(*) \qquad \pi(y^k_{uj}) = \sum_{l+1}^{b_{k+1}} c^{kl}_u \pi(x^{k+1}_{lj}) = 0, \qquad k=1,\ldots,n-1,\ u=1,\ldots,u_k\,,$$
$$j=1,\ldots,b_k\,.$$

Furthermore

$$\pi(y^n_{uj}) = 0,\ u=1,\ldots,u_n,\ j=1,\ldots,b_n\,.$$

Hence the homomorphism

$$\beta : S_m[Z_u^{kl}\colon k, u, l \text{ as above}] \to R$$

given by $\beta | S_m := \pi$ and $\beta(Z_u^{kl}) := c_u^{kl}$ factors through S_{m+1} and we take the induced map to be $\varphi_{m+1}(R, F)$, thereby satisfying property (b).

Since $(b_n, \ldots, b_0)$ obeys the rank condition, there exists a homomorphism $\varphi_{m+1}(\mathbf{Z}, F)$: $S_{m+1} \to \mathbf{Z}$, ensuring us against S_{m+1} being the null ring! —

For future reference we call the construction above *generic exactification*.

If a complex G_m is acyclic over S_m, then (S_m, G_m) is of course a generic free resolution. A priori it is by no means clear that this favourable behaviour can be forced to occur for all types $(b_n, \ldots, b_0)$. It does of course occur in the extremely trivial case in which $n = 1$. Here G_0 is acyclic already. For $n = 2$ it is not hard to show that our construction yields Hochster's universal free resolution:

COROLLARY. *For $n \leqq 2$ every generic free resolution constructed by generic exactification is universal, hence isomorphic to Hochster's universal free resolution.*

PROOF. We have to show that all the maps $\varphi_i(R, F)$ as above are unique. Since $\varphi_0(R, F)$ is certainly unique we may assume that all the maps $\varphi_0(R, F), \ldots, \varphi_m(R, F)$ are unique by induction. $\varphi_{m+1}(R, F)$ is completely determined by $\varphi_m(R, F)$ and the images of the indeterminates Z_u^{kl}. These have to be mapped to coefficients c_u^{kl} satisfying (*). On the other hand, such coefficients are uniquely determined, since the rows of the map a^n are linearly independent.

So we see that the uniqueness property of universal free resolutions is just the uniqueness of the coefficients in a linear comination of linearly independent rows. In Section 2 we will determine all the types $(b_n, \ldots, b_0)$ for which there exist universal free resolutions and see that they are essentially given by the preceding corollary.

Though we can not even offer a mildly convincing plausibility argument, we believe in the following conjecture:

CONJECTURE. *The process of generic exactification can be made eventually stationary for all possible types of finite free resolutions.*

With almost no effort the technique of generic exactification can be used to provide generic models for the objects in certain classes of which finite free resolutions are just a rather special case. We give some examples:

(a) Bounded or unbounded complexes of free modules

$$C : \ldots \to R^{b_n} \xrightarrow{a^n} R^{b_{n-1}} \to \ldots$$

whose homology modules $\mathrm{H}_k(C)$ vanish for certain fixed k. We will find in Section 3 that the underlying ring of a generic exact sequence of type

$$R \to R \to R^2 \to R$$

can not be chosen noetherian.

(b) Perfect free resolutions, i.e. finite free resolutions whose dual is also acyclic. These resolutions occur as resolutions of perfect modules, in particular as resolutions of Cohen–Macaulay residue class rings of regular rings. A celebrated special case is the Hilbert–Burch theorem [4, Theorem 0] which gives an explicit description of the universal perfect resolution

$$0 \to S^n \to S^{n+1} \to S\,.$$

A second famous result is the theorem of Buchsbaum and Eisenbud about the structure of free resolutions

$$0 \to R \xrightarrow{x^3} R^n \xrightarrow{x^2} R^n \xrightarrow{x^1} R$$

which satisfy the condition $x^3 = x^{1*}$ [5]. (Here * denotes transposition.)

