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FUNCTION SPACES AND HURWITZ-RADON NUMBERS

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1. Introduction.

One aim of this paper is to classify the path-components of function spaces in which the target spaces are projective spaces. We consider classification under various equivalence relations ranging from homotopy equivalence to isometry. It is interesting that whenever we know the answers they are the same for all the equivalence relations considered. This was also the outcome in [18], where a similar project was undertaken with spheres as target spaces.

Let $M(X, Y)$ be the space of all continuous maps from a space X to a space Y, equipped with the compact-open topology. Let $\mathbf{R}P^n$, $\mathbf{C}P^n$ denote real, complex projective n-space. We give explicit classifications for the path-components of $M(S, \mathbf{C}P^n)$ where S is any closed connected surface, and of $M(\mathbf{C}P^r, \mathbf{C}P^n)$, $M(\mathbf{R}P^r, \mathbf{C}P^n)$ and $M(\mathbf{R}P^r, \mathbf{R}P^n)$ for certain values of r and n. When the domain is $\mathbf{R}P^r$, there is an interesting connection with Hurwitz-Radon numbers (Propositions 2.3, 2.4).

These results are obtained as corollaries of a more general study. Throughout the paper we shall assume that X is a connected, finite CW-complex. Let ξ be a vector bundle over X and $P\xi$ the associated projective bundle. Instead of the above function spaces, we may consider the space $\Gamma P\xi$ of all cross-sections of $P\xi$. In the special case when ξ is a trivial real or complex $(n+1)$-plane bundle, $\Gamma P\xi$ may clearly be identified with $M(X, \mathbf{R}P^n)$ or $M(X, \mathbf{C}P^n)$. We may try to classify the path-components of $\Gamma P\xi$ under the same equivalence relations as before. The results we get are sufficient to deduce the classifications mentioned in the previous paragraph. These could be obtained within the narrower context of function spaces, but with no less effort. Moreover (see sections 2–4) our approach uses $\Gamma P\xi$ for non-trivial ξ even in studying function spaces.

Part (a) of Proposition 2.1 below answers a question of Graeme Segal, who gave an elegant proof when S is a 2-sphere (private communication). We are grateful to him for raising the question. We are grateful also for the stimulus of papers by Vagn Lundsgaard Hansen on function spaces, and to Ronnie Brown

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for drawing our attention to [14]. Some of our results are similar to those in [14], and we acknowledge our indebtedness to that paper.

Since this paper was written, Jesper Michael Møller has kindly sent us preprints in which he studies questions similar to those in this paper, and in particular he independently proves Proposition 2.2, by methods which are somewhat different from those used here.

2. Results on path-components.

Throughout the paper, $H^*(\cdot)$ denotes cohomology with integer coefficients, and ε (ε_V) denotes a trivial vector bundle over any base space (with fibre the vector space V). We shall use the symbol $\cong$ for an isomorphism between vector bundles, and the (now standard) abbreviation "section" for "cross-section". Since all the spaces we consider are easily seen to be locally path-connected, we may accurately write "component" for "path-component".

In this section we state results relating to the classification of components. A few further results on the homotopy theory of spaces of sections may be found in section 8. At the end of this section we give the layout of the paper.

We begin with the explicit classifications mentioned in section 1. As in [18] our function spaces admit more refined structures. The standard metric on the sphere S^n (or S^{2n+1}) gives rise to a metric on $\mathbf{R}P^n$ (or $\mathbf{C}P^n$). Using this metric on the target and assuming that the domain is compact, we may metrize each of our function spaces by the sup metric. Also, by Eells [5], each may be given the structure of a smooth manifold modelled on a separable Banach space. We shall say that two components are *strongly equivalent* if they are isometric and diffeomorphic.

First let S be a closed connected surface. Let $M_k(S, \mathbf{C}P^n)$ be the component of $M(S, \mathbf{C}P^n)$ consisting of all maps whose degree in integral 2-dimensional cohomology is k. Here k is an integer for S oriented and a mod 2 integer for S non-orientable. The following proposition was known for $n=1$ ([9], [18]).

PROPOSITION 2.1. (a) *If S is oriented then $M_k(S, \mathbf{C}P^n)$ and $M_l(S, \mathbf{C}P^n)$ are strongly equivalent for $k = \pm l$, and not homotopy equivalent otherwise.*

(b) *If S is non-orientable then $M_0(S, \mathbf{C}P^n)$ and $M_1(S, \mathbf{C}P^n)$ are strongly equivalent for n odd and not homotopy equivalent for n even.*

Graeme Segal proved (a) for S a 2-sphere by showing that then $H^{2n}(M_k(S, \mathbf{C}P))$ is cyclic of order $|k|(n+1)$, for $k \neq 0$. This will also be proved in a forthcoming paper by Jesper Michael Møller.

It suggests the next proposition, in which k again denotes degree in integral 2-dimensional cohomology. The result was known for $n=1$ ([20], [13]).

PROPOSITION 2.2. *Let* $1 \leqq r \leqq n$. *Then* $M_k(\mathbf{C}P^r, \mathbf{C}P^n)$ *and* $M_l(\mathbf{C}P^r, \mathbf{C}P^n)$ *are strongly equivalent for* $k = \pm l$ *and not homotopy equivalent otherwise.*

Next we take the domain to be $\mathbf{R}P^r$. Let a_{r+1} denote the order of $[H]-[\varepsilon_{\mathbf{R}}]$ in the Grothendieck group $KO(\mathbf{R}P^r)$, where $[H]$ is the class of the Hopf line bundle. Similarly let b_{r+1} be the order of $[H \otimes \mathbf{C}]-[\varepsilon_{\mathbf{C}}]$ in $K(\mathbf{R}P^r)$.

PROPOSITION 2.3. *Let* $r>0$. *The components of* $M(\mathbf{R}P^r, \mathbf{R}P^n)$ *are strongly equivalent if* $n+1$ *is divisible by* a_{r+1}. *Provided* $r<n-1$, *the two components of* $M(\mathbf{R}P^r, \mathbf{R}P^n)$ *are homotopy equivalent only if* $n+1$ *is divisible by* a_{r+1}.

PROPOSITION 2.4. *Let* $r>1$. *The components of* $M(\mathbf{R}P^r, \mathbf{C}P^n)$ *are strongly equivalent if* $n+1$ *is divisible by* b_{r+1}. *Provided* $r<2n-1$, *the components are homotopy equivalent only if* $n+1$ *is divisible by* b_{r+1}.

The case $r=2$ of Proposition 2.4 follows from Proposition 2.1(b), which originally suggested Proposition 2.4 and thence Proposition 2.3.

As mentioned earlier, Propositions 2.1–2.4 are corollaries of results on spaces of sections. Before stating these we introduce some notation. Let ξ be a real or complex vector bundle over X equipped with a Riemannian or Hermitian metric. We write $S\xi$ for the sphere-bundle associated with ξ, and $\mathbf{R}P\xi$, $\mathbf{C}P\xi$ or simply $P\xi$ for the corresponding projective bundle. One may study the spaces of sections $\Gamma S\xi$, $\Gamma \mathbf{R}P\xi$, $\Gamma \mathbf{C}P\xi$ each equipped with the compact-open topology, and we touch on the homotopy theory of all three. Our methods work best for $\Gamma \mathbf{C}P\xi$, so unless the contrary is stated, from now on ξ will denote a complex $(n+1)$-plane bundle over X equipped with a Hermitian metric, $P\xi$ the associated complex projective bundle, and H_ξ the Hopf line bundle over $P\xi$.

When we study the component in $\Gamma P\xi$ of a section s, it is convenient to assume that s arises from a section of $S\xi$.

LEMMA 2.5. *A section* s *of* $P\xi$ *lifts to a section of* $S\xi$ *if and only if* s^*H_ξ *is trivial.*

We write $N\xi$ for the space of all such sections. Notice that $N\xi$ is a union of components of $\Gamma P\xi$. As on p. 236 of [14], we may always assume that s is in a space of type $N\xi$ provided we are willing to replace ξ by $\lambda \otimes \xi$ for some line bundle λ over X. The study of $\Gamma P\xi$ for a given ξ thus reduces to the study of the spaces $N(\lambda \otimes \xi)$ for all line bundles λ over X. (We present this in detail in section 3.) Because of this, most of our remaining results are stated in terms of $N\xi$.

The advantage of working with $N\xi$ is indicated by the next result, which is a key ingredient in our approach (cf. [14] in the real case).

PROPOSITION 2.6. *Let $G=M(X,S^1)$. Then G is a Lie group (in general infinite-dimensional) which acts on $\Gamma S\xi$ by complex scalar multiplication, and $N\xi$ may be identified with the orbit space $\Gamma S\xi/G$. The resulting projection of $\Gamma S\xi$ to $N\xi$ has the structure of a smooth locally trivial principal G-bundle.*

Next we decribe the components of $N\xi$. As usual, $c_i\xi$ denotes the ith Chern class of ξ with integer coefficients. Proposition 2.7 follows from Proposition 2.6 and well-known facts about $S\xi$, except for the description of π_0 in (b), which we prove in section 7.

