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A NOTE ON τ-CONSTANT FAMILIES OF PLANE CURVES

STEIN ARILD STRØMME

Introduction.

If $\{X_y\}_{y \in Y}$ is an algebraic (or analytic) family of (germs of) isolated plane curve singularities with "constant τ" (i.e. the dimension $\tau(X_y)$ of the base of the miniversal deformation of X_y is independent of $y \in Y$), it seems to be well known (cf. the discussion in [6, p. 667]) that if the parameter space Y is nonsingular, then the family is equisingular (in the sense of Zariski, [8]). The notion of equisingularity has been extended to families over Artinian base schemes by Wahl [7], and he proves that the equisingular deformation functor is smooth. On the other hand, the notion of τ-constant families has an obvious infinitesimal analogue, called equicohomological families in this note. However, the implication equicohomological $\Rightarrow$ equisingular no longer holds; the purpose of this note is to provide a counterexample. The same example shows that the equicohomological stratum in the prorepresentable hull of the deformation functor is singular.

1. Equicohomological deformations.

(1.1). Let $R = k[[X, Y]]$ be the algebra of formal power series over an algebraically closed field k of characteristic 0, let $f \in R$ be a reduced power series of order $r \geqq 1$, that is $f \in (X, Y)^r - (X, Y)^{r+1}$, and put $B = R/(f)$. By definition, B is an algebroid isolated plane curve singularity.

(1.2). Denote by $\boldsymbol{C}$ the category of local k-algebras of finite length, and by $D\colon \boldsymbol{C} \to \boldsymbol{Sets}$ the deformation functor of B, and let (H, ξ) be the prorepresentable hull of D, see [5]. It is described as follows: Let $g_1, \ldots, g_\tau \in R$ induce a k-basis of $R/(f, f_X, f_Y)$, and let $t_1, \ldots, t_\tau$ be variables. Then $H = k[[t_1, \ldots, t_\tau]]$, and if we put

$$\tilde{f} = f + \sum_{i=1}^{\tau} t_i g_i \in R \otimes_k H = R_H ,$$

then $\tilde{B} = R_H/(\tilde{f})$ induces a semiuniversal family for D.

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(1.3). Denote by $H^i(k, B, B)$ the algebra cohomology groups of André [1]; then the tangent space of D (or of H) is canonically isomorphic to $H^1(k, B, B)$, which in turn is (non-canonically) isomorphic to $R/(f, f_X, f_Y)$. Similarly, $H^1(H, \tilde{B}, \tilde{B})$ is isomorphic to $R_H/(\tilde{f}, \tilde{f}_X, \tilde{f}_Y)$.

(1.4). For any morphism $A' \to A$ in $\mathbf{C}$ and any deformation $B_{A'} \in D(A')$, there is a natural base-change map

$$H^1(A', B_{A'}, B_{A'}) \otimes_{A'} A \to H^1(A, B_A, B_A) ,$$

where $B_A = B_{A'} \otimes_{A'} A$. Representing $B_{A'}$ as a quotient of $R_{A'}$ by $f' \in R_{A'}$ and using similar isomorphisms as those in (1.3), we easily see that the base change map is an isomorphism.

(1.5). Definition. A deformation $B_A \in D(A)$ is *equicohomological* if $H^1(A, B_A, B_A)$ is a flat (hence free) A-module. In view of (1.4), these families form a subfunctor EC of D.

(1.6). Let $\tau = \dim_k H^1(k, B, B)$, and denote by $J \subseteq H$ the $(\tau-1)$th Fitting ideal of the finite H-module $H^1(H, \tilde{B}, \tilde{B})$. Put $\bar{H} = H/J$ and $\bar{B} = \tilde{B} \otimes_H \bar{H}$. By [3, Lecture 8, case $n=0$], for any morphism $H \to A$ with A in $\mathbf{C}$, it factors through $\bar{H}$ if and only if $H^1(H, \tilde{B}, \tilde{B}) \otimes_H A$ is flat over A. In view of (1.4) above, this happens exactly when $\tilde{B} \otimes_H A$ is equicohomological. Hence $(\bar{H}, \bar{B})$ is a prorepresentable hull of EC.