(c) Periodic free resolutions

$$\begin{tikzcd} R^{b_n} \arrow[r, "x^n"] & R^{b_{n-1}} \arrow[r] & \ldots \arrow[r] & R^{b_1} \arrow[r, "x^1"] & R^{b_0} \arrow[llll, bend left, "x^0"] \end{tikzcd}$$

of type $(b_n, \ldots, b_0)$. Their generic structure is completely known for $n=1$, $(b_1, b_0) = (m, m)$. In this case the universal periodic complex of the given type is already acyclic, so provides the universal periodic free resolution; for $m=2$ cf. [13], for general m this assertion is a consequence of results of Huneke and Strickland (private communication). On the other hand, for all $n > 1$ and all possible $(b_n, \ldots, b_0)$ with $b_i \geqq 1$ for $i = 0, \ldots, n$ the universal periodic complex is never acyclic!

Eisenbud [7] extensively studied periodic resolutions. His results indicate that periodic resolutions over noetherian rings have period 1 or 2 (after a suitable choice of bases). As a by-product of the non-noetherian example mentioned above we will obtain a generic free resolution

$$\begin{tikzcd} S \arrow[r] & S^2 \arrow[r] & S \arrow[ll, bend left] \end{tikzcd}$$

which is non-split and (necessarily) has S non-noetherian. It is very likely that the rings underlying generic periodic free resolutions of period $n+1$, $n>1$, are never noetherian (except for some trivial exceptions).

(d) Let R be a commutative ring, and $C=(a^k)$ a complex of type $(0, b_n, \ldots, b_0)$. We call C *acyclic in grade t* if the localizations $C \otimes R_\mathfrak{p}$ are acyclic for all prime ideals $\mathfrak{p}$ of R with $\operatorname{grade}\mathfrak{p} \leqq t$. The exactness criterion of Buchsbaum-Eisenbud [3] and Peskine-Szpiro [12] (cf. [11] for the general version needed here) shows that C is acyclic in grade t if and only if

$$\operatorname{grade} \mathrm{I}_{r_k}(a^k) \geqq \min(t+1, k), \qquad k=1, \ldots, n,$$

the integers r_k being the ranks associated with $b := (b_n, \ldots, b_0)$. The condition on $\operatorname{grade} \mathrm{I}_{r_k}(a^k)$ can be expressed by the acyclicity of a suitable truncation of the Koszul complex in the generators of $\mathrm{I}_{r_k}(a^k)$, and therefore the technique of generic exactification can be used to obtain complexes $(S_t(b), G_t(b))$ for every t and every b, such that $(S_t(b), G_t(b))$ is generic for complexes acyclic in grade t. For $t=0, 1$ the truncations of the Koszul complexes to be "exactified" have lengths 1 and 2 respectively. Hence $(S_t(b), G_t(b))$ are universal for $t=0$ and $t=1$. Their structure is explicitely known, cf. section 2.

Instead of considering all commutative rings one can restrict oneself to the class of B-algebras over an arbitrary commutative ring B. After a replacement of $\mathbf{Z}$ by B throughout, the technique of generic exactification leads to a generic object (S_B, G_B) for the class of B-algebras.

2. Universal free resolutions.

Our proof of the non-existence of universal free resolutions for essentially all types $(b_n, \ldots, b_0)$ with $n \geqq 3$ is based on the observation that universal free resolutions already specialize to complexes which are just acyclic in grade 1:

PROPOSITION 1. *Let (S, G) be a complex of type $(0, b_n, \ldots, b_0)$ such that every free resolution of type $(b_n, \ldots, b_0)$ is a universal specialization of (S, G), and let (R, C) be a complex of type $(0, b_n, \ldots, b_0)$ which is acyclic in grade* 1. *Then (R, C) is a universal specialization of (S, G).*

PROOF. Let C be given by the sequence (a^k) of matrices. $C \otimes R_f$ is acyclic for all elements

$$f \in \prod_{k=2}^{n} \mathrm{I}_{r_k}(a^k) =: \mathfrak{b},$$

and the ideal $\mathfrak{b}$ has grade at least two by hypothesis. The uniqueness of all the extensions $S \to R_f$ implies that these homomorphisms are induced by an extension $S \to \Gamma(\operatorname{Spec} R \setminus V(\mathfrak{b}), \mathcal{O})$, $\mathcal{O}$ denoting the structure sheaf of $\operatorname{Spec} R$. Now the proposition follows from the lemma below:

LEMMA 1. *Let R be a commutative ring and $\mathfrak{b}$ an ideal in R with* grade $\mathfrak{b} \geqq 2$. *Then the natural homomorphism $R \to \Gamma(\operatorname{Spec} R \setminus V(\mathfrak{b}), \mathcal{O})$ is an isomorphism.*

PROOF. The assertion is well-known for R noetherian and ultimately rests on the fact that $\mathfrak{b}$ then contains a R-sequence of length 2. This needs not to be true in general. It is however harmless to pass to the polynomial ring $R[X_1, X_2]$. The ideal $\mathfrak{b}R[X_1, X_2]$ contains an $R[X_1, X_2]$-sequence of length 2 [11].

Let R be a commutative ring, and let $a = (a_{ij})$ be a (u, v)-matrix over R. Suppose that $I_{r+1}(a) = 0$. Then for all sequences $i_1, \ldots, i_{r+1}$, $i_k \in \{1, \ldots, u\}$, $j_1, \ldots, j_r$, $j_k \in \{1, \ldots, v\}$, one obtains by expansion of determinants

$$\sum_{t=1}^{r+1} (-1)^t \Delta^{j_1 \ldots j_r}_{i_1 \ldots \hat{i}_t \ldots i_{r+1}}(a) a_{i_t} = 0 \qquad (\wedge : i_t \text{ omitted})$$

where a_{i_t} denotes the i_t-th row of a, and $i_1, \ldots, \hat{i}_t, \ldots, i_{r+1}$ and $j_1, \ldots, j_r$ are the row and column indices of a minor of a. When a occurs as the kth linear map in a complex

$$\to R^{b_n} \to \ldots \to R^{b_k} \xrightarrow{a} R^{b_{k-1}} \to \ldots$$

then we call the k-cycles given by these relations the *determinantal k-cycles*. Note that this definition makes sense in the degenerate case $b_{k-1} = 0$.

Let now (R, F) be a finite free resolution of type $(b_n, \ldots, b_0)$, and let (T, C) be just a complex of type $(0, b_n, \ldots, b_0)$ to which (R, F) specializes. Then the determinantal k-cycles of F specialize to the determinantal k-cycles of C. Since F is acyclic, the determinantal k-cycles of C have therefore to be boundaries for all $k \geqq 1$. If there is a universal free resolution for type $(b_n, \ldots, b_0)$, then it also specializes to complexes which are just acyclic in grade 1, by virtue of Proposition 1. To prove that a type $(b_n, \ldots, b_0)$ does not admit a universal free resolution, it is thus enough to construct a complex of type $(b_n, \ldots, b_0)$ which is acyclic in grade 1, however has a non-boundary determinantal k-cycle for some $k \geqq 1$.

THEOREM 2. *Let $(b_n, \ldots, b_0)$ satisfy the rank condition. Then there is an universal free resolution of type $(b_n, \ldots, b_0)$ if and only if the following holds: For all $k \geqq 2$ one has for the rank r_k that $r_k = b_k$ or $r_k = 0$.*

PROOF. Suppose first that there is a $k \geqq 2$ such that $r_k \neq b_k$ and $r_k \neq 0$. Then necessarily $n \geqq 3$. For $n = 3$ and $(b_3, b_2, b_1, b_0) = (1, 2, 1, 0)$ the complex

$$0 \to R \xrightarrow{(-Y, X)} R^2 \xrightarrow{\binom{X}{Y}} R \to 0$$

over $R = \mathbf{Z}[X, Y]$ is acyclic in grade 1, but has a determinantal 1-cycle, namely $1 \in R$, which is not a boundary.

From this complex one can obtain a complex C for each type under consideration such that C is acyclic in grade 1 and has a non-boundary determinantal k-cycle for some $k \geqq 1$, by successively applying the following simple extension process:
From a complex C' of type $(0, b_n, \ldots, b_0)$ one passes to a complex of type

(a) $(0, 0, b_n, \ldots, b_0)$ by extending C' to the left by $0 \to 0$,

(b) $(0, b_n, \ldots, b_0, 0)$ by doing the same at the right end provided $r_0 = 0$, and

(c) $(0, b_n, \ldots, b_{k+1}, b_k + 1, b_{k-1} + 1, b_{k-2}, \ldots, b_0)$ by taking the direct sum of C' and the complex

$$0 \to 0 \to \ldots \to R \xrightarrow{(1)} R \to 0 \to \ldots \to 0$$

where $R \to R$ is in kth position.

