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APPENDIX TO "PLÜCKER CONDITIONS ON PLANE RATIONAL CURVES": FAMILIES OF RATIONAL PLANE CURVES

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This note is concerned with the relationship between families of parametrized plane rational curves and families of unparametrized plane curves. Let d be a fixed positive integer, and denote by $\underset{\sim}{X}$ the following contravariant functor on the category of k-schemes:

$\underset{\sim}{X}(T)$ = set of finite T-morphisms $\varphi: P^1_T \to P^2_T$ such that for all geometric points $t \to T$, the fiber φ_t of φ is birational onto its image, which is a curve of degree d with only ordinary nodes and cusps as singularities.

Clearly $\underset{\sim}{X}$ is represented by an open subscheme X of projective $(3d+2)$-space. Denote by $\tilde{\varphi}: P^1_X \to P^2_X$ the universal family.

Let A_1 be the Hilbert scheme of plane curves of degree d (i.e. projective $\frac{1}{2}d(d+3)$-space), and let $A \subseteqq A_1$ be the open subscheme corresponding to irreducible curves with only nodes and cusps as singularities. Since $\tilde{\varphi}$ is finite, the formation of $\tilde{\varphi}_*(O_{P^1_X})$ commutes with arbitrary base change on X. Hence the closed subscheme $\tilde{C} \subseteqq P^2_X$ defined by the zeroth Fitting ideal of $\tilde{\varphi}_*(O_{P^1_X})$ defines a morphism $\psi: X \to A$.

THEOREM. *The morphism ψ factors as follows:*

$$\begin{array}{ccc} X & \xrightarrow{\psi} & A \\ {\scriptstyle\varphi}\downarrow & & \uparrow{\scriptstyle i} \\ Y & \xrightarrow[n]{} & R \end{array}$$

where $R \subseteqq A$ is the closed subscheme corresponding to rational curves (with reduced subscheme structure), $n: Y \to R$ is the normalization morphism, and $\varphi: X \to Y$ is a principal PGL (2)*-bundle. Furthermore, Y is nonsingular and n is a homeomorphism.*

COROLLARY. *Put $\lambda = n \circ \varphi: X \to R$. For any subset $U \subseteqq R$, we have $\lambda^{-1}(\bar{U}) = \overline{\lambda^{-1}(U)}$.*

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Remark. One may show that n is an isomorphism precisely over the open subset $R_0 \subseteqq R$ corresponding to curves without cusps. More precisely, if $r \in R$ corresponds to a curve with γ cusps, the germ of R at r is analytically isomorphic to a product of γ ordinary (1-dimensional) cusps and a smooth part (of dimension $3d-1-\gamma$).

Proof of the Theorem. We shall define Y via its functor of points, and later show that it coincides with the normalization of R.

For any A-scheme T, let $C_T \subseteqq P_T^2$ be the pullback of the universal family $C_A \subseteqq P_A^2$. Consider the following functor $\underline{Y}$ on the category of A-schemes:

$\underline{Y}(T)$ = set of subschemes $S \subseteqq C_T$ with the following properties:

(i) S is étale and finite over T of rank $p=\binom{d-1}{2}$.

(ii) $S^{(2)} \subseteqq C_T$, where $S^{(2)}$ is the first infinitesimal neighborhood of S in P_T^2 (defined by the square of the ideal of S in P_T^2).

Note that condition (ii) is equivalent to the condition (ii)′: S is contained in the singular locus of the morphism $C_T \to T$ (defined, for example, by the first Fitting ideal of $\Omega^1_{C_T/T}$).

Clearly $\underline{Y}$ is represented by a locally closed subscheme Y of $\mathrm{Hilb}^p_{P^2} \times A$. I claim that the natural morphism $v\colon Y \to A$ is proper. Indeed, by the valuative criterion for properness, it suffices to complete the following commutative diagram

$$\begin{array}{ccc} T_0 & \to & Y \\ \cap\!\!\shortparallel & \nearrow & \downarrow \\ T & \to & A \end{array}$$

where T is the spectrum of a discrete valuation ring, and $T_0 = T - \{t\}$, $t \in T$ the closed point. So we are given C_T and $S_{T_0} \subseteqq \mathrm{Sing}\,(C_{T_0}/T_0)$. Put S_T = closure of S_{T_0} in P_T^2. Then S_T is flat and finite over T and condition (ii) holds. It remains only to show that the closed fiber S_t is nonsingular. If not, there are local parameters (u, v) of P_t^2 such that $I_{S_t} \subseteqq (u^2, v)$. But then $I_{C_t} \subseteqq (u^4, u^2v, v^2)$, contrary to the assumption that C_t has only ordinary nodes and cusps.

Now let L_Y be the blowing up of C_Y along S_Y. I claim that L_Y is flat over Y, and that for any base change $Y' \to Y$, the pullback $L_{Y'}$ of L_Y coincides with the blowing up of $C_{Y'}$ along $S_{Y'}$. Indeed, the question is local on C_Y (for the étale topology) hence the claim follows from [2, 1.3 and 1.6]. In particular, all the geometric fibers of L_Y are projective lines, and $L_Y \to Y$ is a P^1-bundle. Let $\varphi'\colon X' \to Y$ be the associated principal PGL (2)-bundle [1]. Its functor of points on the category of Y-schemes is $X'(T)$ = set of T-isomorphisms P_T^1

$\to L_T$. Let $\alpha\colon P^1_{X'} \to L_{X'}$ be the universal isomorphism. Then the composed map

$$P^1_{X'} \xrightarrow{\alpha} L_{X'} \to C_{X'} \to P^2_{X'}$$

defines a morphism $\beta\colon X' \to X$.

I claim that β is an isomorphism. Indeed, define a closed subscheme $\tilde{S} \subseteqq P^2_X$ by the first Fitting ideal of $\tilde{\varphi}_*(O_{P^1_X})$. Then $\tilde{S}$ is étale and finite of rank p over X, and defines a morphism $\varphi\colon X \to Y$. Clearly, the map $\tilde{\varphi}\colon P^1_X \to \tilde{C} \subseteqq P^2_X$ coincides with the blowing up of $\tilde{C}$ along $\tilde{S}$. Therefore φ can be lifted to an inverse of β.

Summing up our result so far, we have defined the following part of the diagram of the theorem:

$$\begin{array}{ccc} X & \xrightarrow{\psi} & A \\ {\scriptstyle\varphi}\searrow & & \nearrow {\scriptstyle v = ion} \\ & Y & \end{array}$$

Furthermore, we have shown that φ is a principal PGL (2)-bundle, hence Y is nonsingular. Since any rational plane curve of degree d with ordinary nodes and cusps has a total of p of these singularities, v is injective on geometric points. Since v is proper, it is a birational homeomorphism onto its image R. This also shows that Y is the normalization of R.

REMARK. There is a natural action of PGL (2) on X. Starting in the other end, one may check that this action is free, and construct (Y, φ) as a geometric quotient of this action.

REFERENCES

1. J.-P. Serre, *Espaces fibrés algébriques* in *Anneaux de Chow et Applications*, Seminaire C. Chevalley, 2e Année, 1958, pp. 1–37, Secrétariat Mathématique, Paris, 1958.
2. J. M. Wahl, *Equisingular deformations of plane algebroid curves*, Trans. Amer. Math. Soc. 193 (1974), 143–170.

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