(1.7). Remark. In [2] it is proved that $(\bar{H}, \bar{B})$ actually prorepresents EC, and that EC is the maximal prorepresentable subfunctor of D (in the terminology of that paper, EC is the prorepresentable substratum of D).

(1.8). By Hilbert's syzygy theorem, the R-module $H^1(k, B, B)$ has a free resolution of the form

$$0 \to R^2 \xrightarrow{\varphi} R^3 \xrightarrow{\psi} R ,$$

where ψ is the row vector (f, f_X, f_Y). Furthermore, since coker (ψ) has finite support, the image of ψ is the ideal generated by the maximal minors of φ. The 3×2 matrix $\varphi = (\varphi_{ij})$ will play a key role in the description of the tangent space T_{EC} of EC. By definition, this is EC $(k[\varepsilon])$, where $k[\varepsilon]$ is the ring of dual numbers $(\varepsilon^2 = 0)$. T_{EC} is a subspace of $T_D = H^1(k, B, B)$ and can be described as follows:

(1.9). Proposition. *For any $g \in R$, let $B_g = R[\varepsilon]/(f + \varepsilon g)$. Then the following are equivalent*:

(i) B_g *is equicohomological,* i.e. *in* T_{EC}.
(ii) $g\varphi_{1j}+g_X\varphi_{2j}+g_Y\varphi_{3j} \in (f, f_X, f_Y) \subseteq R \qquad (j=1,2)$.

PROOF. Put $\psi_1 = (g, g_X, g_Y)$ (the vector), then $B_g \in T_{EC}$ iff

$$\operatorname{coker}(\psi+\varepsilon\psi_1) : R[\varepsilon]^3 \to R[\varepsilon]$$

is flat over $k[\varepsilon]$. By a well-known theorem [4] this happens iff φ can be lifted to a matrix $\varphi+\varepsilon\varphi_1$ such that $(\psi+\varepsilon\psi_1)(\varphi+\varepsilon\varphi_1)=0$, that is $\psi\varphi_1+\psi_1\varphi=0$. The existence of such a φ_1 is clearly equivalent to (ii).

2. An example.

(2.1). Put $f=(X^4-Y^4)^2-X^{10}$. This example has been studied by Wahl [7, 6.8]. Let $R/(f, f_X, f_Y) \to T_D$ be the isomorphism induced by the correspondence $g \to B_g$ as in (1.9); then T_D will be identified with $R/(f, f_X, f_Y)$ via this isomorphism. Wahl shows that the tangent space $T_{\overline{ES}}$ of the equisingular deformation functor $\overline{ES}$ is the ideal generated by $(X, Y)^{10}$ and $X^2Y^2(X^4-Y^4)$ in $R/(f, f_X, f_Y)$.

(2.2). PROPOSITION.
(i) *A k-basis for* T_D *is given by* $\{X^iY^j \mid (i,j) \in B\}$, *where*

$$B = \{0,\ldots,5\} \times \{0,\ldots,6\} \cup \{(6,0),(6,1),(6,2),(6,3),(7,3),(8,3)\}.$$

(ii) *A k-basis for* T_{EC} *is given by the following*:

(a) $X^iY^j, \quad (i,j) \in B,\ i+j \geqq 9$,

(b) $X^4-Y^4+4X^6-4X^2Y^4$

X^5-XY^4

$48X^4Y-48Y^5-5X^6Y$

X^5Y-XY^5

$32X^4Y^2-32Y^6-5X^6Y^2$

$X^5Y^2-XY^6$

$X^6Y-X^2Y^5$

$X^6Y^2-X^2Y^6$

X^3Y^5.

In particular, $\tau=48$ and $\dim T_{EC}=18$.