Conversely, if $(b_n, \ldots, b_0)$ satisfies the condition of the theorem, then the existence of a universal resolution follows from the same reason as in the case $n \leqq 2$ treated in the corollary of Theorem 1.

In section 1, example (d) it was pointed out that for $t = 0$ and $t = 1$ there exist pairs $(S_t(b), G_t(b))$ which are universal for complexes acyclic in grade t. The explicit structure of $(S_0(b), G_0(b))$ is given by

$$S_0(b) = \mathbf{Z}[X] \Big/ \sum_{k=2}^{n} \mathrm{I}_1(X^k X^{k-1}) + \sum_{k=1}^{n} \mathrm{I}_{r_k+1}(X^k),$$

X being a sequence of matrices of indeterminates as in the proof of Theorem 1, and

$$G_0(b) : 0 \to S_0(b)^{b_n} \xrightarrow{x^n} S_0(b)^{b_{n-1}} \to \ldots \to S_0(b)^{b_1} \xrightarrow{x^1} S_0(b)^{b_0}.$$

De Concini and Strickland ([6]) proved that $S_0(b)$ is a normal domain, and $G_0(b)$ is acyclic in grade 0 since

$$\operatorname{rank} x^{k+1} + \operatorname{rank} x^k = r_{k+1} + r_k = b_k, \quad k = 1, \ldots, n.$$

Generalizing Hochster's theorem ([9, Theorem 7.2]) it was shown in [2] that the structure of $(S_1(b), G_1(b))$ is determined by the Buchsbaum-Eisenbud factorization theorem ([4, Theorem 3.1]). $S_1(b)$ is a factorial noetherian domain and may be described as the ring of sections in the structure sheaf of $S_0(b)$ over the locus of acyclicity of $G_0(b)$, whereas $G_1(b)$ is simply given by $G_0(b) \otimes S_1(b)$. The complex $G_1(b)$ is acyclic for exactly the types named in Theorem 2. In any case it is the best possible universal "approximation" of a finite free resolution. As an immediate consequence of Proposition 1 we have the following corollary:

COROLLARY. *Let (S, G) be a complex of type $(0, b_n, \ldots, b_0)$ such that every finite free resolution (R, F) of type $b=(b_n, \ldots, b_0)$ is a universal specialization of (S, G). Then every such specialization factors uniquely through $(S_1(b), G_1(b))$.*

3. A non-noetherian example.

In this section we explicitely compute a complex which is generic for the complexes of type

$$(*)\qquad R \xrightarrow{(u)} R \xrightarrow{(y_1, y_2)} R^2 \xrightarrow{\binom{-y_2}{y_1}} R$$

which are exact at R^2. We will see that the underlying ring of the generic complex can not be chosen noetherian.

Let us first assume that R is a noetherian ring with a complex of the species under consideration. Since Koszul complexes are known to be rigid over noetherian rings [1, Proposition 2.4], the element $u \in R$ has to be zero. (It is in fact sufficient that $\bigcap_{n=1}^{\infty} (Ry_1 + Ry_2)_{\mathfrak{p}}^n = 0$ whenever $y_1, y_2 \in \mathfrak{p} \in \operatorname{Spec} R$.) Hence there exists a universal complex of type (*) within the class of noetherian rings which is evidently given by the ring $S_N = \mathbf{Z}[Y_1, Y_2]$ and the exact sequence

$$S_N \xrightarrow{(0)} S_N \xrightarrow{(Y_1, Y_2)} S_N^2 \xrightarrow{\binom{-Y_2}{Y_1}} S_N .$$