PROPOSITION 2.7. (a) *If* $\dim X<2n+1$, *then $N\xi$ is connected.*

(b) *If* $\dim X<2n+2$, *then $N\xi$ is non-empty and* $\pi_0(N\xi)\approx H^{2n+1}(X)/c_n\xi . H^1(X)$.

(c) *If* $\dim X<2n+3$, *then $N\xi$ is non-empty if and only if* $c_{n+1}\xi=0$.

Recalling that ξ has a Riemannian or Hermitian metric, we can give each of our spaces of sections the structure of a metric space and of a smooth manifold just as in the case when ξ is trivial (cf. [6, p. 779]). Strong equivalence then makes sense for spaces of sections. The next proposition lists some easy sufficient conditions for equivalences between spaces of the type $N\xi$. As usual, $\bar{\xi}$ denotes the bundle conjugate to ξ.

PROPOSITION 2.8. (a) *If ξ and η are isomorphic then $N\xi$ and $N\eta$ are strongly equivalent.*

(b) *$N\xi$ and $N\bar{\xi}$ are strongly equivalent.*

(c) *If $\varphi: Y\to X$ is a homotopy equivalence then $N(\varphi^*\xi)$ is homotopy equivalent to $N\xi$.*

Part (a) is immediate. The proofs of (b) and (c) are straightforward and will be omitted.

We shall later need to choose basepoints in spaces of sections. The next proposition follows from more elaborate results stated and proved in section 5.

PROPOSITION 2.9. *For any two points s_0, s_1 in the same component of $\Gamma S\xi$ (or $N\xi$) there is a strong equivalence of $\Gamma S\xi$ (or $N\xi$) which maps s_0 to s_1.*

Now we come to the main results on $N\xi$. First, here is a partial converse to Proposition 2.8(a).

THEOREM 2.10. *Suppose that* $\dim X < 2n-1$ *and that* ξ, ε *are complex* $(n+1)$-*plane bundles over* X *with* ε *trivial. If* $N\xi$ *and* $N\varepsilon$ *are homotopy equivalent then* ξ *is trivial.*

For any complex line bundle λ over X let M_λ be the subspace of $M(X, \mathbf{C}P^n)$ consisting of all maps f such that $f^*H \cong \lambda$, where H is the Hopf line bundle over $\mathbf{C}P^n$. When $\dim X < 2n+1$ the M_λ are precisely the components of $M(X, \mathbf{C}P^n)$. In section 4 we deduce the following corollary of Theorem 2.10 and show how it illuminates the appearance of Hurwitz-Radon numbers in Proposition 2.4.

COROLLARY 2.11. *Let* λ, μ *be complex line bundles over* X. *If* $(n+1)\lambda$ *and* $(n+1)\mu$ *are isomorphic then* M_λ *and* M_μ *are strongly equivalent. If* $\dim X < 2n-1$, *if* μ *is trivial and* M_λ *is homotopy equivalent to* M_μ *then* $(n+1)\lambda$ *is trivial.*

The next theorem is similar to results in [9], [14]. In the statement w_{2n} denotes the mod 2 Stiefel-Whitney class and $[X]$ the fundamental homology class of X.

THEOREM 2.12. *Let* X *be a closed connected* $2n$-*manifold and* ξ *a complex* $(n+1)$-*plane bundle over* X. *Then* $\pi_1(N\xi)$ *is a central extension*

(i) $0 \to \mathbf{Z}/c_n\xi[X] \to \pi_1(N\xi) \to H^1(X) \to 0$ *if* X *is oriented,*

(ii) $0 \to \mathbf{Z}/2/w_{2n}\xi[X] \to \pi_1(N\xi) \to H^1(X) \to 0$ *if* X *is non-orientable.*

In case (i) *the extension is classified by the skew-symmetric map from* $H^1(X) \times H^1(X)$ *to* $\mathbf{Z}/c_n\xi[X]$ *defined by*

$$(a,b) \mapsto (2abc_{n-1}\xi)[X] \mod c_n\xi[X] .$$

In case (ii) $\pi_1(N\xi)$ *is Abelian and the extension is split.*

When $n=1$, the two parts of this theorem are equivalent to Theorems 1 and 2 of [14]. To see this, note first that the oriented 2-plane bundle η on p. 233 of [14] may be represented (uniquely up to isomorphism) by a complex line bundle ζ. Our $S(\eta \oplus \varepsilon_{\mathbf{R}})$ is $S(\eta \oplus \varepsilon^1)$ in the notation of [14], and the section s of $S(\eta \oplus \varepsilon^1)$ in [14] is the constant section 1 of ε^1. We may identify $S(\eta \oplus \varepsilon^1)$ with $P(\zeta \oplus \varepsilon_{\mathbf{C}})$ in such a way that s corresponds to the section s' of $P(\zeta \oplus \varepsilon_{\mathbf{C}})$ which picks out the fibre of $\varepsilon_{\mathbf{C}}$ over each point of X. Clearly s' is in $N(\zeta \oplus \varepsilon_{\mathbf{C}})$ and we get

$$\pi_1(\Gamma S(\eta \oplus \varepsilon^1), s) \approx \pi_1(N(\zeta \oplus \varepsilon_{\mathbf{C}}), s') .$$

(Note that $N(\zeta \oplus \varepsilon_{\mathbf{C}})$ is connected by Proposition 2.7(a).)

For handling situations involving two non-trivial bundles, so that Theorem 2.10 does not apply, we use another partial converse of Proposition 2.8(a).

THEOREM 2.13. *Let X be a closed connected oriented $2m$-manifold and ξ, η complex $(n+1)$-plane bundles over X with $n \geq m$. If $N\xi$ and $N\eta$ are homotopy equivalent then $c_m\xi = \pm c_m\eta$.*

Before turning to $N\xi$ in the real case, we give a result on $\Gamma S\xi$ which will be proved in [4]. In the remaining results, X is again just a connected finite CW-complex.

THEOREM 2.14. *Let ξ, η be real $(n+1)$-plane bundles over X with $n > 2 \dim X + 1$. If $\Gamma S\xi$ and $\Gamma S\eta$ are homotopy equivalent then the stable Thom spaces $X^{-\xi}$ $X^{-\eta}$ are homotopy equivalent, and in particular $S\xi$ is fibre homotopy trivial if and only if $S\eta$ is.*

The real analogues of Lemma 2.5, Propositions 2.6, 2.7 (see section 7), 2.8(a), (c) and 2.9 all hold. However, in place of Theorem 2.10 we have:

PROPOSITION 2.15. *Let $\dim X < n-1$ and let ξ, ε be real $(n+1)$-plane bundles over X, with ε trivial. If $N\xi$ and $N\varepsilon$ are homotopy equivalent then $S\xi$ is stably fibre homotopy trivial at the prime (2).*

For the real case we have the following addition to Proposition 2.8. In the statement, L denotes a product real line bundle with the non-trivial $\mathbf{Z}/2$-action.

PROPOSITION 2.16. *If ξ and η are real vector bundles over X such that $S(L \otimes \xi)$ and $S(L \otimes \eta)$ are $\mathbf{Z}/2$-fibre homotopy equivalent then $N\xi$ and $N\eta$ are homotopy equivalent.*

The proof is immediate. Together with Theorem 2.14 above, Proposition 1.1 of [3] and an easy covering space argument (see Remark 8.9 below) this yields:

THEOREM 2.17. *Let ξ, ε be real $(n+1)$-plane bundles over X with $n > 2 \dim X + 1$ and ε trivial. Then $N\xi$ and $N\varepsilon$ are homotopy equivalent if and only if $S\xi$ is fibre homotopy trivial.*

The layout of the rest of the paper is as follows. In section 3 we explain our approach to sections of projective bundles, in particular proving Lemma 2.5 and Proposition 2.6. In section 4 we illustrate how to deduce our results on function spaces from results on spaces of sections. In section 5 we deal with basepoints, proving Proposition 2.9. Theorem 2.10 and Proposition 2.15 have similar proofs, and we give these in section 6. Theorem 2.12 and Proposition

2.7(b) are treated similarly in section 7. In section 8 we begin to study the homotopy type of $N\xi$ and prove Theorem 2.13.

It is perhaps worthwhile mentioning that our methods have evolved during the preparation of this paper. Initially our technique was closer to methods used in [20], [8], [9], and [10], with ordinary Whitehead products replaced by higher-order Whitehead products of the kind considered in [17]. The present formulation seems to be more general in applicability and less Gothic in detail for the questions we study here.

3. Sections of projective bundles.

In this section we describe in detail the method outlined in section 2 for studying sections of projective bundles. The ideas are well known (see for example [1], [2], [12], [14]) but it is convenient to review them here.