(2.3). COROLLARY. *There exist equicohomological deformations that are not equisingular (not even equimultiple along any section, see* [7] *for definitions).*

(2.4). PROPOSITION. EC *is obstructed, that is* $\bar{H}$ *is singular.*

PROOF OF (2.2). (i) is basically an exercise: First find a monomial base of $R/(f_X, f_Y)$, then express the residue class of f and its multiples in this basis. (In fact $\mu = \dim R/(f_X, f_Y) = 57$, and the annihilator of f modulo (f_X, f_Y) is the ideal (X^3, Y^3).) The multiplication table in T_D should be generated by the relations $X^{10} = X^6 Y^4 = (X, Y)^{12} = 0,\ Y^7 = X^4 Y^3,\ X^7 = X^3 Y^4 + \frac{5}{4} X^5 Y^4$.

(ii) From these computations we may construct the following matrix φ:

$$\varphi = \begin{bmatrix} 40Y^3 & 8X^3 - 10X^5 \\ -4XY^3 & \frac{1}{5}Y^4 - X^4 + X^6 \\ X^4 - 5Y^4 & -\frac{4}{5}X^3 Y + \frac{5}{4}X^5 Y \end{bmatrix}$$

and one checks easily that its minors are f, f_X, and f_Y. With all this, we are in a position to apply the test of (1.9), reducing everything to a system of k-linear equations.

(2.5). REMARK. To make a check on these computations, put for example $g = X^5 - XY^4$. Then

$$\varphi_{11} g + \varphi_{21} g_X + \varphi_{31} g_Y = -2X f_Y$$

$$\varphi_{12} g + \varphi_{22} g_X + \varphi_{32} g_Y = (11 - 20X^2) f + (2X^3 - X) f_X + (\tfrac{5}{2} X^2 Y - \tfrac{7}{5} Y) f_Y$$

hence $f + \varepsilon g$ and its partial derivatives are the maximal minors of $\varphi + \varepsilon \varphi_1$, where

$$\varphi_1 = \begin{bmatrix} 0 & 20X^2 - 11 \\ 0 & X - 2X^3 \\ 2X & \frac{7}{5}Y - \frac{5}{2}X^2 Y \end{bmatrix}.$$

This can of course be verified directly.

PROOF OF (2.4). From the explicit description of the semiuniversal family of (1.2) it is clear that it is algebraizable: it is, in fact, defined over $\mathscr{H}_1 = k[t_1, \ldots, t_n]$. Passing to an affine open neighbourhood $\operatorname{Spec} \mathscr{H} \subseteq \operatorname{Spec} \mathscr{H}_1$ of the origin, $H^1(\mathscr{H}, B_{\mathscr{H}}, B_{\mathscr{H}})$ is a finite $\mathscr{H}$-module, and we may form a quotient $\bar{\mathscr{H}}$ of $\mathscr{H}$ in the same way as in (1.6), the equicohomological stratum. Then $\bar{H}$ is the completion of $\bar{\mathscr{H}}$ at the origin, and it suffices to show that $\bar{\mathscr{H}}$ is singular. Assuming the contrary, we may extend the deformation $f + \varepsilon g$, where $g = X^5$

$-XY^5$ to an equicohomological family defined over a nonsingular curve. The general fiber in this family would be an isolated singularity no worse than that defined by $g=0$, an ordinary 5-ple point. Hence the invariant τ of the general fiber is at most 16, whereas in any equicohomological family, τ is constant, since the formation of H^1 commutes with base change. Since $\tau=48$ for the special fiber, we have the desired contradiction.

(2.6). REMARK. In this example, it happens that $T_{\overline{ES}} \subseteqq T_{EC}$. In the general case, this is not so, as can be seen from the example $X^5+Y^5+\varepsilon X^3Y^3$, which is equisingular but not equicohomological.

NOTE ADDED IN PROOF. G. Pfister recently discovered that the family $X^5+X^2Y^2+Y^4+tX^4$ is τ-constant but not equisingular.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BERGEN
ALLÉGT. 53–55
5014 BERGEN
NORWAY