We prepare the construction of a generic complex of type (*) studying the rings

$$R_{n-1} := \mathbf{Z}[Y_1, Y_2, B_1, \ldots, B_n]/\mathfrak{a}_{n-1}, \qquad n \geqq 1 ,$$

the ideal $\mathfrak{a}_{n-1}$ being generated by the polynomials

$$B_1 Y_2, B_1 Y_1 - B_2 Y_2, B_2 Y_1 - B_3 Y_2, \ldots, B_{n-1} Y_1 - B_n Y_2, B_n Y_1 .$$

One immediately observes the existence of two epimorphisms β_n, γ_n: $R_n \to R_{n-1}$:

$$\beta_n(y_i) = y_i,\ \beta_n(b_{n+1}) = 0,\ \beta_n(b_j) = b_j, \qquad j=1, \ldots, n ,$$
$$\gamma_n(y_i) = y_i,\ \gamma_n(b_1) = 0, \quad \gamma_n(b_j) = b_{j-1}, \qquad j=2, \ldots, n+1 .$$

(Small letters denote residue classes in R_n and R_{n-1} respectively). Obviously $\operatorname{Ker} \beta_n = b_{n+1} R_n$ and $\operatorname{Ker} \gamma_n = b_1 R_n$. Let

$$a_{n-1} = \begin{bmatrix} b_1 & 0 \\ b_2 & b_1 \\ \vdots & \vdots \\ b_n & b_{n-1} \\ 0 & b_n \end{bmatrix}$$

LEMMA 2. (a) *The annihilator* $\operatorname{Ann} y_2$ *of* y_2 *in* R_{n-1} *is generated by* b_1 *and the 2-minors of* a_{n-1}.

(b) $\operatorname{Ann} y_1$ *is generated by* b_n *and the 2-minors of* a_{n-1}.

(c) $\operatorname{Ann} y_1 \cap \operatorname{Ann} y_2$ *is generated by* $y_1^{n-1}b_1 (= y_2^{n-1}b_n)$ *and the 2-minors of* a_{n-1}.

The proof of Lemma 2 rests on

LEMMA 3. R_{n-1} *is a free* $\mathbf{Z}$*-module with basis* $A \cup B \cup C$*, where* A *is the set of monomials in* y_1, y_2*,* B *is the set of monomials in* $b_1, \ldots, b_n$ *and* C *is the set*

$$\begin{aligned} &\{y_1 b_1, y_1 b_2, \ldots, y_1 b_{n-2}, y_1 b_{n-1}\,, \\ &y_1^2 b_1, \ldots, y_1^2 b_{n-2}\,, \\ &\ldots \\ &y_1^{n-1} b_1\}\,. \end{aligned}$$

SKETCH OF THE PROOF OF LEMMA 3. It is not difficult to check that $A \cup B \cup C$ generates R_{n-1} as a $\mathbf{Z}$-module. In an equation

$$\sum_{a \in A} r_a a + \sum_{b \in B} r_b b + \sum_{c \in C} r_c c = 0, \qquad r_a, r_b, r_c \in \mathbf{Z}\,,$$

all the coefficients except the one of $y_1^{n-1}b_1$ can be seen to be zero by the projections onto $\mathbf{Z}[Y_1, Y_2]$, $\mathbf{Z}[B_1, \ldots, B_n]$ and onto R_{n-1} via β_{n-1} and γ_{n-1}. In order to show .the linear independence of $y_1^{n-1}b_1$ one computes a basis of $V \cap \mathfrak{a}_{n-1}$, where V is the $\mathbf{Z}$-submodule of $\mathbf{Z}[Y_1, Y_2, B_1, \ldots, B_n]$ spanned by the monomials of degree n.

Now the proof of Lemma 2 is an easy matter, provided one uses the well-known fact that the 2-minors of a_{n-1} generate the same ideal as the monomials of degree 2 in $b_1, \ldots, b_n$.