As a preliminary we recall conventions about Hopf line bundles. Let V be a complex vector space and PV the corresponding projective space. For any non-zero vector v in V we write $[v]$ for the point in PV which is the line $\mathbf{C}v$ through the origin in V spanned by v. The *dual Hopf bundle* H^* over PV is the bundle whose fibre over a point $[v]$ of PV is the line $\mathbf{C}v$. The *Hopf bundle* H is the dual of H^*. We write H_V^*, H_V when we wish to emphasize the vector space in question. These notions readily extend to vector bundles: if ξ is a complex vector bundle, the Hopf bundle H_ξ and its dual H_ξ^* are both complex line bundles over $P\xi$.

For any complex line bundle λ over X, we define $\Gamma_\lambda P\xi$ to be the (possibly empty) subspace of $\Gamma P\xi$ consisting of all sections s such that $s^*H_\xi \cong \lambda$. Note that only the isomorphism class of λ matters in this definition, and that $N\xi$ in section 2 is $\Gamma_\lambda P\xi$ with $\lambda = \varepsilon_{\mathbf{C}}$. We can now list the steps in our approach to the components of $\Gamma P\xi$.

(1) Under suitable dimensional restrictions (Lemma 3.4) the components of $\Gamma P\xi$ are precisely the spaces $\Gamma_\lambda P\xi$ as λ ranges over the group $\operatorname{Pic} X$ of isomorphism classes of complex line bundles over X.

(2) We replace $\Gamma_\lambda P\xi$ by $N(\lambda\otimes\xi)$ using Corollary 3.3 below.

(3) We study any space of type $N\xi$ via the bundle projection of $\Gamma S\xi$ to $N\xi$ in Proposition 2.6.

Step (2) probably deserves most explanation, and we turn to it immediately. For simplicity let us first consider a single complex vector space V. Suppose that a 1-dimensional complex vector space L is also given. With these data, (cf. [1, p. 45]) there is no canonical isomorphism of V with $L\otimes V$, but there *is* a canonical projective equivalence of PV with $P(L\otimes V)$ given by $[v] \leftrightarrow [l\otimes v]$ for any non-zero vector l in L. We shall identify PV with $P(L\otimes V)$ in this way, but when it is desirable for clarity we name the equivalence θ: $PV \to P(L\otimes V)$. Now $H^*_{L\otimes V}$ over $P(L\otimes V)$ corresponds to $\varepsilon_L \otimes H^*_V$ over PV (with $l\otimes v$ over $[l\otimes v]$ corresponding to $l\otimes v$ over $[v]$). Hence $H_{L\otimes V}$ corresponds to $\varepsilon^*_L \otimes H_V$, so $\varepsilon_L \otimes H_{L\otimes V}$ corresponds to $\varepsilon_L\otimes\varepsilon^*_L\otimes H_V$, which is in turn canonically isomorphic to H_V. To sum up, H_V over PV corresponds to $\varepsilon_L\otimes H_{L\otimes V}$ over $P(L\otimes V)$, i.e. there is a canonical isomorphism between these bundles covering the equivalence θ.

By naturality this all passes to bundles. Let ξ be a complex vector bundle over X and let λ be a complex line bundle over X. Write p for the bundle projection of $P\xi$ or of $P(\lambda\otimes\xi)$. Then there is a canonical equivalence θ of $P\xi$ with $P(\lambda\otimes\xi)$ under which H_ξ corresponds to $p^*\lambda\otimes H_{\lambda\otimes\xi}$, or equivalently $p^*\lambda^*\otimes H_\xi$ corresponds to $H_{\lambda\otimes\xi}$.

Now let s: $X \to P\xi$ be a section. As in [14], s determines a 1-dimensional sub-bundle λ of ξ, whose fibre over a point x of X is the line $s(x)$. The next lemma is rather obvious.

LEMMA 3.1. *With the above notation,* $s^*H^*_\xi \cong \lambda$.

PROOF. We may define a tautologous bundle map from λ to H^*_ξ covering s, by mapping the vector v in the line $s(x)$ (where $s(x)$ is thought of as the fibre of λ over x) to v itself (where $s(x)$ is now thought of as the fibre of H_ξ over the point in $P\xi$ which is again the line $s(x)$).

LEMMA 3.2. *For any line bundles* λ, μ *over* X, $\Gamma_\lambda P\xi$ *is strongly equivalent to* $\Gamma_{\mu^*\otimes\lambda}P(\mu\otimes\xi)$.

PROOF. From the above discussion, there is a $1-1$ onto map from $\Gamma_\lambda P\xi$ to $\Gamma_{\mu^*\otimes\lambda}P(\mu\otimes\xi)$ in which any s in $\Gamma_\lambda P\xi$ is mapped to $\theta\circ s$. Note that

$$(\theta\circ s)^*H_{\mu\otimes\xi} \cong s^*(p^*\mu^*\otimes H_\xi) \cong \mu^*\otimes s^*H_\xi \cong \mu^*\otimes\lambda\,,$$

so $\theta\circ s$ is indeed in $\Gamma_{\mu^*\otimes\lambda}P(\mu\otimes\xi)$. It is straightforward to check that this is a strong equivalence.

COROLLARY 3.3. $\Gamma_\lambda P\xi$ *is strongly equivalent to* $N(\lambda\otimes\xi)$.

PROOF. Apply the lemma with $\mu=\lambda$ and recall that $N(\lambda\otimes\xi)$ is $\Gamma_\varepsilon P(\lambda\otimes\xi)$ where $\varepsilon=\varepsilon_{\mathbf{C}}$.

This completes our account of step (2) above. The relevant lemma for step (1) is well known at least when ξ is trivial, and in general it follows at once from Corollary 3.3 and Proposition 2.7(a).

LEMMA 3.4. *If* $\dim X < 2n+1$ *and* ξ *is a complex* $(n+1)$*-plane bundle over* X, *then* $\Gamma_\lambda P\xi$ *is a single component of* $\Gamma P\xi$ *for any* λ *in* $\operatorname{Pic} X$.

Finally we complete step (3) by proving Proposition 2.6. It is convenient first to prove Lemma 2.5.

PROOF OF 2.5. We write $\pi\colon S\xi \to P\xi$ for the natural projection. If s lifts to a section $\tilde{s}$ of $S\xi$ then $s^*H_\xi = \tilde{s}^*\pi^*H_\xi$, and π^*H_ξ is trivial since it is a line bundle with a nowhere-zero section given by the diagonal map of $S\xi$. Conversely suppose that s^*H_ξ is trivial. By Lemma 3.1 the line bundle λ picked out by s is trivial. Let $\tilde{s}$ be a nowhere-zero section of λ. We may clearly assume that $\tilde{s}(x)$ has unit norm for every x. Then $\tilde{s}$ defines a lift of s to $S\xi$.

PROOF OF 2.6. It is straightforward to check that G is a Lie group. (We analyse its structure in section 8.) Let $\Phi\colon \Gamma S\xi \to N\xi$ be the map given by $s \mapsto \pi \circ s$, where $\pi\colon S\xi \to P\xi$ is again the natural projection. By Lemma 2.5, Φ exists and is onto. Now $G = M(X, S^1)$ acts on $\Gamma S\xi$ by pointwise complex scalar multiplication, and $\Phi(g.s) = \Phi(s)$ for any g in G and s in $\Gamma S\xi$. Conversely, if $\Phi(s_1) = \Phi(s_2)$ then $s_1(x) = \mu_x . s_2(x)$ for some unique complex scalar μ_x of unit norm, and it is easy to check that μ_x varies continuously with x. Thus Φ induces a map $\Psi\colon \Gamma S\xi/G \to N\xi$ which is $1-1$ and onto.

It is now sufficient to prove that $\Gamma S\xi$ is locally G-diffeomorphic to $N\xi \times G$. The proof is analogous to the proof that H^* is locally a product over $\mathbf{C}P^n$.

Given s in $N\xi$, we first find a chart I for a neighbourhood of s in $N\xi$. A lift $\tilde{s}$ of s to $\Gamma S\xi$ gives an orthogonal splitting $\xi = \zeta \oplus \varepsilon_{\mathbf{C}}$, where ζ is a complex n-plane bundle and $\tilde{s}$ corresponds to the constant section 1 of $\varepsilon_{\mathbf{C}}$. Let i embed the total space $E\zeta$ of ζ in $P(\zeta \oplus \varepsilon_{\mathbf{C}}) = P\xi$ by $i(v) = [v, 1]$. Thus i is a fibrewise version of the usual chart $\imath\colon \mathbf{C}^n \to \mathbf{C}P^n$ which maps $(z_1, z_2, \ldots, z_n)$ to $[z_1, z_2, \ldots, z_n, 1]$. Let $\Gamma\zeta$ denote the Banach space of sections of the Hermitian vector bundle ζ. Then i induces an embedding I of $\Gamma\zeta$ in $\Gamma P\xi$, which is pointwise the chart $\imath$. The zero-section of ζ is mapped by i to s, and it is easy to check I maps $\Gamma\zeta$ onto an open subset, U say, of $N\xi$.

We now find a trivialization of $\Gamma S\xi$ over U. An explicit trivialization

$$\theta\colon E\zeta \times S^1 \to \pi^{-1}(i(E\zeta))$$

of $S\xi$ over $i(E\zeta)$ is given by

$$\theta(v, \lambda) = \lambda . (v, 1)/\|(v, 1)\| ,$$

where $\|\cdot\|$ denotes norm. Define $\Theta\colon \Gamma\zeta \times G \to \Phi^{-1}(U)$ by $\Theta(s', f)(x) = \theta(s'(x), f(x))$. Then Θ is G-equivariant since θ is S^1-equivariant, and it is straightforward to check that Θ is a diffeomorphism which gives rise to a local trivialization of $\Gamma S\xi$ as required.

4. Deducing results on function spaces.

First we derive Corollary 2.11. Let ε be a trivial $(n+1)$-plane bundle over X, so that $\lambda\otimes\varepsilon\cong(n+1)\lambda$ and similarly for μ. If $(n+1)\lambda$ and $(n+1)\mu$ are isomorphic then $N(\lambda\otimes\varepsilon)$ and $N(\mu\otimes\varepsilon)$ are strongly equivalent by Proposition 2.8(a), hence so too are $\Gamma_\lambda P\varepsilon$ and $\Gamma_\mu P\varepsilon$ by Corollary 3.3. Since ε is trivial we may identify $\Gamma_\lambda P\varepsilon$, $\Gamma_\mu P\varepsilon$ with M_λ, M_μ. Therefore M_λ and M_μ are strongly equivalent. Conversely, if M_λ and M_μ are homotopy equivalent with μ trivial, then so too are $N(\lambda\otimes\varepsilon)$ and $N(\mu\otimes\varepsilon)$, and by Theorem 2.10, $\lambda\otimes\varepsilon$ is trivial, i.e. $(n+1)\lambda$ is trivial.

Proposition 2.4 follows by taking $X=\mathbf{R}P^r$ in Corollary 2.11, in view of the next two remarks. First, $r<2n+1$ whenever $n+1$ is divisible by b_{r+1}, so in both parts of the proposition the components are M_λ and M_μ where λ is non-trivial and μ is trivial. (Recall that $r>1$, so Pic $(\mathbf{R}P^r)\approx\mathbf{Z}/2$.) Secondly, by definition of b_{r+1} it follows that $(n+1)\lambda$ is trivial if and only if $n+1$ is divisible by b_{r+1}.

Next we compute $\pi_1(M_k(\mathbf{C}P^n,\mathbf{C}P^n))$ using Theorem 2.12. Note that $M_k(\mathbf{C}P^n,\mathbf{C}P^n)$ may be identified with $\Gamma_\lambda P\varepsilon$ where ε is a trivial complex $(n+1)$-plane bundle over $\mathbf{C}P^n$ and $\lambda=H^k$. By Corollary 3.3, $\Gamma_\lambda P\varepsilon$ is strongly equivalent to $N\xi$ where $\xi=H^k\otimes\varepsilon\cong(n+1)H^k$. Then $c_n\xi[\mathbf{C}P^n]=\pm k^n(n+1)$, and Theorem 2.12(i) shows that $\pi_1(M_k(\mathbf{C}P^n,\mathbf{C}P^n))$ is cyclic of order $(n+1)|k|^n$ for $k\neq 0$.

Our other results on function spaces can be derived similarly. We omit details but offer the following guide. Those parts of Propositions 2.1–2.3 asserting existence of strong equivalences follow from Proposition 2.8 and its real analogue. The non-equivalence parts of Propositions 2.1(a) and 2.2 follow from Theorem 2.13, and the analogous parts of Propositions 2.1(b), 2.3 follow from Corollary 2.11, Proposition 2.15 respectively.

5. Basepoints.

We now prove Proposition 2.9 by establishing the following stronger results.

PROPOSITION 5.1. *Let ξ be a real vector bundle with a Riemannian metric over X and let s_0, s_1 be homotopic sections of $S\xi$. Then there is a strong equivalence of $\Gamma S\xi$ which maps s_0 to s_1 and is homotopic to the identity through strong equivalences.*

PROPOSITION 5.2. *Let ξ be a real or complex vector bundle with a Riemannian or Hermitian metric over X, and let s_0, s_1 in $\Gamma S\xi$ be such that their projections*

$[s_0], [s_1]$ in $N\xi$ are homotopic. Then there is a strong equivalence of $\Gamma S\xi$ which maps s_0 to s_1, and a strong equivalence of $N\xi$ which maps $[s_0]$ to $[s_1]$ and is homotopic to the identity through strong equivalences.

Each strong equivalence in these propositions may in fact be induced by an isometric automorphism of ξ. Proposition 5.1 follows quickly from the next lemma.

LEMMA 5.3. *Under the hypotheses of* 5.1, *there is an isometric automorphism f of ξ, homotopic to the identity through such automorphisms and such that $f \circ s_0 = s_1$.*

PROOF. Let s_t $(0 \leqq t \leqq 1)$ be a homotopy from s_0 to s_1. Suppose first that for each t the section $s_0 + s_t$ of ξ is nowhere-zero. Take f_t to be (fibrewise) the composition of reflection in the hyperplane perpendicular to s_0 and reflection in the hyperplane perpendicular to $s_0 + s_t$. Then $f_t \circ s_0 = s_t$ and $f_0 = 1$. We may take $f = f_1$ in this case.

In general, since X is compact, there is a real number $\delta > 0$ such that $s_t + s_u$ is nowhere-zero whenever $|t-u| < \delta$, and f can be taken to be the product of a finite number of automorphisms of the above type. Clearly f induces a strong equivalence of $\Gamma S\xi$ of the kind required in Proposition 5.1.

To prove Proposition 5.2 in the real case, we note that when $[s_0]$ is homotopic to $[s_1]$, there is an element b in $G = M(X, S^0)$ such that $T(b) \circ s_0$ is homotopic to s_1, where $T(b)$ is the isometric automorphism of $S\xi$ given by (fibrewise) multiplication by b. We apply Proposition 5.1 to $T(b) \circ s_0$ and s_1, to get an isometric automorphism f of ξ which is homotopic to the identity and maps $T(b) \circ s_0$ to s_1. Then $s \mapsto f \circ T(b) \circ s$ is a strong equivalence of $\Gamma S\xi$ mapping s_0 to s_1, and $[s] \to Pf \circ [s]$ is a strong equivalence of $N\xi$ which maps $[s_0]$ to $[s_1]$ and is homotopic to the identity through strong equivalences.

For the complex case of Proposition 5.2, we need one modification of the above proof. Given a in $G = M(X, S^1)$ and a nowhere-zero section s of ξ, we define the complex reflection $R(s;\, a)$ to be the (unitary) automorphism of ξ which fixes vectors orthogonal to s and multiplies s by a. We now re-define

$$f_t = R(s_0 + s_t;\, a_t) \circ R(s_0;\, -1)\,,$$

where

$$a_t = -(1 + \overline{\langle s_0, s_t \rangle})/(1 + \langle s_0, s_t \rangle)\,,$$

and $\langle \cdot, \cdot \rangle$ is the Hermitian inner product. (Note that this reduces to the previous f_t in the real case.) Then again $f_t \circ s_0 = s_t$ and $f_0 = 1$. The rest of the proof follows exactly as in the real case.

REMARK. We could prove that the homotopy type of $\Gamma S\xi$ or $N\xi$ as a pointed space is independent of choice of basepoint within a fixed component just by verifying that any basepoint in $\Gamma S\xi$ or $N\xi$ is non-degenerate. However, we believe that the strong homogeneity results in this section are also of interest.

6. Proofs of Theorem 2.10 and Proposition 2.15.

We deduce Theorem 2.10 from the following proposition, which is very similar to work of Arunas Liulevicius (see for example [15]). We include a proof for completeness.

PROPOSITION 6.1. *Let ξ, η be complex $(n+1)$-plane bundles over X and suppose that there exists a map ψ: $P\xi \to P\eta$ over X satisfying $\psi^* H_\eta \cong H_\xi$. Then ξ and η are stably isomorphic.*

PROOF. Recall from [1] that $K(P\xi)$ is a free $K(X)$-module on generators $1, [H_\xi], [H_\xi]^2, \ldots, [H_\xi]^n$ and that

$$\sum_{i=0}^{n+1} (-1)^i [\lambda^i \xi][H_\xi]^i = 0 . \tag{6.2}$$

Since $\lambda^{n+1}\xi$ is a line bundle, we may solve (6.2) for $[H_\xi]^{n+1}$:

$$[H_\xi]^{n+1} = [\lambda^{n+1}\xi]^{-1} \sum_{i=0}^{n} (-1)^{n-i} [\lambda^i \xi][H_\xi]^i .$$

There is an entirely similar equation with ξ replaced by η. Now $\psi^*[H_\eta] = [H_\xi]$, so $\psi^*[H_\eta]^i = [H_\xi]^i$ for all i. Also, ψ is a map over X, so $\psi^*[\lambda^i\eta] = [\lambda^i\eta]$. Hence

$$\begin{aligned}[\lambda^{n+1}\xi]^{-1} \sum_{i=0}^{n} (-1)^{n-i} [\lambda^i \xi][H_\xi]^i &= [H_\xi]^{n+1} = \psi^*[H_\eta]^{n+1} \\ &= [\lambda^{n+1}\eta]^{-1} \sum_{i=0}^{n} (-1)^{n-i} [\lambda^i \eta][H_\xi]^i .\end{aligned}$$

Equating coefficients of the generator 1 we get $[\lambda^{n+1}\xi] = [\lambda^{n+1}\eta]$, and then equating coefficients of $[H_\xi]$ we get $[\xi] = [\eta]$ as required.