Let

$$S_0 := \mathbf{Z}[Y_1, Y_2, B_{01}]/\mathfrak{b}_0\,,$$

$\mathfrak{b}_0$ being generated by $B_{01}Y_1$, $B_{01}Y_2$ and put $u := b_{01} \in S_0$. The rings S_n are inductively defined by

$$S_n := S_{n-1}[B_{n1}, \ldots, B_{nn+1}]/\mathfrak{b}_n ,$$

$\mathfrak{b}_n$ being generated by

$$\begin{aligned}
&b_{n-1,1} - B_{n1}Y_1, B_{n1}Y_2 \\
&b_{n-1,2} - B_{n2}Y_1, b_{n-1,1} - B_{n2}Y_2 \\
&\ldots \\
&b_{n-1,n} - B_{nn}Y_1, b_{n-1,n-1} - B_{nn}Y_2 \\
&B_{nn+1}Y_1, b_{n-1,n} - B_{nn+1}Y_2
\end{aligned}$$

We take $\alpha_{n-1}\colon S_{n-1} \to S_n$ to be the natural homomorphism, and G_n as the complex

$$S_n \xrightarrow{(u)} S_n \xrightarrow{(y_1, y_2)} S_n^2 \xrightarrow{\binom{-y_2}{y_1}} S_n .$$

Finally we put $S := \varinjlim S_n$ and $G := \varinjlim G_n$.

Proposition 2. (a) (S, G) *is a generic complex of type* (*).
(b) *In* S *the element* u *is linearly independent over* $\mathbf{Z}$, *in particular* $u \neq 0$.

Proof. After all, it suffices for (a) to prove that the direct systems (S_n) and (G_n) are obtained from (S_0, G_0) by generic exactification. The polynomials defining $\mathfrak{b}_n$ are chosen to kill the cycles

$$c_1 := (b_{n-1,1}, 0), \quad c_2 := (b_{n-1,2}, b_{n-1,1}), \ldots ,$$
$$c_n := (b_{n-1,n}, b_{n-1,n-1}), \quad c_{n+1} := (0, b_{n-1,n}) .$$

So we have to show that $c_0 := (y_1, y_2)$ and these cycles generate the kernel of $S_{n-1}^2 \to S_{n-1}$. The substitution $Y_i \to Y_i$, $B_i \to B_{n-1i}$ induces an isomorphism $\delta_{n-1}\colon R_{n-1} \to S_{n-1}$. Therefore we simply write b_i for b_{n-1i}.

Let $(d_1, d_2) \in \operatorname{Ker}(S_{n-1}^2 \to S_{n-1})$. The annihilator of y_1 modulo y_2 is evidently generated by $y_2, b_1, \ldots, b_n$. After subtraction of a suitable linear combination of the cycles $c_0, \ldots, c_{n+1}$ above we may then assume $d_2 = 0$ and $d_1 y_2 = 0$. By virtue of Lemma 2, (a), each element $(d_1, 0)$ with $d_1 y_2 = 0$ is a linear combination of $c_1, \ldots, c_{n+1}$.

The element u is linearly independent in S since it is linearly independent in all the rings S_{n-1}.

Part (b) of the preceding proposition demonstrates that there is no noetherian generic complex of type (*). (S, G) has been constructed such that G

is exact at S^2. Unintentionally we have obtained a sequence with much better exactness properties:

PROPOSITION 3. (a) *G is an exact sequence. Hence (S, G) is generic for exact sequences*

$$R \xrightarrow{(v)} R \xrightarrow{(x_1, x_2)} R^2 \xrightarrow{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}} R\,.$$

(b) *Even the periodic sequence*

$$\begin{tikzcd} S \arrow[r, "{(y_1, y_2)}"] & S^2 \arrow[r, "{\begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}}"] & S \arrow[ll, bend left, "(u)"] \end{tikzcd}$$

is exact and, hence, generic for such sequences.

PROOF. As above we identify S_{n-1} and R_{n-1}, S_n and R_n. The kernel of α_{n-1} contains all the 2-minors of the matrix a_{n-1}. Therefore u generates the intersection of the annihilators of y_1 and y_2 in S by virtue of Lemma 2, (c). This proves (a). On the other hand y_1 and y_2 generate the annihilator of u in S since $S/Sy_1 + Sy_2$ is naturally isomorphic to $\mathbf{Z}$ and u is linearly independent over $\mathbf{Z}$, as noted above.

Again the noetherian situation is significantly simpler as we have seen already. In particular, every periodic resolution as considered in part (b) of Proposition 3 is a split-exact sequence bent into a circle when the underlying ring is noetherian.

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