Now in preparation for the proof of Theorem 2.10, let ξ, ε be complex $(n+1)$-plane bundles over X, with ε trivial and $\dim X < 2n-1$. Choose a Hermitian metric on ξ. For dimensional reasons, $\pi_i(\Gamma S\xi) = 0$ for $0 \leqq i \leqq 2$, so the homotopy sequence of the fibring in Proposition 2.6 tells us that $N\xi$ is connected, that $\pi_1(N\xi) \approx \pi_0(G) \approx H^1(X)$, and that $\pi_2(N\xi) \approx \pi_1(G) \approx \mathbf{Z}$. In the following diagram, each vertical map is evaluation ev_x at a point x in X.

$$\begin{array}{ccccc} G & \to & \Gamma S\xi & \to & N\xi \\ \downarrow & & \downarrow & & \downarrow \\ S^1 & \to & S\xi_x & \to & P\xi_x . \end{array}$$

The associated homotopy ladder shows that $ev_x: N\xi \to P\xi_x$ induces an isomorphism of π_2.

PROOF OF THEOREM 2.10. Suppose we are given a homotopy equivalence $f: N\varepsilon \to N\xi$. We shall construct a map $\psi: P\varepsilon \to P\xi$ satisfying the hypotheses of Proposition 6.1, and the required result will follow. Let $c: \mathbf{C}P^n \to N\varepsilon$ denote the inclusion of the constant section of $P\varepsilon$. Since $\pi_2(N\varepsilon) \approx \mathbf{Z}$ and c has a left inverse, c induces an isomorphism of π_2. Together with the previous paragraph, this shows that for any x in X, the composition

$$\mathbf{C}P^n \xrightarrow{c} N\varepsilon \xrightarrow{f} N\xi \xrightarrow{ev_x} P\xi_x \tag{6.3}$$

induces an isomorphism of π_2. Let us consider the adjoint of $f \circ c$ as a map $\theta: P\varepsilon \to P\xi$ over X. For any x in X, the restriction $\theta_x: P\varepsilon_x \to P\xi_x$ is essentially the composition (6.3), so θ_x induces an isomorphism of π_2. Now

$$\operatorname{Pic}(P\varepsilon) \approx \operatorname{Pic}(X \times \mathbf{C}P^n) \approx \operatorname{Pic} X \oplus \mathbf{Z} ,$$

(the group structure being given by tensor product of line bundles) so $\theta^* H_\xi \cong \lambda \otimes H^k$ for some line bundle λ over X and some integer k. But λ must be trivial, since for fixed z_0 in $\mathbf{C}P^n$, $\theta \mid X \times \{z_0\}$ defines a section $s = f(c(z_0))$ of $P\xi$ which is in $N\xi$, so $s^* H_\xi$ is trivial. Also, for any x in X, we have seen that

$$\theta_x: \{x\} \times \mathbf{C}P^n \to P\xi_x$$

induces an isomorphism of π_2, hence of cohomology $H^2(\cdot)$, so $k = \pm 1$. If $k = 1$ we take $\psi = \theta$ and if $k = -1$ we take ψ to be the composition of θ with complex conjugation on $P\varepsilon$. In either case ψ satisfies the hypotheses of Proposition 6.1 as required.

SKETCH PROOF OF PROPOSITION 2.15. This is similar to the proof of Theorem 2.10. Under the hypotheses of Proposition 2.15 we again get a map $\psi: \mathbf{R}P\varepsilon \to \mathbf{R}P\xi$ over X such that $\psi^* H_\xi \cong H_\varepsilon$. Hence ψ lifts to a map over X from $S\varepsilon$ to $S\xi$ of odd degree on each fibre, and the result follows.

7. Proofs of Theorem 2.12 and Proposition 2.7(b).

In the proof of Theorem 2.12, we concentrate on the case when X is oriented, omitting the obvious modifications needed in the (easier) case when X is non-orientable.

Under the given dimensional assumptions, $\Gamma S\xi$ and hence $N\xi$ are connected. We choose a basepoint s_0 in $\Gamma S\xi$. The fibring in Proposition 2.6 gives an exact sequence

$$(7.1) \qquad \ldots \to \pi_1(G) \xrightarrow{i_*} \pi_1(\Gamma S\xi) \xrightarrow{\Phi_*} = \pi_1(N\xi) \xrightarrow{\partial} \pi_0(G) \to 0\,.$$

The proof of Theorem 2.12 proceeds by applying obstruction theory to investigate this sequence. We shall use some facts about Euler classes, and we recall these in an appendix to this section. Note that as in section 6, $\pi_0(G) \approx H^1(X)$ and $\pi_1(G) \approx \mathbf{Z}$.

Our first goal is to construct an isomorphism

$$\theta\colon \pi_1(\Gamma S\xi) \to \mathbf{Z}$$

and show that the image of $\theta \circ i_*$ is generated by $c_n\xi[X]$. Let D^2 denote the unit 2-disc and write $p_2\colon D^2 \times X \to X$ for the projection. Given any element σ of $\pi_1(\Gamma S\xi)$, consider a loop representing σ and based at s_0. The adjoint of this loop is a partial section, s say, of $S(p_2^*\xi)$ over $S^1 \times X$. Let $e(p_2^*\xi, s)$ be the relative Euler class in $H^{2n+2}(D^2 \times X, S^1 \times X)$ which is the obstruction to extending s over $D^2 \times X$ (see appendix). We define $\theta(\sigma)$ to be the integer obtained by evaluating $e(p_2^*\xi, s)$ on the fundamental homology class of $(D^2 \times X, S^1 \times X)$. It follows from classical obstruction theory (see [7]) that θ is an isomorphism.

We now evaluate $\theta \circ i_*$. The image under i_* of a generator of $\pi_1(G)$ is represented by the loop $f\colon S^1 \to \Gamma S\xi$ such that $f(z)(x) = z.s_0(x)$ for any x in X and z in S^1 (where S^1 is viewed as the unit circle in $\mathbf{C}$). The adjoint of f is the partial section s of $p_2^*\xi$ over $S^1 \times X$ given by the lift $(z,x) \mapsto z.s_0(x)$ of $p_2\colon S^1 \times X \to X$ to $S\xi$. We wish to evaluate the relative Euler class of s.

As before, s_0 gives a splitting $\xi = \zeta \oplus \varepsilon_{\mathbf{C}}$. By multiplicativity of Euler classes (see appendix),

$$(7.2) \qquad e(p_2^*\xi, s) = e(\zeta).u\,,$$

where $e(\zeta)$ is the Euler class of ζ in $H^{2n}(X)$ and u is a generator of $H^2(D^2, S^1)$. Hence evaluating $e(p_2^*\xi, s)$ on the fundamental class of $(D^2 \times X, S^1 \times X)$ gives the same answer as evaluating $e(\zeta)$ on the fundamental class of X. Since $e(\zeta) = c_n\xi$, the image of $\theta \circ i_*$ is generated by $c_n\xi[X]$ as required.

Finally, we show that the extension

$$0 \to \mathbf{Z}/c_n\xi[X] \to \pi_1(N\xi) \xrightarrow{\partial} H^1(X) \to 0$$

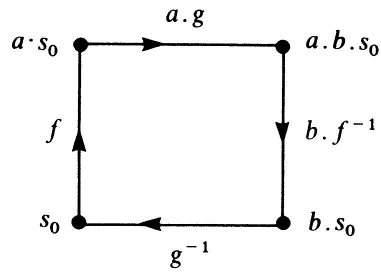
is central and is classified as stated in Theorem 2.12. It is sufficient to prove: given a, b in $H^1(X)$ and any α, β in $\pi_1(N\xi)$ satisfying $\partial\alpha = a$, $\partial\beta = b$, we have

$$\alpha\beta\alpha^{-1}\beta^{-1} = \Phi_*(\sigma) \quad \text{where } \theta(\sigma) = (2abc_{n-1}\xi)[X]\,.$$

For, applying this with $b = 0$ we see that the extension is central, and noting that $H^1(X)$ is finitely generated free Abelian and $\mathbf{Z}/c_n\xi[X]$ is Abelian, we see that the extension is classified as required.

We represent a, b by maps from X to S^1 also called a, b. In view of Proposition 2.6 we may represent α, β by maps $f, g\colon I \to \Gamma S\xi$ with $f(0)=g(0)=s_0$ and $f(1)=a.s_0$, $g(1)=b.s_0$. The commutator $\alpha\beta\alpha^{-1}\beta^{-1}$ is then represented by the loop σ in $\Gamma S\xi$ shown in diagram (7.3).

(7.3)



To determine $\theta(\sigma)$, we need to evaluate the obstruction to extending σ from S^1 (the boundary of the square in (7.3)) to D^2 (the interior of this square), or equivalently the relative Euler class of the corresponding partial section, s say, of $p_2^*\xi$ over $S^1 \times X$.

For this evaluation, we use the following device. Identify opposite edges of the square to form a torus $T = S_1^1 \times S_2^1$, where the circle S_1^1 corresponds to the vertical sides and S_2^1 to the horizontal sides. Let $\pi\colon (D^2, S^1) \to (T, S_1^1 \vee S_2^1)$ be the identification map. We construct a bundle over $T \times X$ whose pull-back under $\pi \times 1$ is $p_2^*\xi$. First consider the line bundle η_a over $S_1^1 \times X$ got by applying the mapping torus construction to $a\colon X \to S^1$, so the total space of η_a is the quotient space

$$I \times X \times \mathbf{C}/(0,x,z) \sim (1,x,a(x).z)\,,$$

where I is the unit interval and the indicated identifications are made for all x in X, all z in $\mathbf{C}$. Let η_a denote also the pull-back of this bundle by the projection of $T \times X$ on $S_1^1 \times X$. Define η_b analogously as the pull-back of a line bundle over $S_2^1 \times X$. It is straightforward to check that $p_2^*\xi$ is isomorphic to $(\pi \times 1)^*(\eta_a \otimes \eta_b \otimes \xi)$, and s is the pull-back of a partial section, s' say, of $S(\eta_a \otimes \eta_b \otimes \xi)$ over $(S_1^1 \vee S_2^1) \times X$. The obstruction in $H^{2n+2}(D^2 \times X, S^1 \times X)$ to extending s over $D^2 \times X$ is the image under the isomorphism $(\pi \times 1)^*$ of the obstruction in $H^{2n+2}(T \times X, (S_1^1 \vee S_2^1) \times X)$ to extending s' over $T \times X$. The advantage of this change of viewpoint is that the restriction of $H^{2n+2}(T \times X, (S_1^1 \vee S_2^1) \times X)$ to $H^{2n+2}(T \times X)$ is an isomorphism. It is therefore enough to calculate the (absolute) Euler class of $\eta_a \otimes \eta_b \otimes \xi$, or equivalently its Chern class c_{n+1}.

Now $c_1\eta_a = ua$ where u generates $H^1(S_1^1)$ and $c_1\eta_b = vb$ where v generates $H^1(S_2^1)$. Hence

$$c_1(\eta_a \otimes \eta_b) = ua + vb ,$$

and

$$c_{n+1}(\eta_a \otimes \eta_b \otimes \xi) = c_{n+1}\xi + c_n\xi(ua+vb) + + c_{n-1}\xi(ua+vb)^2 + \ldots + (ua+vb)^{n+1} .$$

But on dimensional grounds, $c_{n+1}\xi = 0$, $ac_n\xi = 0 = bc_n\xi$, and $(ua+vb)^i = 0$ for $i > 2$ while

$$(ua+vb)^2 = -2abuv .$$

Hence

$$c_{n+1}(\eta_a \otimes \eta_b \otimes \xi) = -2abuvc_{n-1}\xi$$

and the required result follows (the sign is a matter of convention).

PROOF OF PROPOSITION 2.7(b). This is similar to the proof of Theorem 2.12. In the exact sequence of pointed sets

$$\pi_0(G) \to \pi_0(\Gamma S\xi) \to \pi_0(N\xi) \to 0$$

arising from Proposition 2.6, the group $\pi_0(G)$ acts on $\pi_0(\Gamma S\xi)$ and $\pi_0(N\xi)$ is the quotient set. We apply obstruction theory to analyse this. Since ξ is a complex $(n+1)$-plane bundle and $\dim X \leqq 2n+1$, the difference construction d of [7] gives a bijection

$$\pi_0(\Gamma S\xi) \to H^{2n+1}(X)$$

defined by $s \mapsto d(s, s_0)$ for any section s representing a component of $\Gamma S\xi$. Let $a\colon X \to S^1$ represent an element of $\pi_0(G)$. To compute the action of $\pi_0(G)$ on $\pi_0(\Gamma S\xi)$ we need to compare $d(a.s, s_0)$ with $d(s, s_0)$. We have

$$d(a.s, s_0) = d(a.s, a.s_0) + d(a.s_0, s_0) ,$$

and it is classical (see appendix) that $d(a.s, a.s_0) = d(s, s_0)$. It is now enough to prove that $d(a.s_0, s_0) = ac_n\xi$, where on the right-hand side a is viewed as representing a class in $H^1(X) \approx \pi_0(G)$.

We can calculate $d(a.s_0, s_0)$ as follows. Let $p_2\colon I \times X \to X$ be the projection and consider the partial section s_1 of $p_2^*\xi$ over $\dot{I} \times X$ defined by $a.s_0$ on $\{0\} \times X$ and by s_0 on $\{1\} \times X$. Then $d(a.s_0, s_0)$ maps to $e(p_2^*\xi, s_1)$ under the obvious isomorphism of $H^{2n+1}(X)$ with $H^{2n+2}(I \times X, \dot{I} \times X)$. As in the proof of Theorem 2.12 and with similar notation, $e(p_2^*\xi, s_1)$ can be calculated using

$c_{n+1}(\eta_a \otimes \xi)$, and the latter is $uac_n\xi$ where u generates $H^1(S^1)$. This gives the required answer for $d(a.s_0, s_0)$.

PROPOSITION 2.7(b) IN THE REAL CASE. We consider the real analogue of Proposition 2.7(b) worth putting on record. Let ξ be a real $(n+1)$-plane bundle over X, and suppose that $\dim X \leqq n$. As before, $\pi_0(G)$ acts on $\pi_0(\Gamma S\xi)$ and $\pi_0(N\xi)$ is the quotient set. Again we choose a basepoint s_0 in $\Gamma S\xi$, and let ζ be the complementary bundle to s_0 in ξ. This time the difference construction gives a bijection

$$\theta \colon \pi_0(\Gamma S\xi) \to H^n(X; \mathbf{Z}(\xi)),$$

where $\mathbf{Z}(\xi)$ denotes the integers twisted by the orientation bundle of ξ. Also, $\pi_0(G) \approx H^0(X; \mathbf{Z}/2) \approx \mathbf{Z}/2$ (recall that X is connected); let t generate this group. The Chern class $c_n\xi$ is replaced by the Euler class e of ζ (thus $e \in H^n(X; \mathbf{Z}(\xi))$, where $\mathbf{Z}(\zeta)$ is appropriately identified with $\mathbf{Z}(\xi)$). We can now describe the action of $\pi_0(G)$ on $\pi_0(\Gamma S\xi)$, identifying the latter group with $H^n(X; \mathbf{Z}(\xi))$ via θ.

PROPOSITION 2.7(b)$_{\mathbf{R}}$. *With the above hypotheses and notation, t acts by the affine involution*

$$t(x) = \begin{cases} -x-e & \text{if } n \text{ is even}, \\ x-e & \text{if } n \text{ is odd}. \end{cases}$$

(Note that $2e=0$ when n is odd.)

The proof is as in the complex case.

Appendix to section 7.

For the reader's convenience we recall facts about Euler classes used above. Let Y be a subcomplex of a finite CW-complex X and let ξ be an oriented real $(2n+2)$-plane (or complex $(n+1)$-plane) bundle over X. Let $E\xi$ be the total space of ξ and $E_0\xi$ the subspace of all non-zero vectors. As usual we have a Thom class U_ξ in $H^{2n+2}(E\xi, E_0\xi)$. Let $s \colon Y \to E_0\xi$ be a section over Y. By for example Lemma 1.4.1 of [2], s extends to a section $\tilde{s} \colon X \to E\xi$, and (by the same lemma applied to $(X \times I, X \times \dot{I} \cup Y \times I)$) $\tilde{s}$ is unique up to homotopy through sections each of which restricts to s on Y. The relative Euler class $e(\xi, s)$ is by definition $\tilde{s}^* U_\xi$. Note that if Y is empty, we may take $\tilde{s}$ to be the zero-section, and the definition reduces to that of the Euler class $e(\xi)$.

Now suppose that we have two sets of data as above, (X_i, Y_i, ξ_i, s_i) for $i=1,2$. By the previous paragraph, the homotopy class of

$$s = \tilde{s}_1 \times s_2 \cup s_1 \times \tilde{s}_2 : X_1 \times Y_2 \cup Y_1 \times X_2 \to E_0(\xi_1 \times \xi_2)$$

is well-defined, and has $\tilde{s}_1 \times \tilde{s}_2$ as an extension over $X_1 \times X_2$. It is easy to check that

$$(7.4) \qquad e(\xi_1 \times \xi_2, s) = e(\xi_1, s_1).e(\xi_2, s_2).$$

In particular this applies if, say, Y_2 is empty.

To get (7.2), set $(X_1, Y_1) = (D^2, S^1)$ and $(X_2, Y_2) = (X, \varnothing)$. We may think of $p_2^*\xi = p_2^*(\zeta \oplus \varepsilon_{\mathbf{C}})$ as $\varepsilon_{\mathbf{C}} \times \zeta$ over $D^2 \times X$. Note that under the splitting $\xi = \zeta \oplus \varepsilon_{\mathbf{C}}$, the lift $(z, x) \mapsto z.s_0(x)$ of p_2 to $E_0\xi$ corresponds to the lift $(z, x) \mapsto 0 + z.1$ of p_2 to $E_0(\zeta \oplus \varepsilon_{\mathbf{C}})$. The partial section s of (7.2) is then $s_1 \times \tilde{s}_2$, where

$$s_1 : S^1 \to S^1 \times (\mathbf{R}^2 \setminus \{0\})$$

is given by $z \mapsto (z, z)$ and $\tilde{s}_2$ is the zero-section. It is easy to check that $e(\varepsilon_{\mathbf{C}}, s_1)$ is a generator u of $H^2(D^2, S^1)$, and (7.2) now follows from (7.4).

To check the equation $d(a.s, a.s_0) = d(s, s_0)$ used in the proof of Proposition 2.7(b), consider again the vector bundle $p_2^*\xi$ over $I \times X$ and let s' be the partial section of $S(p_2\xi)$ over $\dot{I} \times X$ defined by s on $\{0\} \times X$ and s_0 on $\{1\} \times X$. Then $d(s, s_0)$, $d(a.s, a.s_0)$ correspond to $e(p_2^*\xi, s')$, $e(p_2^*\xi, a.s')$ as in the proof of Proposition 2.7(b). We now replace $(I \times X, \dot{I} \times X)$ by any finite CW-pair (X, Y) with X connected, let ξ be a complex $(n+1)$-plane bundle over X, s a section of $S\xi$ over Y, and prove that for any map $a\colon X \to S^1$ we have $e(\xi, a.s) = e(\xi, s)$. For if $\tilde{s}$ is an extension of s, then $a.\tilde{s}$ may be taken as extension of $a.s$, and we get the commutative diagram

$$\begin{array}{ccccc} & \overset{\tilde{s}^*}{\nwarrow} & H^{2n+2}(E\xi, E_0\xi) & \xrightarrow{\approx} & H^{2n+2}(\mathbf{R}^{2n+2}, \mathbf{R}^{2n+2} \setminus \{0\}) \\ H^{2n+2}(X, Y) & & \uparrow a^* & & \uparrow a^* \\ & \underset{(a.\tilde{s})^*}{\swarrow} & H^{2n+2}(E\xi, E_0\xi) & \xrightarrow{\approx} & H^{2n+2}(\mathbf{R}^{2n+2}, \mathbf{R}^{2n+2} \setminus \{0\}). \end{array}$$

The horizontal isomorphisms are given by restriction to a point of X, and the right-hand vertical a^* is clearly the identity isomorphism of infinite cyclic groups. Hence $a^*U_\xi = U_\xi$ and $e(\xi, a.s) = (a.\tilde{s})^*U_\xi = \tilde{s}^*a^*U_\xi = \tilde{s}^*U_\xi = e(\xi, s)$ as required.

8. The homotopy type of $N\xi$.

As usual ξ is a complex $(n+1)$-plane bundle $(n > 0)$ with a Hermitian metric over X. We assume that $\Gamma S\xi$ is non-empty, choose a basepoint s_0 in $\Gamma S\xi$ and take its image $[s_0]$ as basepoint in $N\xi$. In this section we study the homotopy type of the pointed space $N\xi$ (which is independent of choice of s_0 within a fixed component of $\Gamma S\xi$, cf. section 5). At the end of the section we prove Theorem 2.13.

We begin by mentioning a qualitative result on the homotopy groups of $N\xi$. Recall that X is a finite complex.

PROPOSITION 8.1. *With notation as above,* $\pi_1(N\xi)$ *is finitely generated nilpotent. For* $i>1$, $\pi_i(N\xi)$ *is finitely generated.*

This follows by the method of [11, Chapter II.2]. We omit the details. In general, if s is a section of a locally trivial bundle with nilpotent fibre over a finite complex then the path-component of the space of sections containing s is a nilpotent space.

For our proof of Theorem 2.13 we shall use a classifying space for $G = M(X,S^1)$. We therefore study G more closely. Let us write $G(X)$ to emphasize its dependence on X.

LEMMA 8.2. *There is an isomorphism*

$$G(X) \approx V(X)\times H^1(X)\times S^1$$

for some Banach space $V(X)$.

PROOF. Choose a basepoint $*$ in X and let 1 be the basepoint in S^1. Let $F(X) \subset G(X)$ be the subgroup of all basepoint-preserving maps, and let us use a superscript zero to denote the component of the constant map to 1. Using the fact that $H^1(X)$ is finitely generated free Abelian, we may lift a set of generators to $F(X)$ and hence show

$$F(X) \approx F(X)^0\times H^1(X)\,.$$

Also, since S^1 is an Abelian group we have

$$G(X) \approx F(X)\times S^1\,.$$

Now any map in $F(X)^0$ lifts uniquely to a map $(X,*)\to(\mathbf{R},0)$. The set $V(X)$ of all maps of the latter type forms a Banach space with the sup norm, and the vector space addition in $V(X)$ corresponds to the pointwise multiplication in $F(X)^0$. This completes the proof.

We can now describe a classifying space for $G(X)$. Write $T(X)$ for the torus $H^1(X)\otimes\mathbf{R}/H^1(X)$ (the "Picard variety" of X). Then $T(X)$ is a classifying space for $H^1(X)$.

COROLLARY 8.3. *We may take* $BG(X)$ *to be* $T(X)\times\mathbf{C}P^\infty$.

This follows from Lemma 8.2 since $V(X)$ is contractible. We now fix a basepoint in $\mathbf{C}P^\infty$, so that as pointed space it classifies S^1-bundles over pointed complexes with a given trivialization over the basepoint. From Corollary 8.3 we get:

PROPOSITION 8.4. *For any pointed finite CW-complex K, we have*

$$[K; BG(X)] \approx [K; T(X)] \times [K; CP^\infty] \approx (H^1(K) \otimes H^1(X)) \oplus H^2(K) .$$

In particular, $\pi_1(BG(X)) \approx H^1(X)$, $\pi_2(BG(X)) \approx \mathbf{Z}$ *and* $\pi_i(BG(X)) = 0$ *for* $i > 2$.

Next we want to consider a classifying map for the bundle Φ: $\Gamma S\xi \to N\xi$. To prepare for this we note:

LEMMA 8.5. *The space* $N\xi$ *is paracompact and has the homotopy type of a CW-complex.*

PROOF. Since $N\xi$ is metrizable it is paracompact. Also, $N\xi$ is easily seen to be ELCX in the sense of [16] and hence by [16] it has the homotopy type of a CW-complex.

In fact one may use Proposition 8.1 and the criterion of Wall [19] to show that $(N\xi)^0$ is of finite type; we shall not require this for present purposes.

We may now let σ (or when appropriate $\sigma(\xi)$) denote a classifying map for the bundle Φ: $\Gamma S\xi \to N\xi$ of Proposition 2.6. It will be important for the proof of Theorem 2.13 to have a more concrete description of the induced function σ_*: $[K; N\xi] \to [K; BG(X)]$ for finite pointed CW-complexes K.

LEMMA 8.6. *Let K be a connected finite pointed CW-complex and let* f: $K \to N\xi$ *represent a class* $[f]$ *in* $[K; N\xi]$. *Let* s: $K \times X \to P\xi$ *be the lift determined by f of the projection* p_2: $K \times X \to X$. *Then* s^*H_ξ *is trivial over* $\{k\} \times X$ *for each k, so is classified by say* α *in* $(H^1(K) \otimes H^1(X)) \oplus H^2(K) \subset H^2(K \times X)$; α *corresponds to* $\sigma_*[f]$ *under the correspondence in Proposition* 8.4.

PROOF. The first assertion is clear by definition of $N\xi$. Now consider the two natural transformations

$$[K; N\xi] \to [K; BG(X)] \approx (H^1(K) \otimes H^1(X)) \oplus H^2(K)$$

given by σ_* and by the construction in the statement of the lemma. We wish to show that they coincide.

To show this on the component $H^2(K)$ it is enough to look at the restriction from X to a point, in other words to check it for $X = *$. But it is easy to see that in this case the two constructions are identical.

For the component $H^1(K) \otimes H^1(X)$ we use naturality and note that it is enough to check the case $K = S^1$. In this case there is an element a in $G(X)$ such that s^*H_ξ is isomorphic to the line bundle η_a over $S^1 \times X$ given by the mapping torus construction as in section 7. This line bundle is classified by $[a] \in H^1(X)$.

On the other hand, the principal $G(X)$-bundle over S^1 induced by $f: S^1 \to N\xi$ from the bundle $\Phi: \Gamma S\xi \to N\xi$ is isomorphic to the bundle whose fibre over a point z in S^1 is the space of sections of $S(\eta_a | \{z\} \times X)$; this is obtained by the mapping torus construction on α and is classified by $[a] \in H^1(X) \approx [S^1; BG(X)]$ as required.

From now on we assume that $m = \dim X < 2n+1$ so that $\Gamma S\xi$ and $N\xi$ are connected.

PROPOSITION 8.7. *With hypotheses as above,*

$$\sigma_* : \pi_i(N\xi) \to \pi_i(BG(X))$$

is bijective if $i < 2n - m + 1$ *and surjective if* $i \leqq 2n - m + 1$.

PROOF. We use the homotopy sequence of the fibring $\Phi: \Gamma S\xi \to N\xi$. As usual, the boundary operator of this sequence may be identified with σ_* via the transgression isomorphism of $\pi_{i-1}(G(X))$ with $\pi_i(BG(X))$. The result follows since $\pi_i(\Gamma S\xi) = 0$ for $i < 2n - m + 1$ on dimensional grounds.

We apply Proposition 8.7 to get the next lemma, which is used in the proof of Theorem 2.13.

LEMMA 8.8. *Let* ξ *and* η *be complex* $(n+1)$*-plane bundles over* X *with* $\dim X < 2n - 1$. *For any map* $f: N\xi \to N\eta$, *there is a map* $f_\infty: BG(X) \to BG(X)$ *(unique up to homotopy) such that the following diagram homotopy commutes:*

$$\begin{array}{ccc} N\xi & \xrightarrow{\sigma(\xi)} & BG(X) \\ \downarrow f & & \downarrow f_\infty \\ N\eta & \xrightarrow{\sigma(\eta)} & BG(X) . \end{array}$$

PROOF. Recall that $N\xi$ and $N\eta$ have the homotopy type of CW-complexes, and so has $BG(X)$ by Corollary 8.3. Using Propositions 8.4 and 8.7 the result now follows by standard obstruction theory, viewing $\sigma(\xi)$ as an inclusion, and examining the obstruction to extending $\sigma(\eta) \circ f$ over $BG(X)$.

By analysing further the homotopy classes of self-maps of $BG(X)$ one can prove that if $N\xi$ and $N\eta$ are homotopy equivalent then one can get an S^1-equivariant homotopy equivalence from $\Gamma S\xi$ to either $\Gamma S\eta$ or $\Gamma S\bar{\eta}$ (assuming still that $\dim X < 2n - 1$). We shall not use this result, so we merely prove the easier real analogue, which we do need (see section 2).

REMARK 8.9. *Let* ξ, η *be real* $(n+1)$*-plane bundles over* X *with* $\dim X < n - 1$. *Then any homotopy equivalence of* $N\xi$ *with* $N\eta$ *lifts to a* $\mathbf{Z}/2$*-equivariant homotopy equivalence of* $\Gamma S\xi$ *with* $\Gamma S\eta$.

PROOF. As we mentioned in section 2, $N\xi$ is locally path-connected. We observe that $\Gamma S\xi$ is connected, $\pi_1(N\xi) \approx \mathbf{Z}/2$ and the map from $\Gamma S\xi$ to $N\xi$ (which is the real analogue of Φ in Proposition 2.6) is a universal cover; and similarly with ξ replaced by η. The result readily follows.

We now come to the key lemma for the proof of Theorem 2.13.

LEMMA 8.10. *Let ξ, η be complex $(n+1)$-plane bundles over X with $\dim X = m < 2n-1$, and suppose that $N\xi$ and $N\eta$ are homotopy equivalent. Let μ be any complex line bundle over a finite pointed CW-complex K such that $\dim K < 2n-m+3$ and $H^1(K)=0$. Then at least one of $c_{n+1}(\mu\otimes\xi)$, $c_{n+1}(\mu\otimes\bar{\xi})$ vanishes if and only if at least one of $c_{n+1}(\mu\otimes\eta)$, $c_{n+1}(\mu\otimes\bar{\eta})$ vanishes.*

PROOF. Since $H^1(K)=0$, it follows from Proposition 8.4 that $[K; BG(X)] \approx H^2(K)$. Now we prove that the following conditions are equivalent:

(i) $c_{n+1}(\mu\otimes\xi)=0$,
(ii) the class $[\mu]$ lies in the image of

$$\sigma_*\colon [K; N\xi] \to [K; BG(X)] \approx H^2(K) .$$

For each of these is equivalent to:
(iii) $N(\mu\otimes\xi)$ is non-empty.
The equivalence of (i) and (iii) follows immediately from Proposition 2.7(c) applied to $K \times X$. It is straightforward to check that the equivalence of (ii) and (iii) follows from Corollary 3.3 (applied to bundles over $K \times X$) and Lemma 8.6.

Let $f\colon N\xi \to N\eta$ be a homotopy equivalence. From Lemma 8.8 we have a commutative diagram of set functions, for which we use the same names as the maps inducing them:

$$\begin{array}{ccc} [K; N\xi] & \xrightarrow{\sigma(\xi)} & [K; BG(X)] \approx H^2(K) \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_\infty}\downarrow \\ [K; N\eta] & \xrightarrow{\sigma(\eta)} & [K; BG(X)] \approx H^2(K) . \end{array} \tag{8.11}$$

From the proof of Lemma 8.8 it is clear that f_∞ here is multiplication by ± 1. Lemma 8.10 may now be deduced as follows. Suppose that $c_{n+1}(\mu\otimes\xi)=0$ or $c_{n+1}(\mu\otimes\bar{\xi})=0$. Note that the latter condition is equivalent to $c_{n+1}(\bar{\mu}\otimes\xi)=0$. By the equivalence of (i) and (ii) we get that $\pm[\mu]$ lies in the image of $\sigma(\xi)$. Commutativity of (8.11) now yields that $\pm[\mu]$ is in the image of $\sigma(\eta)$, and again by equivalence of (i) and (ii), either $c_{n+1}(\mu\otimes\eta)=0$ or $c_{n+1}(\mu\otimes\bar{\eta})=0$. Clearly this argument reverses to complete the proof.

PROOF OF THEOREM 2.13. We show first that the case $m=n$ can be settled by looking at fundamental groups. By Theorem 2.12(i), $c_m\xi=0$ if and only if

$$\operatorname{rank}\pi_1(N\xi) = \operatorname{rank}H^1(X)+1 ,$$

and if $c_m\xi$ is non-zero, then the number of elements of finite order in $\pi_1(N\xi)$ is $|c_m\xi[X]|$. Similar statements hold for η. But if $N\xi$ and $N\eta$ are homotopy equivalent then certainly their fundamental groups are isomorphic, and it follows that $c_m\xi = \pm c_m\eta$.

Assume now that X is a closed oriented $2m$-manifold with $m<n$. Theorem 2.13 will be established in this case by applying Lemma 8.10 with various (K,μ). Fix a prime power $q>1$. Let $\pi\colon S \to \mathbf{C}P^{n-m+1}$ be the projection of the S^1-bundle associated with the tensor power H^q, so that S is a lens space of dimension $2(n-m)+3$. Let μ be the line bundle π^*H over S. For K we take (a finite complex of dimension $2(n-m+1)$ homotopy equivalent to) the complement of an embedded open $(2(n-m)+3)$-disc in S. Then $H^{2(n-m+1)}(K)$ is cyclic of order q, generated by $(c_1\mu)^{n-m+1}$. Also,

$$H^{2(n+1)}(K\times X) \approx H^{2(n-m+1)}(K)\otimes H^{2m}(X)$$

and

$$c_{n+1}(\mu\otimes\xi) = (c_1\mu)^{n-m+1}.c_m\xi .$$

Hence by Lemma 8.10, $c_m\xi[X]\equiv 0 \pmod q$ if and only if $c_m\eta[X]\equiv 0 \pmod q$. (Note that $c_m\bar{\xi} = \pm c_m\xi$.) Since this holds for all q, $c_m\xi[X] = \pm c_m\eta[X]$.

REMARKS. (a) The group of kth roots of unity acts on $S\xi$ for any complex vector bundle ξ. The orbit space of this action is a bundle of lens spaces over X. The methods of this paper may be used to study its space of sections.

(b) Considerations of length have prevented our treating also spaces of basepoint-preserving maps here, but our methods give some results on them, too. It is perhaps worthwhile mentioning that, in contrast to the classical cases when Y is an H-space or X is a co-H-space, in general distinct path-components of a space $F(X,Y)$ of basepoint-preserving maps need not be homotopy equivalent. An example is given by $F(\mathbf{R}P^r, \mathbf{R}P^n)$ for $1<r<n$ and n even: one component has π_{n-1} finite and the other has π_{n-1} infinite.

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